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Quasinormed Cones and Bicompletion Isometries

Manal Yagoub Ahmed Juma
Department of mathematic, College of Science, Qassim University, Buraidah, Saudi Arabia.
M.juma@qu.edu.sa

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Abstract:
Since (X, e_(p_j)) is an extended quasi-metric cone, we demonstrate how any quasi-norm p_jon an actual cancellative cone X naturally implies an extended quasi-metric e_(p_j) on that cone. We demonstrate that bicompletion respects the structure of a quasi-normalized cone under bijective isometries. In fact, we find that isometries are not generally injective in this case. In addition, a few situations are shown.

Key words: Bicompletion, quasi-cone, calculative, bijective isometry, injective.

1. Introduction and Preliminary Information

The letters $\mathbb{R}^+$, ω, and N will be used for the sets of nonnegative real numbers, nonnegative integer numbers, and positive integer numbers, respectively, throughout this essay. The letters $\mathbb{R}^+$, ω, and N will be used for the sets of nonnegative real numbers, nonnegative integer numbers, and positive integer numbers, respectively, throughout this essay. Remind that a semigroup $(X, \mathbb{R}^+)$ is a monoid if its neutral element is 0 [2]. For each $x, y\ in\ X$ and $r, s\ in\ \mathbb{R}^+$,the cone (on $\mathbb{R}^+$) is defined as a triple $((X, \mathbb{R}^+, \cdot)$ where $(X, +)$ is an Abelian monoid and $\cdot$ is a function from $\mathbb{R}^+ \times X$ to X.

(a) $r \cdot (s \cdot x) = (rs) \cdot x$;

(b) $r \cdot (x + y) = (r \cdot x) + (r \cdot y)$;
(c) $(r + s) \cdot x = (r \cdot x) + (s \cdot x)$;
$(d) 1 \cdot x = x$.

Every element $x \in X$ that allows an inverse is distinct and is denoted by -x, as is normal. When Y is a part of X and $+|_Y$ and $\cdot|_Y$are the limits of + and to Y, respectively, the cone $(Y, +|_Y, \cdot|_Y))$ is said to be a subcone of a cone $(X, +, \cdot)$.Assume that $(X, +, \cdot)$ is a cone. as long as $f_j(x + y) \leq f_j(x) + f_j(y)$

Definition (1.1): The function $f_j : X \to \mathbb{R}$ is considered subadditive for any $x, y \in X$. For any x in X and $r\ in$, $\mathbb{R}^+$, a quasi-norm on a cone (X,+,·) is a subadditive function $p_j : X \to \mathbb{R}^+$ such that :

(a) $x = 0$ if and only if $-x \in X$ and $p_j(x) = p_j(-x) = 0$,and

(b) $p_j(r \cdot x) = rp_j(x)$.

Definition (2.1): A quasi-norm p on X is a norm on a cone $(X, +, \cdot)$ if it satisfies the following needs:

$p_j(x) = 0$ if only if $x = 0$.

Definition (3.1): The cancellative cone $(X, \mathbb{R}^+, \cdot)$is defined as follows: $x, y, z \in X, z + x = z + y$ implies $x = y$ for any $x, y, z \in X$.

One relatively direct way of interpreting any linear space X, R+, · as cone consists in the fact that the operation · has to be confined to R+ × X. As is well documented, every norm on a space X generates a metric on X. We continue this useful result by showing how quasi-metrics can naturally arise from quasi-norms on cancellative cones. In this work, we analyze the bicompletion of these structures for bijective isometries. This we prove to be a noninjective isometry between quasi-normed cones. We note that the standard Sorgenfrey line on R+ can be obtained through the extended quasi-metric created by a norm on R+ itself.

Moreover, we extend the approach to complexity functions, giving a rather remarkable example of a space documented in different parts of Theoretical Computer Science (see Example 3.2 below). The notion of quasi-metric space is specially introduced and developed in [1].

