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A Generalized Study of Zero Divisor Graphs of Boolean Rings $\mathbb{Z}2n = \mathbb{Z}2 \times \mathbb{Z}2 \times \cdots \times \mathbb{Z}2$

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Abstract:
The Zero-divisor graph of a commutative ring R is defined as a graph in which the vertices represent the non-Zero Zero divisors of R, and two vertices x and y are connected if and only if x× y = 0. In this study, we focus on examining the Zero divisor graphs of Boolean rings and deriving insights from their graphical representations.

Keywords: Zero-divisor graph, Commutative Ring, Boolean Ring $\mathbb{Z}2n = \mathbb{Z}2 \times \mathbb{Z}2 \times \mathbb{Z}2 \times \ldots \times \mathbb{Z}2$, Girth, Diameter.

1 Introduction

The roots of graph theory can be traced back to 1735 when the Swiss mathematician Leonhard Euler provided a groundbreaking solution to the Konigsberg bridge problem,[1] introducing a novel conceptual framework. Euler's subsequent theorem marked a seminal moment in the field, paving the way for the development of Eulerian graphs.

The exploration of cycles on polyhedra by the Revd. Thomas Penyngton Kirkman (1806-95) and Sir William Rowan Hamilton (1805-65) led to the conception of Hamiltonian graphs. The foundational notion of a tree, defined as a connected graph devoid of cycles, first emerged implic- itly in the work of Gustav Kirchho (1824-87), who applied graph-theoretical principles to analy$\mathbb{Z}$e currents in electrical networks. Later, Arthur Cayley (1821-95), James Joseph Sylvester (1806- 97), Georg Polya (1887-1985), and others introduced trees in connection with the enumeration of specific chemical structures.

I. Beck introduced the concept of a Zero-divisor graph in 1988.[2] Denoted by $\Gamma(R)$, the Zero divisor graph of a ring R is a simple graph whose vertices represent elements of R, with two vertices x and y being adjacent if and only if their product equals Zero. Beck's research focused on colorings of R.[2] This perspective originated in a paper by D.F. Anderson and P.S. Livingston[4] and has since seen further development.[6, 12, 13] For instance, it is established in[15] that all Zero-divisor graphs are connected, meaning a path exists between any pair of vertices.

A ring R is classified as Boolean if every element r in R satisfies the condition $r^2 = r$. In particular, a Boolean ring is invariably commutative with a characteristic of 2. A graph is deemed Boolean if it exhibits isomorphism to the Zero-divisor graph of a Boolean ring.[10–12, 16, 17].

2 Preliminaries

Graph theory is a branch of mathematics concerned with the study of graphs, which consist of nodes and edges, often used to represent mathematical concepts visually. The field explores the connections between the vertices and edges within these structures. Below, we will delve into fundamental definitions to facilitate comprehension of key terminologies.

A Zero-divisor graph is an undirected graph representing the Zero-divisors of a commutative ring. Its vertices correspond to elements of the ring, and its edges connect pairs of elements whose product equals Zero.

In this context, two vertices u and v within the graph G are considered connected if there exists a path from u to v. This notion of connection forms an equivalence relation on the vertex set V. Consequently, the vertex set V can be partitioned into nonempty subsets $V_1, V_2, ..., V_w$, where two vertices u and v are connected if and only if they both belong to the same subset V_i. The subgraphs

$G[V_1], G[V_2], ..., G[V_w]$ are referred to as the components of G. If G contains only one component,

it is termed connected; otherwise, it is classified as disconnected.

Additionally, a graph is considered simple when it does not contain loops, meaning there are no edges connecting a vertex to itself, and when each pair of vertices is connected by at most one edge, ensuring that no two edges join the same pair of vertices. (A self-loop is an edge that joins a single endpoint to itself) [Refer Figure1]

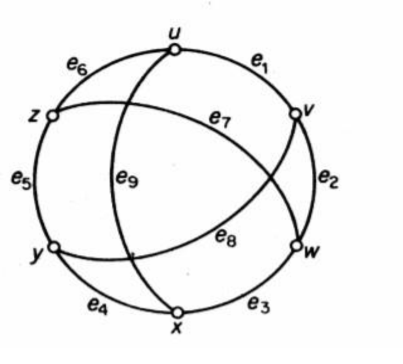


Fig 1: SIMPLE GRAPH

Graph theory describes a walk-in graph G as a finite, non-null sequence denoted by $W = v_0e_1v_1e_2v_2...e_kv_k$. This sequence alternates between vertices and edges, where for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. This sequence is called a walk from v_0 to v_k, or a (v_0, v_k) walk. The vertices v_0 and v_k are respectively termed the origin and terminus of the walk, while $v_1, v_2, . . . , v_{k-1}$ are its internal vertices. The integer k is the length of W. Example:uavfyfvgyhwbv [Refer Figure 2]

