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On Sombor Energy of Graphs with self-loops

S. H. Pathan[1] **and S. C. Patekar**[2]

[1,2] Department of Mathematics, Savitribai Phule Pune University, Pune (India) [1]samrinpathan508@gmail.com, [2]shri82patekar@gmail.com

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Abstract:

The goal of this paper is to broaden the concept of Sombor Energy from a simple graph to one containing self-loops. Let G be a simple nth-order graph, and G_s be the graph generated by adding σ self-loops to G. Sombor matrix of G_s is defined as

$$A_{SO}(G_s) = (a_{ij}) = \begin{cases} \sqrt{d_i^2 + d_j^2}; & if\, v_i \text{ and } v_j are\ adjacent \\ \sqrt{2}d_i; & if\ i = j \\ 0; & if\, v_i \text{ and } v_j \text{ are not adjacent}. \end{cases}$$

If $\lambda_1(G_s), \lambda_2(G_s), \dots, \lambda_n(G_s)$ are eigenvalues of $A_{SO}(G_s)$, then Sombor Energy of G_s is defined as

$$E_{SO}(G_s) = \sum_{i=1}^{n} \left|\lambda_i(G_s) - \frac{\sqrt{2}\sum_{j=1}^{\sigma} d_j}{n}\right|$$

where d_j is the degree of vertex v_j with self loop

Keywords: Eigenvalue, Sombor Energy, Self-loops.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, $|V(G)| = n$. If the vertices $u, v \in V(G)$ are adjacent, then edge with end points are u and v is denoted by uv. The neighbor of u is denoted by $N(u)$ that is the set of vertices adjacent to u, $|N(u)|$ is total number of adjacent vertices to u and is called degree of u and denoted by $d_G(u)$.

Let $A(G)$ be the adjacency matrix of a simple graph G with vertices $v_1, v_2, \dots, v_n$, elements of adjacency matrix are defined by

$$A(G) = \left(a_{ij}\right) = \begin{cases} 1; & if\ v_i \text{ and } v_j\ are\ adjacent \\ 0; & if\ v_i \text{ and } v_j\ are\ not\ adjacent. \end{cases}$$

and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of matrix $A(G)$.

The energy of simple graph, introduced by I. Gutman [3], is defined as

$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|$.

Let S be a subset of V(G). The number of element of S will be denoted by σ. Let G_S is graph with σ self-loops. Because graphs containing self-loops are useful in chemistry, [heteroatoms,heteroconjugated,chemistry,molecules].

In 2021, Gutman defined the adjacency matrix $A(G_S)$ [9] of the graph G_S.

$$A(G_S) = (a_{ij}) = \begin{cases} 1; & if\ v_i\ and\ v_j\ are\ adjacent \\ 1; & if v_i \in S \\ 0; & if\ v_i\ and\ v_j\ are\ not\ adjacent. \end{cases}$$

Definition 1. *[9] Let* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be the eigenvalues of matrix* $A(G_S)$ *such that* $\sum_{i=1}^{n} \lambda_i = \sigma$ *then energy of graph with* σ *self loop is given by*

$$E(G_S) = \sum_{i=1}^{n} \left|\lambda_i - \frac{\sigma}{n}\right|.$$

Definition 2. *[7] Let* G *be a graph. If* $u, v \in V(G)$ *and* $uv \in E(G)$*, then Sombor index of graph* G *is defined by*

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}$$

Definition 3. *[8] The Sombor matrix of G is defined by*

$$S(G) = (s_{ij})_{n\times n} = \begin{cases} \sqrt{d_u^2 + d_v^2} & \text{; if u and vare adjacent} \\ 0 & \text{; otherwise} \end{cases}$$

We denote the eigenvalues of S(G) by μ_i,s such that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. The set of all eigenvalues of S(G) is called Sombor spectrum and μ_1 is the Sombor spectral radius of G. The Sombor energy [8] is defined by

$$E_{SO}(G) = \sum_{i=1}^{n} |\mu_i|.$$

The sum of squares of eigenvalues of S(G) satisfies following equation is given by [2]

$$2F = \mu_1^2 + \mu_2^2 + \mu_3^2 + \ldots + \mu_n^2 \tag{1}$$

where

$$F = F(G) = \sum_{i=1}^{n} d_{v_i}^3 = \sum_{v_i \sim v_j} (d_i^2 + d_j^2)$$

is forgotten topological index of G [1].