A quasi-metric X → ℝ + explains if the set X describes a nonnegative real number element in the set X. However, this definition must be true for all sets of elements taken from elements set x, y, and z, Elements set X:

(a) $d(x, y) = d(y, x) = 0$ if and only if $x = y$ and

(b) $d(x, z) \leq d(x, y) + d(y, z)$:

We'll also talk about extended quasi-metric. Except for the fact that $d(x, y) = +\infty$.is permitted, they adhere to the first three axioms.A (n extended) quasi-metric space is a pair (X, d) in which X is a (nonempty) set and d is a (n extended) quasi-metric on X. With the family of open d-balls $\{B_d(x, \rho) : x \in X, \ \rho > 0\}$, as its basis, every extended quasi-metric d on a set X yields a T_0 topology $\mathcal{T}(d)$on X; for all $x \in X$ and $\rho > 0$, $B_d(x, \rho) = \{y \in X : d(x, y) < \rho\}$

$d^s(x, y) = \max\{d(x, y), d(y, x)\}$, defined on $X \times X$, is a (n extended) metric on X if d is a (n extended) quasi-metric on a set X.

When d^s is a complete extended metric on a set X, then d on X is a bicomplete extended quasi-metric.

2. Producing Extended Quasi Metrics

An extended quasi-metric d on a cone $(X, +, \cdot)$is considered constant if $d(x + z, y + z) = d(x, y)$ and $d(\rho x, \rho y) = \rho d(x, y)$, as in [3].assuming $\rho \in \mathbb{R}^+$ and $x, y \ and \ z \in X$.

Definition (2.1): A pair (X, d) is considered to be an extended quasi-metric cone if X is a cone and d is an invariant extended quasi-metric on X.
Assume $(X, +, \cdot)$ is a cone. For each $x \in X$, the formula $x + X = \{x + y : y \in X\}$ is defined.

Proposition (2.2): Consider that on the cancellative cone $(X, +, \cdot)$. p is a quasi-norm. e_{p_j}defined on $X \times X$ is an invariant extended quasi-metric on X If $x \in X$ and $y \in x + X$ with $y = x + a$, and $e_{p_j}(x,y) = +\infty$ if $x \in X$ and $y \notin x + X$, respectively. Consequently, $\left(X, e_{p_j}\right)$ is a stretched quasi-metric cone. Also, for every x in X, ρ in $\mathbb{R}^+$, and every $\mu > -1$, the translations are $T\left(e_{p_j}\right)$--open, and $B_{e_{p_j}}(x, \mu + 1) = \delta x + \{y \in X : p_j(y) < \rho(\mu + 1)\}$

Proof:

If $e_{p_j}(x,x) = p_j(0) = 0$, then For any $x \, in \, X$. Allow $e_{p_j}(x,y) = e_{p_j}(y,x) = 0$at this point.Hence, $y = x + a$, $x = y + b$, occur when $a, b \in X$. Given that $a + b = 0$ and X is cancellative, we can deduce that b=-a.Thus, $a = 0$ since $p_j(a) = p_j(-a) = 0$. Therefore, $x = y$.

Furthermore, we show that for each $x, y, z \, in \, X$, $e_{p_j}(x,z) e_{p_j}(x,y) + e_{p_j}(y,z)$,Think about just the case when $y \in x + X$ and $z \in y + X$. At that point, $e_{p_j}(x,y) = p_j(a)$ and $e_{p_j}(y,z) = p_j(b)$. have the properties $y = x + a$, $z = y + b$ for each $a, b \in X$. Because of this, $z = x + a + b$, and so

$$e_{p_j}(x,z) = p_j(a + b) \leq p_j(a) + p_j(b) = e_{p_j}(x,y) + e_{p_j}(y,z).$$

From this, we conclude that e_{p_j} is an extended quasi-metric on X. We then verified that e_{p_j} is invariant. Let x, y, and z be part of X. If $e_{p_j}(x + z, y + z) = +\infty$., then $e_{p_j}(x,y) = +\infty$ as X is cancellative. Otherwise, assume that an is such that $e_{p_j}(x + z, y + z) = p_j(a)$.

If $+z = x + z + a \;\; then \; y = x + a$, and $e_{p_j}(x,y) = p_j(a)$.

