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Quasi D-Limits Of Bounded Sequences And Regular Matrices

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Abstract:

In this paper, we explore the concept of quasi-D-limits for bounded sequences through the nonnegative regular matrix transformation D. Quasi D-limits extend spaces of D-limits notions, providing a framework to analyse the convergence properties of sequences that exhibit specific bounded behaviour. We establish criteria for quasi-D-limits, demonstrate their existence via regular matrix transformations, and apply these results to various mathematical and applied contexts.

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1. Introduction:

The study of limits in sequence convergence has been a foundational aspect of analysis. In particular, bounded sequences whose terms are confined within a specific range often require advanced techniques to elucidate their limiting behaviour. quasi-D-limits offer an innovative approach, enabling us to extend conventional limit concepts. The idea of Quasi almost convergence in normed space was featured by Hajdukovic[4]. Further the concept of quasi Banach limit is introduced by Das and Mishra in [3] and the concept of quasi-invariant limit is recently explained by Mishra[5] in the space of real bounded sequences which yields quasi-invariant convergent sequences.

This paper aims to investigate the quasi-D-limits of bounded sequences utilizing matrix transformations. We first introduce the necessary background on quasi-D-limits and matrix transformations, followed by an exploration of their interrelations.

2. Preliminaries:

Let m be the space of real bounded sequences $x = \{x_n\}$ normed by $\|x\| = \sup_n |x_n|$. Let m^* denote the set of all continuous linear functional on m. Throughout this work, we have used an infinite matrix D whose scalar entries d_{nk} are in $m.$ Consider a sequence x in m. Let Dx be the transformed sequence whose general term is written as

$$(Dx)_n = \sum_{k=0}^{\infty} d_{nk} x_k$$

and it is convergent for each $n \geq 0$.

Let us recall the definition of conservative matrix (due to Stieglitz [8]).

Definition 1: A matrix $D = (d_{nk})$ is said to be conservative matrix iff it satisfies the following properties:

(a) $\|D\| < \infty$ i.e., $\sup_n \sum_{k=0}^{\infty} |d_{nk}| < \infty$.

(b) $\lim_n d_{nk} = d_k$ for every fixed k.

(c) $\lim_n \sum_{k=0}^{\infty} d_{nk} = d$.

If $d_k = 0$ and $d = 1$, then D is called regular (see [9] p.64).

Let $D = (d_{nk})$ be the a fixed regular matrix with $\|D\| < 1$(This is assumed throughout the paper).

Definition 2: A linear functional $\psi \in m^*$ is said to be an D-mean or D-limit if and only if the following properties hold good:

(i) For $x = \{x_n\}$, $\psi(x) \geq 0$ if $x_n \geq 0$ for all n.

(ii) $\psi(e) = 1$ for $e = (1,1,1, \dots .)$.

(iii) $\psi(Dx) = \psi(x)$ for all $x \epsilon m$.

Let $D = B$ be the translation matrix i.e. $(Bx)_n = x_{n+1}$ then it is called a B -limit and is often called as a Banach limit [1]. Some inequalities are shown in [6] between sublinear functionals emerging from Banach limits of some sequences and their matrix transformations using conservative and regular matrices.

Definition 3: A matrix $C \in m$ is called a D-Invariant matrix if $C(D - I)$ behaves as a zero map and maps every bounded sequence to null sequence. (I is the Identity matrix.)

The concept of D-limits were first introduced by Bell[2], where he assumed that D is a positive matrix such that $\|D\| = 1$. From definition of D-limit , it follows that

$$x \leq y \Rightarrow \psi(x) \leq \psi(y) .$$

and from (i) and (ii) that $\|\psi\| = 1$. Here the product of D with itself p times is denoted by D^p.

Let Q_D be the set of all bounded sequences having equal D-means. So, we can write

$$Q_D = \{x \in m\colon \lim_p h_{pn}(x) = l \text{ uniformly in } n, l = D - lim x\}$$

where for $p \geq 0, n > 0$.

$$h_{pn}(x) = \frac{x_n + (Dx)_n + \cdots + (D^p x)_n}{p+1} . \qquad (2.1)$$

We know that a sublinear functional P on m generates D-limit if $\psi \in m^*$and $\psi < P \Rightarrow \psi$ is a D-limit. Here $\psi < P$ $means$ $\psi(x) \leq P(x)$ for all $x \in m$. P is said to dominate D-limit if every D-limit

$\psi < P$. i.e. $\psi \in Q_D \Rightarrow \psi < P$, where Q_D is the set of all D-limits. It is provided that Q_D is a closed convex set. It is proved that $\psi \in m^*$ is a D-limit if and only if $\psi(x) \leq t(x)$,where $t(x)$ is a sublinear functional on m defined by

$$t(x) = \lim_p sup \sup_n h_{pn}(x).$$

i.e.
$$t(x) = \lim_p sup \sup_n \frac{1}{p+1} \sum_{k=0}^{p} (D^k x)_n. \quad (2.2)$$

We now take the idea from above and purpose the following definition for a new family of functionals of the kind of D-mean called quasi D-limit.