Definition 4. *Let S be a subset of V(G). The number of element of S will be denoted by* σ*. The Sombor matrix of graph G with* σ *self loop is defined by*

$$A_{SO}(G_s) = (a_{ij}) = \begin{cases} \sqrt{d_i^2 + d_j^2} & if v_i\ and\ v_j are\ adjacent \\ \sqrt{2} d_i & if\ i = j \\ 0 & if v_i\ and\ v_j are\ not\ adjacent \end{cases}$$

If $\lambda_1(G_s), \lambda_2(G_s), \ldots\ \lambda_n(G_s)$ are eigenvalues of $A_{SO}(G_s)$, then the Sombor energy of G_s (which is analogous to the energy of any matrix with a non-zero diagonal [Laplace energy, Laplace energy and radiac energy, New spectral]) must be defined as

$$E_{SO}(G_s) = \sum_{i=1}^{n} \left|\lambda_i(G_s) - \frac{\sqrt{2}\sum_{j=1}^{\sigma} d_j}{n}\right|$$

where d_j is the degree of vertex v_i with self loop.

2. Main Results

Proposition 1. Let G be graph with n vertices and S be any subset of V(G) with σ elements. If $\sigma = 0$, then $E_{SO}(G_S) = E_{SO}(G)$.

Proof. It is trivially obvious, since for $\sigma = 0$, the graphs G_S and G coincide then Sombor matrix of G_S and G are same and hence Sombor energy are also same.

Let $S(G)$ be Sombor matrix of G and $D(G_S)$ is the diagonal matrix with diagonal entries degree of vertex having self loop then Sombor matrix of G_S is equal to

$$A_{SO}(G_S) = S(G) + \sqrt{2}D(G_S) \tag{2}$$

Proposition 2. *Let G be a graph with n vertices and m edges. If $S \subset V$ with σ elements, then eigenvalues $\lambda_1(G_s), \lambda_2(G_s), \ldots,\ \lambda_n(G_s)$ of $A_{SO}(G_S)$ satisfy,*

1. $\sum_{i=1}^{n} \lambda_i^2(G_s) = 2F + 4\sum_{v_i \in S} d_i^2$
2. $\sum_{i=1}^{n} [\lambda_i(G_s) - \frac{\sqrt{2}\sum_{j=1}^{\sigma} d_j}{n}]^2 = 2F + 4\sum_{v_i \in S} d_i^2 - \frac{2(\sum_{v_i \in S} d_i)^2}{n}$.

Proof. 1. From equation (2) $A_{SO}(G_S) = S(G) + \sqrt{2}D(G_S)$

Therefore,

$$\sum_{i=1}^{n} \lambda_i^2(G_s) = \sum_{i=1}^{n} [(S(G) + \sqrt{2}D(G_S))^2]_{ii}$$

$$= \sum_{i=1}^{n} [(S(G)^2)_{ii} + 2\sqrt{2}[S(G)D(G_S)]_{ii} + 2[(D(G_S))^2]_{ii}]$$

From equation (1), $2F = \sum_{i=1}^{n} (S(G)^2)_{ii}$ where $F = \sum_{v_i \sim v_j} (d_i^2 + d_j^2)$

As the diagonal entries of diagonal matrix are nonzero for σ self loop and diagonal entries of Sombor matrix is zero. Hence, diagonal entries of $S(G)D(G_S)$ are zero.

Therefore,

$$\sum_{i=1}^{n}[S(G)D(G_S)]_{ii}=0$$

Also,

$$\sum_{i=1}^{n}[D(G_S)^2]=2\sum_{v_i\in S}d_i^2$$

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i^2(G_s)&=\sum_{i=1}^{n}(S(G)^2)_{ii}+2\sqrt{2}\sum_{i=1}^{n}[S(G)D(G_S)]_{ii}+2\sum_{i=1}^{n}(D(G_S)^2)_{ii}\\&=2F+2(2\sum_{v_i\in S}d_i^2)\\&=2F+4\sum_{v_i\in S}d_i^2\end{aligned}$$

2. $\sum_{i=1}^{n}[\lambda_i(G_s)-\frac{\sqrt{2}\sum_{j=1}^{\sigma}d_j}{n}]^2=\sum_{i=1}^{n}[\lambda_i(G_S)^2-2\sqrt{2}\lambda_i(G_s)\frac{\sqrt{2}\sum_{j=1}^{\sigma}d_j}{n}+2\frac{(\sum_{j=1}^{\sigma}d_j)^2}{n^2}]$

$$=\sum_{i=1}^{n}[\lambda_i^2(G_S)-\frac{2\sqrt{2}}{n}\sum_{v_i\in S}d_i\sum_{i=1}^{n}\lambda_i(G_S)+2n\frac{(\sum_{j=1}^{\sigma}d_j)^2}{n^2}]$$

As $\sum_{i=1}^{n}\lambda_i(G_S)=\sqrt{2}\sum_{v_i\in S}d_i$,

$$=2F+4\sum_{v_i\in S}d_i^2-\frac{2\sqrt{2}}{n}\sum_{v_i\in S}d_i\sqrt{2}\sum_{v_i\in S}d_i+2\frac{(\sum_{j=1}^{\sigma}d_j)^2}{n}$$

$$=2F+4\sum_{v_i\in S}d_i^2-\frac{4}{n}(\sum_{v_i\in S}d_i)^2+2\frac{(\sum_{j=1}^{\sigma}d_j)^2}{n}$$

$=2F+4\sum_{v_i\in S}d_i^2-\frac{2}{n}(\sum_{v_i\in S}d_i)^2$ $\quad\left(since\sum_{v_i\in S}d_i=\sum_{j=1}^{\sigma}d_j\right)$.