The same as we derive that $e_{p_j}(\rho x, \rho y) = \rho e_{p_j}(x,y)$.) for all $x, y \in X$ and $\rho \in \mathbb{R}^+$. Similarly, for any $x, y \in X$ and $\rho \in \mathbb{R}^+$, we find that $e_{p_j}(\rho x, \rho y) = \rho e_{p_j}(x,y)$. Finally, recall that for each x in X and $\rho \; in \; \mathbb{R}^+$, $e_{p_j}(0,x) = p_j(x)$ .Thus, for any $\mu > -1$,we get rB_(e_(p_j)) (0,δ)=B_(e_(p_j)) (0,ρδ) and $B_{e_{p_j}}(0, \mu + 1) = \{x \in X : p_j(x) < \delta\}$ .It is obvious that for each $x \; in \; X$ and each δ > 0,

$$\rho B_{e_{p_j}}(x, \delta) = \rho x + B_{e_{p_j}}(0, \rho\delta),$$

$\mathcal{T}\left(e_{p_j}\right)$ −open is the translation for + and · as a result.

Example (2.3): For every x in $\mathbb{R}^+$, find a quasi-norm p such that $p_j(x) = 0$, given the standard addition and product on $\mathbb{R}^+$.The Alexandr off extended quasi-metric on $\mathbb{R}^+$ is the extended quasi-metric in this instance, or $e_{p_j}(x,y) = 0$ if $x \leq y$ and $e_{p_j}(x,y) = +\infty$ otherwise.

Example (2.4): For each x in $\mathbb{R}^+$, $p_j(x) = x$, therefore let $p_j : \mathbb{R}^+ \to \mathbb{R}^+$. p_j is clearly a norm on $\mathbb{R}^+$, and if $x \leq y$, then $e_{p_j}(x,y) = +\infty$ and $y - x$. The Sorgenfrey topology on $\mathbb{R}^+$ is thus the topology generated by e_{p_j} , since e_{p_j} is the Sorgenfrey extended quasi-metric on $\mathbb{R}^+$.

Remark (2.5):Because p_j is a (quasi) norm on a linear space $(X,+,\cdot)$, the (extended) quasimetric $p_j$of Proposition 2.2 is the classical (quasi)metric on X that p_j generates. $e_{p_j}(x,y) = p_j(y-x)$,

for all x, y belonging to X.Since there is a good solution to the bicompletion problem in the context of quasi-metric spaces ([5]), we will focus on quasi-norms defined on cancellative cones. However, some instances of spaces that naturally emerge from modeling particular processes in Theoretical Computer Science can be viewed as extended cancellative quasi-metric cones (see Example 3.2) below). Consequently, we propose the following notion.

Definition (2.6): Quasi-normed cones are pairs (X, p_j) in which X is a cancellative cone and p_j is a quasi-norm on X.

3. The Semi-Normal Cone's Bicompletion

Remember that $f_j : X \to Y$ is a linear function from a cone $(X,+,\cdot)$ to a cone $(Y,\oplus,\otimes)$. such that $f_j(\alpha \cdot x + \beta \cdot y) = \alpha \otimes f_j(x) \oplus \beta \otimes f_j(y)$.

Definition (3.1):The quasi-normed cones (X, p_j) and (Y, q_j). are isometric to a linear function $f_j : X \to Y$. It guarantees that for every $x \, in \, X$, $q_j\left(f_j(x)\right) = p_j(x)$. The following illustration shows that, unlike the quasi-metric case, there are non-injective isometries between quasi-normed cones.

Example (3.2): Taking cue from [6]'s applications for program and algorithmic complexity analysis, [4] introduces and investigates the idea of the so-called dual complexity space, which consists of the pair$(\mathcal{C}^*, d_{\mathcal{C}^*})$, where

$$\mathcal{C}^* = \left\{ f_j \in (\mathbb{R}^+)^\omega : \sum_j \left(\sum_{n=0}^{\infty} 2^{-n} f_j(n) < +\infty \right) \right\},$$

and $d_{\mathcal{C}^*}$ is the quasi-metric on $\mathcal{C}^*$ given by

$$d_{\mathcal{C}^*}(f_j, g_j) = \sum_j \sum_{n=0}^{\infty} 2^{-n} \left[\left(g_j(n) - f_j(n) \right) \vee 0 \right].$$