Definition 4: The linear functional $\psi \in m^*$ is a quasi D-mean or quasi D-limit or quasi matrix invariant limit if $\psi(x) \leq r(x)$, $x \in m$, where $r(x)$ is a sublinear functional on m defined by

.
$$r(x) = \lim_p sup \sup_n \frac{1}{p+1} \sum_{k=0}^{p} (D^k x)_{np} \ . \quad (2.3)$$

. Quasi D-limit need not exist for every regular matrix.

Example 1: Consider

$$d_{nk} = \begin{cases} 1 & , \quad if\ k = 2n \\ 0 & , \quad if\ k > 2n \\ \frac{1}{n} & , \ if\ k\ is\ even\ 1 \leq k \leq 2n \\ \frac{-1}{n} & , \ if\ k\ is\ odd, 1 \leq k \leq 2n \end{cases}$$

Here D is regular. So, for $x = \{1,0,1,0, \dots \dots\}$, $\quad Dx = \{-1,-1,-1,-1, \dots \dots\}$.

If ψ is a quasi D-limit then $0 \leq \psi(x) = \psi(Dx) = -1$ which is impossible. So quasi D-limit does not exist.

We now propose the following definitions for new kind of D-convergent sequences, called as quasi D-convergent sequence. For this we define it as follows

Definition 5: A sequence $x \in m$ is called a quasi D-convergent sequence if $-r(-x) = r(x)$.

The following theorem is known.

Theorem A [10]: For all $x \in m$, the sublinear functional $t(x)$ both generates and dominates D-limit

$\psi(x)$ if and only if $\psi(x) \leq t(x)$.

Applying the above theorem, we will prove some new results for the sublinear functional $r(x)$, where these are based on the idea of quasi Banach Limit defined in [3] , quasi invariant limits and their convergence suggested by Mishra[5] and Nuray[7].

3. Main Results:

Now we are going to prove a theorem on existence o D-limit and then establish some inequalities on quasi D-limit .

Theorem 1: (Existence Theorem) If there is a nonnegative regular and D-invariant matrix then D-limits exist or quasi D-limits exist.

Proof: Suppose C is such a nonnegative regular D-invariant matrix. Define a nonnegative homogeneous sublinear functional p on m by $p(x) = \limsup Cx$ .If x is a convergent sequence, then by regularity of C we get $\lim x = p(x)$. Therefore, extending the above limit to ψ we have $\psi(x) \le p(x)$ for all $x \in m$.

Thus, $-p(-x) \le -\psi(-x) = \psi(x) \le p(x)$ for all $x \in m$. Since C is nonnegative ,we have $0 \le \liminf Cx \le \psi(x)$.This implies ψ is nonnegative. Here we have $C(D-I)x = 0$.Therefore

$-p(-(D-I)x) = 0 = p\big((D-I)x\big)$ which implies $\psi\big((D-I)x\big) = 0$.

Hence, ψ is a D-limit. Considering $p(x)$ as equal to the sublinear functional in (2.3),we can say that

ψ is a quasi D-limit .

Example 2: Cesaro matrix is a nonnegative regular B -Invariant matrix. So, its quasi Banach limit[3] exists. If we define

$$d_{nk} = \begin{cases} \frac{1}{n} & , if\ 1 \le k \le n \\ 0 & , elsewhere \end{cases} \tag{3.1}$$

then the matrix $D = (d_{nk})$ is a Cesaro matrix of order one which is a translation matrix and its quasi D-limit [3] or quasi Banach limit exists as it is a nonnegative, regular and $D(=B)$-Invariant matrix.

Theorem 2: For all $x \in m,$ the sublinear functional $r(x)$ generates D-limit .

Proof: We first proceed to prove that the sublinear functional $\psi \in m^*$ generates D-mean.

Let $\psi \in m^*$ and satisfies the inequality

$$\psi(x) \le r(x) \quad \text{for all } x \in m \tag{3.2}$$

So by linearity of $\psi(x)$and sub-linearity of $r(x)$ we have from (3.2), that is

$$-r(-x) \le \psi(x) \le r(x) \tag{3.3}$$

This indicates that for $x \ge 0$ implies $r(x) \ge 0$ and $-r(-x) \ge 0$ and so from (3.3), we get

$$\psi(x) \ge 0, \text{for all } x \ge 0.$$

Also $-r(-e) = r(e) = 1$, where $e = (1,1,1,1,\dots,1)$. Hence from (3.3), we get $\psi(e) = 1$.