Lemma 1. *Let $G=K_n$ be a complete graph with n vertices, If G_S is a graph obtained from G by adding n-1 self loops, then Sombor eigenvalues of G_S are*

$\lambda_1=\frac{(n^2-1)\sqrt{2}+\sqrt{2n^4+8n^3-12n^2+8n-6}}{2}$, $\lambda_2=\frac{(n^2-1)\sqrt{2}-\sqrt{2n^4+8n^3-12n^2+8n-6}}{2}$, and $\lambda_i=0\ for\ i\geq 3$.

Proof. Let J be the $n\times n$ matrix with all entries one and 0 is the matrix with all zero entries.

Sombor matrix of G_S is

$$A_{SO}(G_S)=\begin{bmatrix}(n+1)\sqrt{2}J_{n-1\times n-1} & \sqrt{2(n^2+1)}J_{n-1\times 1}\\ \sqrt{2(n^2+1)}J_{1\times n-1} & 0\end{bmatrix}_{n\times n}$$

Then,

$$det(\lambda I - A_{SO}(G_S)) = det\begin{bmatrix}\lambda I_{n-1\times n-1} - (n+1)\sqrt{2}J_{n-1\times n-1} & -\sqrt{2(n^2+1)}J_{n-1\times 1}\\ -\sqrt{2(n^2+1)}J_{1\times n-1} & \lambda I_{1\times 1}\end{bmatrix}_{n\times n}$$

If M is non-singular square matrix

$$det\begin{bmatrix}M & N\\ P & Q\end{bmatrix} = det(M)det(Q - PM^{-1}N)$$

Here $M = \lambda I_{n-1\times n-1} - (n+1)\sqrt{2}J_{n-1\times n-1}, N = -\sqrt{2(n^2+1)}J_{n-1\times 1}, P = -\sqrt{2(n^2+1)}J_{1\times n-1}$ and $Q = \lambda I_{1\times 1}$,

then $det(M) = \lambda^{n-2}(\lambda - (n^2-1)\sqrt{2})$

$$M^{-1} = \frac{1}{\lambda^{n-2}(\lambda - (n^2-1)\sqrt{2})}(\lambda^{n-3}(\lambda - (n^2-1)\sqrt{2}I_{n-1\times n-1}) + \lambda^{n-3}(n + 1)\sqrt{2}J_{n-1\times n-1})$$

$$PM^{-1} = -\sqrt{2(n^2+1)}J_{1\times n-1} \times \frac{1}{\lambda^{n-2}(\lambda - (n^2-1)\sqrt{2})}(\lambda^{n-3}(\lambda - (n^2-1)\sqrt{2}I_{n-1\times n-1}) + \lambda^{n-3}(n+1)\sqrt{2}J_{n-1\times n-1})$$

$$= -\sqrt{2(n^2+1)}J_{1\times n-1} \times [\frac{1}{\lambda}I_{n-1\times n-1} + \frac{(n+1)\sqrt{2}}{\lambda(\lambda - (n^2-1)\sqrt{2})}J_{n-1\times n-1}]$$

$$= \frac{-\sqrt{2(n^2+1)}}{\lambda}J_{1\times n-1} - \frac{\sqrt{2(n^2+1)}(n+1)\sqrt{2}}{\lambda(\lambda - (n^2-1)\sqrt{2})} \times (n-1)J_{1\times n-1}$$

$$= \frac{-\sqrt{2(n^2+1)}}{\lambda}J_{1\times n-1} - \frac{2(n^2-1)\sqrt{(n^2+1)}}{\lambda(\lambda - (n^2-1)\sqrt{2})} \times J_{1\times n-1}$$

$$= [\frac{-\sqrt{2(n^2+1)}}{\lambda} - \frac{2(n^2-1)\sqrt{(n^2+1)}}{\lambda(\lambda - (n^2-1)\sqrt{2})}]J_{1\times n-1}$$

$$= [\frac{-(\lambda - (n^2-1)\sqrt{2})\sqrt{2(n^2+1)} - 2(n^2-1)\sqrt{(n^2+1)}}{\lambda(\lambda - (n^2-1)\sqrt{2})}]J_{1\times n-1}$$