There are several $d_{\mathcal{C}^*}$properties discussed in [4]. Note specifically that T_1 topology is not caused by $d_{\mathcal{C}^*}$. With the neutral element $f_{0_j} \in \mathcal{C}^*$ given by $f_{0_j}(n) = 0$ for all $n \in \omega$, and $\cdot$ being the operation specified by $\left(\lambda \cdot f_j\right)(n) = \lambda f_j(n)$ for all $n \in \omega$, $(\mathcal{C}^*,+,\cdot)$ is clearly a cancellative cone.
Let's say that

$$\sum_j p_j(f_j) = \sum_j \left(\sum_{n=0}^{\infty} 2^{-n} f_j(n) \right)$$

Suppose that $p_j : \mathcal{C}^* \to \mathbb{R}^+$.The fact that p_j is a quasi-norm on $\mathcal{C}^*$ is well accepted. Next, the induced extended quasi-metric p_j on $\mathcal{C}^*$ is given by

$$\sum_j e_{p_j}(f_j, g_j)^* = \sum_j \left(\sum_{n=0}^{\infty} 2^{-n} \left(g_j(n) - f_j(n) \right) \right)$$

if $f_j \leq g_j$, and $e_{p_j}(f_j, g_j) = +\infty$ otherwise.

Let $X = \{f_j \in \mathcal{C}^* : f_j(0) > 0\} \cup \{f_{j_0}\}$. It is frequently observed that X is a subclone of $\mathcal{C}^*$

$q(f_j) = f_j(0)$ is the definition of $f_j : X \to \mathbb{R}^+$. There is no doubt that q_j is a quasi-norm on X. Let $F(f_j)(0) = f_j(0)$ and $F(f_j)(n) = 0$ define $F : X \to \mathcal{C}^*$ for each $f_j \in X$and $n \in \mathbb{N}$. F is obviously linear from $(X, +, \cdot)$ to $(\mathcal{C}^*, +, \cdot)$. Moreover, for any $f_j in\, X$,

$$\sum_j p_j \left((F(f_j) \right) = \sum_j \left(\sum_{n=0}^{\infty} 2^{-n} F \left(f_j(n) \right) \right) = \sum_j f_j(0)_j = \sum_j q_j(f_j)$$

F, thus, is an isometry between (X, q_j) and $(\mathcal{C}^*, p_j)$.

But F is not injective if $f_j, g_j \in X$ satisfy $f_j(0) = g_j(0)$ and $f_j(1) \neq g_j(1)$.This is because we get $F(f_j) = F(g_j)$.F, thus, is an isometry between (X, q_j) and $(\mathcal{C}^*, p_j)$. However, if $f_j, g_j \in X$ complete $f_j(0) = g_j(0)$ and $f_j(1) \neq g_j(1)$, then F is not injective. The reason for this is that we obtain $F(f_j) = F(g_j)$.

Definition (3.3): It is contended that two quasi-normed cones (X, f_j) and (Y, q_j), are isometric if there is a bijective isometry $f_j : X \to Y$ between them.

Proposition (3.4): The quasi-metric spaces (X, p_j) and (Y, q_j) are also isometric to f_j if a (bijective) isometry f_j isometric to(X, e_{p_j}) and (Y, e_{q_j})quasi-normed cones.

Proof: Let x, y are in X. $e_{q_j}\left(f_j(x), f_j(y)\right) = e_{p_j}(x, y) = +\infty$. is equal to e_(q_j). whether $f_j(y) \in f_j(x) + Y$. For some $z\, in\, Y$, $e_{q_j}\left(f_j(x), f_j(y)\right) = q_j(z)$if $f_j(y) = f_j(x) + z$. Only one $a \in X$ can be covered by $f_j(a) = z$ since f is bijective. Because of this

$f_j(y) = f_j(x) + f_j(a) = f_j(x + a)$.

Since $y = x + a$, $e_{p_j}(x, y) = p_j(a)$. follows. (X, e_{p_j}) and (Y, e_{q_j}) are isometrically represented by f, we find.

Definition (3.5): An extended quasi-metric e_{p_j} is bicomplete on X, and a quasi-normed cone (X, p_j) is bicomplete if it is.

Definition (3.6): Consider the quasi-normed cone (X, p_j) .If (X, p_j) .is isometric to a dense subspace of (Y, q_j) in the extended metric space $\left(y, \left(e_{q_j}\right)^s\right)$, then (X, p_j) is a bicompletion in terms of a bicomplete quasi-normed cone (Y, p_j).