Taking D as a nonnegative regular matrix, we have

$$\limsup_{p} \sup_{n} \frac{1}{p+1}\sum_{k=0}^{p}(D^{k+1}-D^{k})x_{np} = \limsup_{p} \sup_{n} \frac{(D^{p+1}-I)x_{np}}{p+1}$$

$$\leq \|x\| \limsup_{p} \frac{1}{p+1} = 0.$$

This shows $r(Dx - x) = 0$.

Similarly, we also obtained $r(x - Dx) = 0$.

Hence from (3.3) we have $\psi(Dx - x) = 0$. Since ψ is linear, $\psi(Dx) = \psi(x), x \in m$.

This proves that

$\psi(x) \leq r(x) \Rightarrow \psi$ is a D-limit (3.4)

So $r(x)$ also generates D –limit as required.

Note: From above we say that $r(x)$ also generates quasi D -limit.

Next consider ψ is a quasi D-limit. Then

$$\psi(x_{np}) = \psi(Dx)_{np} = \psi(D^2x)_{np} = \cdots \ldots = \psi(D^p x)_{np}$$

This implies

$$\psi(x) = \psi\left(\frac{x_{np}+(Dx)_{np}+(D^2x)_{np}+\cdots\ldots+(D^p x)_{np}}{p+1}\right)$$

$$\leq \limsup_{p} \sup_{n} \left(\frac{x_{np}+(Dx)_{np}+(D^2x)_{np}+\cdots\ldots+(D^p x)_{np}}{p+1}\right) = r(x)$$

i.e., $r(x)$ dominates quasi D –limit.

Theorem 3: Let $t(x)$ and $r(x)$ be two sublinear functionals defined in (2.2) and (2.3) respectively.Then

$$r(x) \leq t(x) \quad \text{for all } x \in m \quad .$$

Proof: Combining the given condition of (3.4) and the inequality of Theorem A, we will get that for all $\psi \in m^*$.

$$\psi(x) \leq r(x) \Rightarrow \psi(x) \leq t(x) \quad (3.5)$$

claim :

$$r(x) \leq t(x) \text{ for all } x \in m \,. \quad (3.6)$$

Suppose to the contrary, that (3.6) is false. Then there exists a sequence $y \in m$ such that

$$r(y) > t(y) \,. \quad (3.7)$$

Since r is sublinear, by Hahn-Banach Theorem, there exist a linear functional g on m such that

$$g(y) = r(y) .$$

From (3.7), it follows that

$$g(y) > t(y) .$$

However, from the given condition (3.5), we know that for all $\psi \in m^*$

$$\psi(x) \leq r(x) \Rightarrow \psi(x) \leq t(x)$$

Substituting $g(y)$ into this condition, we have: $g(y) \leq t(y)$

which contradicts our earlier assertion $g(y) > t(y)$

Thus, our assumption that $g(y) > t(y)$ is untenable. Therefore, we conclude that:

$$r(x) \leq t(x) \text{ for all } x \in m. \tag{3.8}$$

Hence proved the result.

Next, we are going to prove an important corollary of above theorem. From the definition and from above theorem, the set of D-convergent sequences Q_D also can be written as:

$$Q_D = \{x \in m\colon -t(-x) = t(x)\} . \tag{3.9}$$

This identifies the structure of sequences x for which $-t(-x) = t(x)$ which are also referred to define quasi D-convergent sequences. We denote the set of quasi D-convergent sequences as $Q_D{}^*$ where

$$Q_D{}^* = \{x \in m\colon -r(-x) = r(x)\} . \tag{3.10}$$

Then we now prove the following corollary.

Corollary: $Q_D \subseteq Q_D{}^*$

where Q_D is the set of D-convergent sequences, and $Q_D{}^*$ is the set of quasi D-convergent sequences.

Proof: Since $t(x)$ and $r(x)$ are sublinear, it follows from above theorem that

$$-t(-x) \leq -r(-x) \leq r(x) \leq t(x) \text{ for all } x \in m.$$

If $x \in Q_D$ then by definition,

$$-t(-x) = t(x)$$

Substituting this condition into the chain of inequalities above, we have:

$$-t(-x) = t(x) \Rightarrow -r(-x) = r(x).$$

This implies that $x \in Q_D{}^*$, where $Q_D{}^*$ represents the set of quasi D-convergent sequences.

Thus, every $x \in Q_D$ (D-convergent sequence) is also an element of ${Q_D}^*$(quasi D-convergent sequence).

Therefore, we conclude that: $Q_D \subseteq {Q_D}^*$

This completes the proof.

4. Conclusion:

This study presented the quasi D-limit for bounded sequences and demonstrated their existence using matrix transformations. The findings indicate that matrix transformations are an effective tool for examining the limiting behaviour of various types of sequences within their respective spaces.

Furthermore, the inclusions for quasi D-limit convergent and D-convergent sequences pave the way for more in-depth sequence analysis. These findings have major implications for a variety of mathematical topics, including functional analysis, topology, and numerical analysis, while also laying the groundwork for future study into the interactions between sequence spaces and their transformations.

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