$$= [\frac{-\sqrt{2(n^2+1)}\lambda}{\lambda(\lambda - (n^2-1)\sqrt{2})}]J_{1\times n-1}$$

$$= [\frac{-\sqrt{2(n^2+1)}}{(\lambda - (n^2-1)\sqrt{2})}]J_{1\times n-1}$$

$$PM^{-1}N = [\frac{-\sqrt{2(n^2+1)}}{(\lambda-(n^2-1)\sqrt{2})}]J_{1\times n-1} \times -\sqrt{2(n^2+1)}J_{n-1\times 1}$$

$$= \frac{-\sqrt{2(n^2+1)} \times -\sqrt{2(n^2+1)}(n-1)}{(\lambda-(n^2-1)\sqrt{2})}J_{1\times 1}$$

$$= \frac{2(n^2+1)(n-1)}{(\lambda-(n^2-1)\sqrt{2})}J_{1\times 1}$$

$$Q - PM^{-1}N = \lambda I_{1\times 1} - \frac{2(n^2+1)(n-1)}{(\lambda-(n^2-1)\sqrt{2})}J_{1\times 1}$$

Therefore ,

$$det(Q-PM^{-1}N) = \lambda - \frac{2(n^2+1)(n-1)}{(\lambda-(n^2-1)\sqrt{2})}$$

$$det(\lambda I - A_{SO}(G_S)) = det(M)det(Q-PM^{-1}N)$$

$$= \lambda^{n-2}(\lambda-(n^2-1)\sqrt{2}) \times [\lambda - \frac{2(n^2+1)(n-1)}{(\lambda-(n^2-1)\sqrt{2})}]$$

$$= \lambda^{n-2}(\lambda-(n^2-1)\sqrt{2})(\lambda - \frac{2(n^2+1)(n-1)}{(\lambda-(n^2-1)\sqrt{2})})$$

$$= \lambda^{n-2}(\lambda^2-(n^2-1)\sqrt{2}\lambda-2(n^2+1)(n-1))$$

$$= \lambda^{n-2}(\lambda - \frac{(n^2-1)\sqrt{2}-\sqrt{2n^4+8n^3-12n^2+8n-6}}{2})(\lambda - \frac{(n^2-1)\sqrt{2}+\sqrt{2n^4+8n^3-12n^2+8n-6}}{2})$$

Hence, eigenvalues of G_S are

$\lambda_1 = \frac{(n^2-1)\sqrt{2}+\sqrt{2n^4+8n^3-12n^2+8n-6}}{2}$, $\lambda_2 = \frac{(n^2-1)\sqrt{2}-\sqrt{2n^4+8n^3-12n^2+8n-6}}{2}$, and $\lambda_i = 0 \ for \ i \geq 3$.

Theorem 3. *Let* $G = K_n$ *be a complete graph with n vertices. If* G_S *is a graph obtained from* G *by adding* $n-1$ *self loops, then Sombor energy of* G_S *is* $E_{SO}(G_S) = (n-2)\sqrt{2}\frac{n^2-1}{n} + \sqrt{2n^4+8n^3-12n^2+8n-6}$.

Proof. By definition Sombor energy of matrix with self loop is ,

$$E_{SO}(G_s) = \sum_{i=1}^{n} |\lambda_i(G_s) - \frac{\sqrt{2}\sum_{j=1}^{\sigma} d_j}{n}|$$

By lemma 1, for complete graph with n-1 selfloop

$\sum_{j=1}^{\sigma} d_j = n^2-1$ and its eigenvalues are $\lambda_1 = \frac{(n^2-1)\sqrt{2}+\sqrt{2n^4+8n^3-12n^2+8n-6}}{2}$, $\lambda_2 = \frac{(n^2-1)\sqrt{2}-\sqrt{2n^4+8n^3-12n^2+8n-6}}{2}$, and $\lambda_i = 0 \ for\ i \geq 3$.

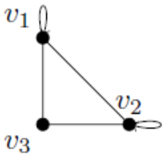
$$E_{SO}(G_s) = |\lambda_1 - \frac{(n^2-1)\sqrt{2}}{n}| + |\lambda_2 - \frac{(n^2-1)\sqrt{2}}{n}| + \sum_{i=3}^{n} |-\frac{(n^2-1)\sqrt{2}}{n}|$$

$$= |\frac{(n^2-1)\sqrt{2}+\sqrt{2n^4+8n^3-12n^2+8n-6}}{2} - \frac{(n^2-1)\sqrt{2}}{n}| + |\frac{(n^2-1)\sqrt{2}-\sqrt{2n^4+8n^3-12n^2+8n-6}}{2} - \frac{(n^2-1)\sqrt{2}}{n}| + (n-2)\frac{(n^2-1)\sqrt{2}}{n}$$

$$= \frac{(n^2-1)\sqrt{2}(n-2)}{2n} + \frac{\sqrt{2n^4+8n^3-12n^2+8n-6}}{2} + [-\frac{(n^2-1)\sqrt{2}(n-2)}{2n} + \frac{\sqrt{2n^4+8n^3-12n^2+8n-6}}{2}] + (n-2)\frac{(n^2-1)\sqrt{2}}{n}$$

$$= \sqrt{2n^4+8n^3-12n^2+8n-6} + (n-2)\frac{(n^2-1)\sqrt{2}}{n}.$$

Example 1. *Consider $G = K_3$ is complete graph with 3 vertices and G_S is the graph obtained by adding 2 loops to graph $G = K_3$.*