One bicompletion of each quasi-normed cone (X, p_j) is $(\tilde{X}, \tilde{p}_j)$ which is isometric to any bicompletion of (X, p_j) .This is what we shall demonstrate.

$\left(X, e_{p_j}\right)$ is the symbol for the extended quasi-metric space created by (X, p_j).In the extended metric space $\left(X, \left(e_{p_j}\right)^s\right)$, $\widetilde{X}$ is the sum of all Cauchy sequences. Consider that for each

$\mu > -1$, there exists n_0 ∈N such that $\left(e_{p_j}\right)^s (x_n, x_m) < \mu + 1$,and for any $m, n \geq n_0$,

$x_{\ m} \in x_n + X$ if˜ $x := (x_n)_{n\in\mathbb{N}} \in \tilde{X}$

R is a relation of $\tilde{X}$ that has the following definition: For each $x := (x_n)_{n\in\mathbb{N}}$ and $y := (y_n)_{n\in N}$ in $\hat{X}$, set $Ry \Leftrightarrow \lim_{n\to\infty} \left(e_{p_j}\right)^s (x_n, y_n) = 0$. R is an equivalency relation on Xˆin this situation. Show the quotient of $\hat{X}/R$ by $\hat{X}$.For any x in $\hat{X}$ therefore,

$\hat{X} = \{[x] : x \in \hat{X}\}$,where$[x] = \{y \in \hat{X} : xRy\}$.

Insert $[x] + [y] = [x + y]$ and $a \cdot [x] = [ax]$ for each $x := (x_n)_{n\in\mathbb{N}}$ and $y := (y_n)_{n\in\mathbb{N}}$ in $\hat{X}$ and every $a \in \mathbb{R}^+$, where $+y = (x_n + y_n)_{n\in\mathbb{N}}$, and $ax = (ax_n)_{n\in\mathbb{N}}$. These processes are simple to comprehend due to their precise specification. This is the result that comes next.

Lemma (3.7): Take the case of a quasi-normed cone (X, p_j). A cancellative cone is therefore $(\tilde{X}, +, \cdot)$

Proof:

$(\tilde{X}, +)$ is a cancellative Abelian semigroup with neutral element $[0] \in \widetilde{X}$, as we can easily conclude from $(X, +, \cdot)$ being a cancellative cone. Furthermore, the previously defined operation ·, which was defined as $\mathbb{R}^+ \times \tilde{X}$ to $\tilde{X}$, satisfies for any $[x], [y] \in \tilde{X}$and $r, s \in \mathbb{R}^+$:

(a) $r \cdot (s[x]) = (rs) \cdot [x]$;

(b) $r \cdot ([x] + [y]) = (r \cdot [x]) + (r \cdot [y])$;
(c) $(r + s) \cdot [x] = (r \cdot [x]) + (s \cdot [x])$;
(d) $1 \cdot [x] = [x]$.

The proof of the following result on a general monoid context can be found in [99].

Lemma (3..8): Assume a quasi-normed cone (X, p_j) and that $x := (x_n)_{n\in\mathbb{N}} \in \hat{X}$ then:

(a) $\lim_{n\to\infty} p_j(x_n)$ exists and is finite.
(b) $\lim_{n\to\infty} p_j(x_n) = \lim_{n\to\infty} p_j(y_n)$ for all $y \in [x]$.
We may define a function $\widetilde{p_J} : \tilde{X} \to \mathbb{R}^+$ given the earlier lemma, which is defined as

$\widetilde{p_j}([x]) = \lim_{n\to\infty} p_j(x_n)$ for all $x \in \hat{X}$.

We will show how $(\tilde{X}, \widetilde{p_j})$ is a bicomplete quasi-normed cone in the following Lemma 3:9.

Lemma (3.9): Consider the following two cones $(X, +, \cdot)$ and $(Y, \oplus, \otimes)$ If $f_j : A \to Y$ is a linear function and A is a sub cone of X, then $f_j(A)$ is a sub cone of Y.

Lemma (3.10): A quasi-normed cone (X, p_j) is considered. Then the following statements are exact:

(a) The quasi-normed cone $(\tilde{X}, \tilde{p}_j)$bicomplete.