The Sombor Energy of graph G_S is given by

$$E_{SO}(G_S) = \sum_{i=1}^{3} |\lambda_i(G_s) - \frac{\sqrt{2}\sum_{j=1}^{\sigma} d_j}{3}|$$

Sombor matrix of G_S is $A_{SO}(G_S) = \begin{bmatrix} 4\sqrt{2} & 4\sqrt{2} & 2\sqrt{5} \\ 4\sqrt{2} & 4\sqrt{2} & 2\sqrt{5} \\ 2\sqrt{5} & 2\sqrt{5} & 0 \end{bmatrix}$

Here $\lambda_1 = 10\sqrt{2}, \lambda_2 = -2\sqrt{2}, \lambda_3 = 0$

$$E_{SO}(G_S) = |10\sqrt{2} - \frac{8\sqrt{2}}{3}| + |-2\sqrt{2} - \frac{8\sqrt{2}}{3}| + |0 - \frac{8\sqrt{2}}{3}| = 20.741798915$$

In above theorem putting n=3 we get,

$$E_{SO}(G_S) = (3-2)\sqrt{2}\frac{3^2-1}{3} + \sqrt{2(3)^4 + 8(3)^3 - 12(3)^2 + 8(3) - 6}$$

$$= \sqrt{2}\tfrac{8}{3} + \sqrt{288} = \sqrt{2}\tfrac{8}{3} + 12\sqrt{2} = 20.741798915.$$

Lemma 2. *Let $G = K_n$ be a complete graph with n vertices. If G_S is a graph obtained from G by adding σ self loops, then Sombor eigenvalues of Sombor matrix G_S are* 0 with multiplicity $\sigma - 1$, $-(n-1)\sqrt{2}$ with multiplicity $n - \sigma - 1$, $\frac{\sqrt{2}((n-1)^2+2\sigma)+\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2}$ with multiplicity 1, and $\frac{\sqrt{2}((n-1)^2+2\sigma)-\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2}$ with multiplicity 1.

Proof. Let J be the $n \times n$ matrix with all entries one, I be $n \times n$ identity matrix.

Sombor matrix of complete graph G_S with σ self loop is

$$A_{SO}(G_S) = \begin{bmatrix} (n+1)\sqrt{2}J_{\sigma\times\sigma} & \sqrt{2(n^2+1)}J_{\sigma\times n-\sigma} \\ \sqrt{2(n^2+1)}J_{n-\sigma\times\sigma} & (n-1)\sqrt{2}(J-I)_{n-\sigma\times n-\sigma} \end{bmatrix}_{n\times n}$$

Then,

$$\det(\lambda I - A_{SO}(G_S))$$
$$= det\begin{vmatrix} \lambda I_{\sigma\times\sigma} - (n+1)\sqrt{2}J_{\sigma\times\sigma} & -\sqrt{2(n^2+1)}J_{\sigma\times n-\sigma} \\ -\sqrt{2(n^2+1)}J_{n-\sigma\times\sigma} & \lambda I_{n-\sigma\times n-\sigma} - (n-1)\sqrt{2}(J-I)_{n-\sigma\times n-\sigma} \end{vmatrix}_{n\times n}$$

$$\det(\lambda I - A_{SO}(G_S)) =$$
$$\begin{vmatrix} \lambda I_{\sigma\times\sigma} - (n+1)\sqrt{2}J_{\sigma\times\sigma} & -\sqrt{2(n^2+1)}J_{\sigma\times n-\sigma} \\ -\sqrt{2(n^2+1)}J_{n-\sigma\times\sigma} & (\lambda + (n-1)\sqrt{2})I_{n-\sigma\times n-\sigma} - (n-1)\sqrt{2}J_{n-\sigma\times n-\sigma} \end{vmatrix}_{n\times n}$$

If M is non-singular square matrix,

$$det\begin{bmatrix} M & N \\ P & Q \end{bmatrix} = det(M)det(Q - PM^{-1}N)$$

here, $M = \lambda I_{\sigma\times\sigma} - (n+1)\sqrt{2}J_{\sigma\times\sigma}, N = -\sqrt{2(n^2+1)}J_{\sigma\times n-\sigma}, P = -\sqrt{2(n^2+1)}J_{n-\sigma\times\sigma}$ and $Q = (\lambda + (n-1)\sqrt{2})I_{n-\sigma\times n-\sigma} - (n-1)\sqrt{2}J_{n-\sigma\times n-\sigma},$

Then, $det(M) = \lambda^{\sigma-1}(\lambda - (n+1)\sqrt{2}\sigma)$. Therefore

$$M^{-1} = \frac{1}{\lambda^{\sigma-1}(\lambda-(n+1)\sqrt{2}\sigma)}(\lambda^{\sigma-2}(\lambda-(n+1)\sqrt{2}\sigma I_{\sigma\times\sigma}) + \lambda^{\sigma-2}(n+1)\sqrt{2}J_{\sigma\times\sigma})$$

$PM^{-1} = -\frac{\sqrt{2(n^2+1)}}{(\lambda-(n+1)\sqrt{2}\sigma)}J_{n-\sigma\times\sigma}$ and $PM^{-1}N = \frac{2(n^2+1)\sigma}{(\lambda-(n+1)\sqrt{2}\sigma)}J_{n-\sigma\times n-\sigma}$. So

$$Q - PM^{-1}N = (\lambda+(n-1)\sqrt{2})I_{n-\sigma\times n-\sigma} - (n-1)\sqrt{2}J_{n-\sigma\times n-\sigma} - \frac{2(n^2+1)\sigma}{(\lambda-(n+1)\sqrt{2}\sigma)}J_{n-\sigma\times n-\sigma}$$

The eigenvalues of $(\lambda+(n-1)\sqrt{2})I_{n-\sigma\times n-\sigma}$ are $(\lambda+(n-1)\sqrt{2})$ with n-σ multiplicity and eigenvalues of $(n-1)\sqrt{2}J_{n-\sigma\times n-\sigma} + \frac{2(n^2+1)\sigma}{(\lambda-(n+1)\sqrt{2}\sigma)}J_{n-\sigma\times n-\sigma}$ are 0 with multiplicity $n-\sigma-1$ and $\frac{(\lambda(n-1)\sqrt{2}+4\sigma)(n-\sigma)}{\lambda-(n+1)\sqrt{2}\sigma)}$ with multiplicity 1.

Hence, the eigenvalues of $Q-PM^{-1}N$ are $(\lambda+(n-1)\sqrt{2}) - \frac{(\lambda(n-1)\sqrt{2}+4\sigma)(n-\sigma)}{\lambda-(n+1)\sqrt{2}\sigma)}$ with multiplicity 1 and $\left(\lambda+(n-1)\sqrt{2}\right)$ with multiplicity $n-\sigma-1$.

Then

$$det(Q-PM^{-1}N) = (\lambda+(n-1)\sqrt{2})^{n-\sigma-1}(\frac{\lambda^2-\sqrt{2}\lambda((n-1)^2+2\sigma)-\sigma(2(n^2-1)+4(n-\sigma))}{\lambda-(n+1)\sqrt{2}\sigma)})$$

$$det(M)det(Q-PM^{-1}N) = \lambda^{\sigma-1}(\lambda+(n-1)\sqrt{2})^{n-\sigma-1}(\lambda^2-\sqrt{2}\lambda((n-1)^2+2\sigma)-\sigma(2(n^2-1)+4(n-\sigma)))$$

Hence, eigenvalues of $A_{SO}(G_S)$ are 0 with multiplicity $\sigma-1$, $-(n-1)\sqrt{2}$ with multiplicity $n-\sigma-1$, $\frac{\sqrt{2}((n-1)^2+2\sigma)+\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2}$ with multiplicity 1, and

$\frac{\sqrt{2}((n-1)^2+2\sigma)-\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2}$ with multiplicity 1.

Theorem 4. *Let $G = K_n$ be a complete graph with n vertices. If G_S is a graph obtained from G by adding σ self loops, then Sombor energy of G_S is* $E_{SO}(G_S) = \sqrt{2}\frac{(n+1)(\sigma-1)\sigma+(n-\sigma-1)(n(n-1)+(n+1)\sigma)}{n} + \sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}$.

Proof. By definition, the Sombor energy of matrix with σ self loop is ,

$$E_{SO}(G_s) = \sum_{i=1}^{n} |\lambda_i(G_s) - \frac{\sqrt{2}\sum_{j=1}^{\sigma} d_j}{n}|$$

By lemma 2, for complete graph with σ selfloop has $\sum_{j=1}^{\sigma} d_j = (n+1)\sigma$ and eigenvalues are 0 with multiplicity $\sigma - 1$, $-(n-1)\sqrt{2}$ with multiplicity $n - \sigma - 1$,

$\frac{\sqrt{2}((n-1)^2+2\sigma)+\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2}$ with multiplicity 1,

$\frac{\sqrt{2}((n-1)^2+2\sigma)-\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2}$ with multiplicity 1.