(b) (X, p_j) is isometric to a dense subspace of $(\tilde{X}, \tilde{p}_j)$ in the metric space $\left(\tilde{X}, \left(e_{\tilde{p}_j}\right)^s\right)$.

isometric to a dense subspace of the metric space

Proof:

(a) The cancellative condition of (X˜,p˜_j) is obtained from Lemma (3.7).

Let $x := (x_n)_{n\in\mathbb{N}}$ be an element of Xˆsuch that – $[x] \in \tilde{X}$ and $\tilde{p}_j([x]) = \tilde{p}_j(-[x]) = 0$. Since $\lim_{n\to\infty} p_j(x_n) = 0 = \lim_{n\to\infty} p_j(-x_n)$ follows. $\lim_{n\to\infty} \left(e_{p_j}\right)^s (0, x_n) = 0$ because, finally, $e_{p_j}(0, x_n) = p_j(x_n)$and $e_{p_j}(x_n, 0) = p_j(-x_n)$

Consequently, $[x] = [0]$.

Lemma (3.7) yields the cancellative condition of $(\tilde{X}, \tilde{p}_j)$.

Consider an element of $\hat{X}$, $x := (x_n)_{n\in\mathbb{N}}$, such that – $[x] \in \tilde{X}$ and $\tilde{p}_j([x]) = \tilde{p}_j(-[x]) = 0$. As a result, $\lim_{n\to\infty} p_j(x_n) = 0 = \lim_{n\to\infty} p_j(-x_n)$.

$\lim_{n\to\infty} \left(e_{p_j}\right)^s (0, x_n) = 0$ since $e_{p_j}(0, x_n) = p_j(x_n)$ and $e_{p_j}(x_n, 0) = p_j(-x_n)$ at last.

As a result, $[x] = [0]$.

We have

$$\tilde{p}_j(a \cdot [x]) = \lim_{n\to\infty} p_j(ax_n) = \lim_{n\to\infty} ap_j(x_n) = a \lim_{n\to\infty} p_j(x_n) = a\tilde{p}_j([x]).$$

given $x := (x_n)_{n\in\mathbb{N}} \in \hat{X}$ and $a \in \mathbb{R}^+$.

Let $x := (x_n)_{n\in\mathbb{N}}$ and $y := (y_n)_{n\in\mathbb{N}}$ be two of $\hat{X}$.elements.

$p_j(x_n + y_n) \le p_j(x_n) + p_j(y_n)$ is taken into consideration in order to explain the triangle inequality; therefore,

$\lim_{n\to\infty} p_j(x_n + y_n) \le \lim_{n\to\infty} p_j(x_n) + \lim_{n\to\infty} p_j(y_n)$.

$\tilde{p}_j([x] + [y]) \le \tilde{p}_j([x]) + \tilde{p}_j([y])$,

for this. So $\widetilde{p_j}$ is a quasi-norm for $\tilde{X}$

It well known that the bicompletion of the quasi-metric space $\left(X, e_{p_j}\right)$ is a quasi-metric space $\left(X^b, \left(e_{p_j}\right)^b\right)$, where $X^b = \{[x] : x$ is a Cauchy sequence in the metric space $\left(X, \left(e_{p_j}\right)^s\right)\}$, $\left(e_{p_j}\right)^b([x],[y]) = \lim_{n\to\infty} e_{p_j}(x_n, y_n)$ for all $[x],[y] \in X^b$, and for each Cauchy sequence

$x := (x_n)_{n\in\mathbb{N}}\ in\ \left(X, \left(e_{p_j}\right)^s\right)$,

$[x] = \left\{y := (y_n) : y \text{ is a Cauchy sequence in } \left(X, \left(e_{p_j}\right)^s\right) \text{ and } \lim_{n\to\infty} \left(e_{p_j}\right)^s (x_n, y_n) = 0\right\}$.

$X^b = \tilde{X}$ and $\left(e_{p_j}\right)^b = e_{\tilde{p}_j}$, are the results.

(b) The constant sequence $x, x, \dots x, \dots$. for every x in X is defined by $\hat{x}$. provided that $\left(X^b, \left(e_{p_j}\right)^b\right)$ is the bicompletion of $\left(X, e_{p_j}\right)$, $i(X)$ is dense in $\left(\tilde{X}, \left(e_{\tilde{p}_j}\right)^s\right)$, where i is the one-to-one function from $X\ to\ \tilde{X}$ provided by $i(x)$=[$[\hat{x}]$] for every $x \in X$.