Then Somber energy becomes

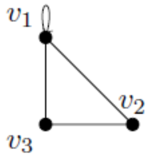
$$\begin{aligned}
E_{SO}(G_s) = {} & \sum_{i=1}^{\sigma-1} |0 - \frac{\sqrt{2}(n+1)\sigma}{n}| + \sum_{i=1}^{n-\sigma-1} | - (n-1)\sqrt{2} - \frac{\sqrt{2}(n+1)\sigma}{n}| \\
& + |\frac{\sqrt{2}((n-1)^2+2\sigma)+\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2} \\
& - \frac{\sqrt{2}(n+1))\sigma}{n}| \\
& + |\frac{\sqrt{2}((n-1)^2+2\sigma)-\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2} \\
& - \frac{\sqrt{2}(n+1))\sigma}{n}|
\end{aligned}$$

$$\begin{aligned}
= {} & (\sigma-1)\frac{\sqrt{2}(n+1)\sigma}{n} + (n-\sigma-1)((n-1)\sqrt{2} + \frac{\sqrt{2}(n+1)\sigma}{n}) \\
& + \frac{\sqrt{2}((n-1)^2+2\sigma)+\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2} \\
& - \frac{\sqrt{2}(n+1))\sigma}{n} \\
& - \frac{\sqrt{2}((n-1)^2+2\sigma)-\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2} \\
& + \frac{\sqrt{2}(n+1))\sigma}{n}
\end{aligned}$$

$$\begin{aligned}
= {} & (\sigma-1)\frac{\sqrt{2}(n+1)\sigma}{n} + (n-\sigma-1)((n-1)\sqrt{2} + \frac{\sqrt{2}(n+1)\sigma}{n}) \\
& + \frac{\sqrt{2}((n-1)^2+2\sigma)+\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2} \\
& - \frac{\sqrt{2}((n-1)^2+2\sigma)-\sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))}}{2}
\end{aligned}$$

$$= \sqrt{2}\,\frac{(n+1)(\sigma-1)\sigma+(n-\sigma-1)(n(n-1)+(n+1)\sigma)}{n} + \sqrt{2((n-1)^2+2\sigma)^2+4\sigma(2(n^2-1)+4(n-\sigma))} \,.$$

Example 2. *Consider* $G = K_3$ *is complete graph with 3 vertices and* G_S *is the graph obtained by adding 1 loops to graph* $G = K_3$.



Sombor matrix of G_S is

$$A_{SO}(G_S) = \begin{bmatrix} 4\sqrt{2} & 2\sqrt{5} & 2\sqrt{5} \\ 2\sqrt{5} & 0 & 2\sqrt{2} \\ 2\sqrt{5} & 2\sqrt{2} & 0 \end{bmatrix}$$

Here$\lambda_1 = 10.7234, \lambda_2 = -2.82843, \lambda_3 = -2.2381$ The Sombor Energy of graph G_S is given by

$$E_{SO}(G_S) = \sum_{i=1}^{3} |\lambda_i(G_s) - \frac{\sqrt{2}\sum_{j=1}^{\sigma} d_j}{3}|$$

$$= |10.7234 - \frac{4\sqrt{2}}{3}| + |-2.82843 - \frac{4\sqrt{2}}{3}| + |-2.2381 - \frac{4\sqrt{2}}{3}|$$

$$= 17.675548083$$

In above theorem putting $\sigma = 1$ and n=3 we get,

$$E_{SO}(G_S) = \sqrt{2}\frac{(3+1)(1-1) + (3-1-1)(3(3-1) + (3+1)1)}{3}$$

$$+ \sqrt{2((3-1)^2 + 2)^2 + 4(2(3^2-1) + 4(3-1))} = 10\frac{\sqrt{2}}{3} + \sqrt{72 + 96}$$

$$= 17.675526605.$$

Theorem 5. *Let G be the complete graph of order n and* G^l *be the graph obtained from G by adding a loop on each vertex of G then Sombor Energy* $E_{SO}(G \cup \quad G^l) = \frac{2n}{n-1} E_{SO}(G)$

Proof. Let $H_n = (G \cup \quad G^l)$. The graph H_n contains 2n vertices and n loops. The Sombor matrix of H_n is given by:

$$A_{SO}(H_n) = \left[(n-1)\sqrt{2}(J-I)_{n\times n} \quad [0]_{n\times n} \qquad n \times n\middle|(n+1)\sqrt{2}(J)_{n\times n}\right]$$

The characteristics polynomial of above matrix is given by:

$$\phi(H_n:x) = \left[xI-(n-1)\sqrt{2}(J-I)_{n\times n} \quad [0]_{n\times n} \quad n\times n \middle| xI-(n+1)\sqrt{2}(J)_{n\times n}\right]$$

If $\lambda_1,\lambda_2,\ldots,\lambda_n$ are eigenvalues of $A_{SO}(H_n)$, then

$$\lambda_1=(n-1)^2\sqrt{2}, \lambda_i=-(n-1)\sqrt{2} \quad for \quad i=2,3,\ldots,n, \lambda_{n+1}=n(n+1)\sqrt{2}, \lambda_j = 0 \quad for \quad j=n+2,n+3,\ldots,2n$$

Sombor Energy of H_n is given by ,

$$E_{SO}(G_s)=\sum_{i=1}^{2n}|\lambda_i(G_s)-\frac{\sqrt{2}\sum_{j=1}^{\sigma}d_j}{n}|$$

Here $\sum_{j=1}^{\sigma} d_j = n(n+1)$

Hence,

$$E_{SO}(G_s)=\sum_{i=1}^{2n}|\lambda_i(G_s)-\frac{\sqrt{2}n(n+1)}{2n}|$$

$$= |(n-1)^2\sqrt{2}-\frac{\sqrt{2}(n+1)}{2}|+(n-1)|-(n-1)\sqrt{2}-\frac{\sqrt{2}(n+1)}{2}|+|n(n+1)\sqrt{2} -\frac{\sqrt{2}(n+1)}{2}|+(n-1)|0-\frac{\sqrt{2}(n+1)}{2}|$$

$$= [2(n-1)^2-(n+1)]\frac{\sqrt{2}}{2}+(n-1)(2(n-1)+(n+1))\frac{\sqrt{2}}{2}+[2n(n+1)-(n +1)]\frac{\sqrt{2}}{2}+(n-1)(n+1)\frac{\sqrt{2}}{2}$$

$$=\frac{1}{\sqrt{2}}[2n^2-4n+2-n-1+(n-1)(2n-2+n+1)+(2n^2+2n-n-1)+n^2 -1]$$

$$=\frac{1}{\sqrt{2}}(8n^2-8n)$$

$$=4\sqrt{2}n(n-1)$$

Therefore,

$$E_{SO}(G_s)=4\sqrt{2}n(n-1) \tag{3}$$

The Sombor matrix of graph $G=K_n$ is ,

$$A_{SO}(G)=\left[(n-1)\sqrt{2}(J-I)_{n\times n}\right]$$

Eigenvalues of $A_{SO}(G)$ are$(n-1)^2\sqrt{2}$ with multiplicity 1 and $-(n-1)\sqrt{2}$ with $n-1$ mutilplicity.

Hence Sombor Energy of G is given by,

$$E_{SO}(G) = \sum_{i=1}^{n} |\lambda_i|$$

$$= (n-1)^2\sqrt{2} + (n-1)(n-1)\sqrt{2}$$

$$= 2(n-1)^2\sqrt{2}$$

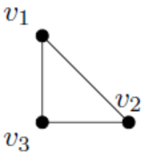
Therefore,

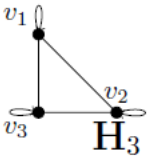
$$E_{SO}(G) = 2(n-1)^2\sqrt{2} \tag{4}$$

Hence, by equation (3) and (4),

$$E_{SO}(H_n) = \frac{2n}{(n-1)} E_{SO}(G)$$

Example 3. *Consider the graph* $H_3 = K_3 \cup \quad K_3^l$ *and* $G = K_3$. *The graph* H_3 *contains 6 vertices and three loops. It is known that* $E_{SO}(G) = 8\sqrt{2}$





The Sombor matrix of H_3 is

$$A_{SO}(H_3) = \begin{bmatrix} 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 0 \\ 2\sqrt{2} & 0 & 2\sqrt{2} & 0 & 0 & 0 \\ 2\sqrt{2} & 2\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\sqrt{2} & 4\sqrt{2} & 4\sqrt{2} \\ 0 & 0 & 0 & 4\sqrt{2} & 4\sqrt{2} & 4\sqrt{2} \\ 0 & 0 & 0 & 4\sqrt{2} & 4\sqrt{2} & 4\sqrt{2} \end{bmatrix}$$

The eigen values of H_3 are $4\sqrt{2}, 12\sqrt{2}, \left(-2\sqrt{2}\right)$ with multiplicity 2, and 0 with multiplicity 2.

Hence,

$E_{SO}(H_3) = |4\sqrt{2} - 2\sqrt{2}| + 12\sqrt{2} - 2\sqrt{2}| + 2| - 2\sqrt{2} - 2\sqrt{2}| + 2|0 - 2\sqrt{2}| = 24\sqrt{2}.$

Therefore, $E_{SO}(H_3) = \frac{6}{2} E_{SO}(G)$.

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