For each $x\ in\ X$, note that $[\hat{x}]$ is the collection of all sequences in X that converge to x in the metric space $\left(X, \left(e_{p_j}\right)^s\right)$.Since it is standard procedure to confirm that i is a linear function, the preceding Lemma states that $i(X)$ is a semi linear subspace of $\tilde{X}$. We may deduce that $\left(X, p_j\right)$ and $\left(i(X), \tilde{p}_j|_{i(X)}\right)$ are isometric quasi-normed cones because, for all $x \in X$, $\tilde{p}_j\left(i(x)\right) = \tilde{p}_j([\hat{x}]) = p_j(x)$. The proof is complete.

Lemma (3.11): Take a quasi-normed cone $\left(X, p_j\right)$, a bicomplete quasi-normed cone $\left(Y, q_j\right)$,and a sub cone A of X to Y. If A is dense in $\left(X, \left(e_{p_j}\right)^s\right)$, then f is a one-to-one isometry. From $\left(X, p_j\right)$ to $\left(Y, q_j\right)$, f_j then extends uniquely to a one-to-one isometry.

Proof:A contains a sequence $(x_n)_{n\in\mathbb{N}}$such that, for all $x \in X\backslash A$, $\lim_{n\to\infty} \left(e_{p_j}\right)^s (x, x_n) = 0$. In the metric space $\left(X, \left(e_{p_j}\right)^s\right)$, the sequence $(x_n)_{n\in\mathbb{N}}$ (related to $x \in X\backslash A$ is a Cauchy sequence, and for any $m, n \geq n_0$, there exists $n_0 \in \mathbb{N}$ such that $\left(e_{p_j}\right)^s (x_n, x_m) < \mu + 1$.for any value of $\mu > -1$. Proposition (3.2.9) states that for any $m, n \geq n_0$, $\left(e_{p_j}\right)^s (f(x_n), f(x_m) < \mu + 1$. The metric $\left(e_{q_j}\right)^s (f(x_n)$, is therefore a Cauchy sequence in the metric space $\left(Y, \left(e_{q_j}\right)^s\right)$, and converges to a point $x^* \in Y$.

Define $f_j^* : X \to Y$ for each x in A, where $f_j^*(x) = f_j(x)$, and for each $x\ in\ X\backslash A$,where

$f_j^*(x) = x^{**}$. It should be noted that the definition of $f_j^*$is not changed by the sequences $(x_n)_{n\in\mathbb{N}}$In fact, if$(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$) are sequences in A that converge to a point $x\ in\ X\backslash A$ with respect to the metric $\left(e_{q_j}\right)^s$; additionally, if we designate by x^* and $y^*$the limit points in

$\left(Y,\left(e_{q_j}\right)^s\right)$ of $\lim_{n\to\infty}\left(e_{q_j}\right)^s f_j(x_n), f_j(y_n) = 0$,since $\lim_{n\to\infty}\left(e_{p_j}\right)^s (x_n, y_n) = 0$.Therefore, $x^* = y^*$.

We conclude that f extends uniquely to f^*based on Lemma (3.12) of [7], where ${f_j}^*$is a one-to-one function such that $q_j\left({f_j}^*(x)\right) = p_j(x), \left(X, p_j\right)$ and $\left(Y, q_j\right)$ are quasi-normed cones. We simply show that ${f_j}^*$ is linear on X after that. Consider x and y to be in X.

Only the case where $x, y \in X\backslash A$, which is linear on A, is considered. Assume that the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be in A converge to x and y, respectively, in the metric space $\left(X,\left(e_{p_j}\right)^s\right)$.

Subsequently, with respect to $\left(e_{q_j}\right)^s$, ${f_j}^*$converges to $\left(f_j(ax_n + by_n)\right)_{n\in\mathbb{N}}$. That's because, with considering $\left(e_{q_j}\right)^s$, $(ax_n + by_n)_{n\in N}$ converges to $ax + by$. With respect to $\left(e_{q_j}\right)^s$, the sequence $\left(af_j(x_n) + bf_j(y_n)\right)_{n\in\mathbb{N}}$ converges to ${f_j}^*(ax + by)$since f is linear on A.On the other hand, $\left(f_j(x_n)\right)_{n\in\mathbb{N}}$ converges to ${f_j}^*(x)$ and $\left(f_j(y_n)\right)_{n\in\mathbb{N}}$ converges to ${f_j}^*(y)$ with respect to $\left(e_{q_j}\right)^s$, as per the definition of ${f_j}^*$. For the metric $\left(e_{q_j}\right)^s$, *then* $\left(\left(af_j(x_n) + bf_j(y_n)\right)\right)_{n\in\mathbb{N}}$ *converges to a* $a{f_j}^*(x) + b{f_j}^*(y)$.*Consequently,* ${f_j}^*(ax + by) = a{f_j}^*(x) + b{f_j}^*(y)$.

Lemma (3.2.12): A quasi-normed cone $\left(X, p_j\right)$ has bicompletions of all kinds that are isometric to $\left(\tilde{X}, \tilde{p}_j\right)$.

Proof: A bicompletion $\left(Y, q_j\right)$ is assumed to exist for $\left(X, p_j\right)$. Suppose that i is the one-to-one isometry from (X, p) to $\left(\tilde{X}, \tilde{p}_j\right)$ is defined in Lemma (3.2.14). Furthermore, because X is dense in the metric space $\left(Y,\left(e_{\mathrm{q}_j}\right)^s\right)$, the prior Lemma leads to the conclusion that f has a unique one-to-one isometry extension f^* to $\left(Y, q_j\right)$.

It remains to be shown that ${f_j}^* : Y \to \tilde{X}$ is an onto mapping. Let x actually stand for any random $\tilde{X}$ point. For a sequence $(x_n)_{n\in\mathbb{N}}$ in X, $\lim_{n\to\infty} e^s_{\tilde{p}_j}\left(x, f_j(x_n)\right) = 0$, given that $f_j(X)$ is dense in $\left(\tilde{X},\left(e_{\tilde{p}_j}\right)^s\right)$. A Cauchy sequence is thus found in $\left(\tilde{X},\left(e_{\tilde{p}_j}\right)^s\right)$, $\left(f_j(x_n)\right)_{n\in\mathbb{N}}$. In $\left(Y,\left(e_{\mathrm{q}_j}\right)^s\right)$, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence since ${f_j}^*$is an isometry. For every $y\ in\ Y$,if $\lim_{n\to\infty}\left(e_{\mathrm{q}_j}\right)^s (y, x_n) = 0$, $\lim_{n\to\infty}\left(e_{\tilde{p}_j}\right)^s\left({f_j}^*(y), {f_j}^*(x_n)\right) = 0$. For ${f_j}^*(y) = x$. This completes the proof. From the above stated lemmas, the following can be deduced immediately.

Theorem (3.13): For any quasi-normed cone $\left(X, p_j\right)$, there is only one bicompletion (up to bijective isometry).

Corollary (3.2.19): $\left(X, e_{p_j}\right)$ and $\left(Y, e_{q_j}\right)$ quasi-metric spaces that are isometric by f_j if $\left(X, p_j\right)$ and $\left(Y, q_j\right)$ are isometric quasi-normed cones by a (bijective) isometry f_j.

Proof: Suppose that $x, x + 2(\mu + 1) \in X$. $f_j(x + 2(\mu + 1)) = e_{p_j}(x, x + 2(\mu + 1)) = +\infty$. if $f_j(x + 2(\mu + 1)) \in f_j(x) + Y$.

For $z \in Y$ such that $f_j(x + 2(\mu + 1) = f_j(x) + x + \mu + 1$, otherwise

$e_{q_j}\left(f_j(x), f_j(x + 2(\mu + 1))\right) = q_j(x + \mu + 1)$.

There is a single $a \in X$ with $f_j(a) = x + \mu + 1$ since f_j is bijective. Consequently, $f_j(x + 2(\mu + 1)) = f_j(x) + f_j(a) = f_j(x + a)$. Because $\mu = \frac{a}{2} - 1$, $e_{p_j}(x, x + 2(\mu + 1) = p_j(a)$. The isometry of $\left(X, e_{p_j}\right)$ and $\left(Y, e_{q_j}\right)$ by f_j,is what we deduce.

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