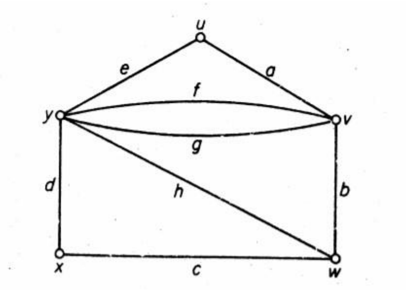


Fig 2: WALK

If the edges $e_1, e_2, ..., e_k$ of a walk W are distinct, W is called a trail.

Example: wcxdyhwbvgy

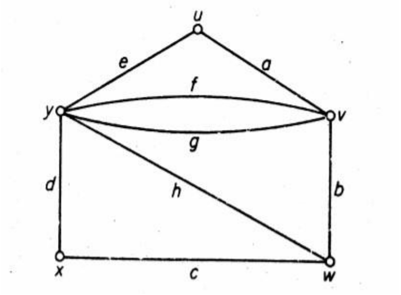
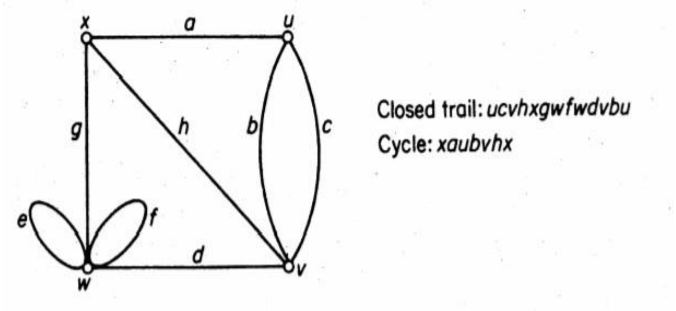


Fig 3: TRAIL and PATH

A path is a trail in which no vertices (except possibly the end vertices) are repeated i.e. the vertices $v_0, v_1, ..., v_k$ are distinct, W is called a path.

Example: xcwhyeuav A circuit is a closed trail (that is end vertices are the same) with at least one edge known as Circuit.



Closed trail: ucvhxgwfwdvbu
Cycle: xaubvhx

Fig 4: CYCLE

A walk is closed if it has a positive length and its origin and terminus are the same. A closed trail whose origin and internal vertices are distinct is a cycle. The graph's diameter is the maximum distance between the pair of vertices. It can also be defined as the maximal distance between the pair of vertices.

Example: $BC \rightarrow CF \rightarrow FG$ A radius of the graph exists only if it has a diameter. The minimum among all the maximum distances between vertexes to all other vertices is considered as the radius of Graph G. It is denoted as r(G).

Example: $BC \rightarrow CA$ The girth of a graph is the length of the shortest cycle contained in the graph. If the graph does not contain any cycles (i.e., it's an acyclic graph), its girth is defined as infinity.

- a 4-cycle (square) has girth 4. A grid also has a girth 4, and a triangular mesh has a girth 3. A graph with a girth of four or more is triangle-free.

3 Main results

3.1 Theorem

Statement: Let $\mathbb{Z}_2{}^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times ... \times \mathbb{Z}_2$ be a Boolean ring and $\mathbb{Z}_2{}^n$ has $(2^n - 2)$ Zero divisors i.e $|\mathbb{Z}_2{}^n| = 2^n$ and $|\mathbb{Z}(\mathbb{Z}_2{}^n)| = 2^n - 2$, then n vertices in Zero divisor graph $\Gamma(\mathbb{Z}^n)$ have 2^{n-1} – degree.

Proof Let $\mathbb{Z}_2{}^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times ... \times \mathbb{Z}_2$, be a Boolean ring and $\mathbb{Z}_2{}^n$ has $(2^n - 2)$ Zero divisors i.e. Zero divisors and $\mathbb{Z}_2{}^n$ *is* given by,

$\mathbb{Z}_2{}^n = \{(b_1, b_2, b_3, ..., b_n) \mid b_i's \in \{0, 1\}, i = 1, 2, ..., n\}$

Since, each b_is has two choices 0 and 1, so there are 2^n elements in $\mathbb{Z}_2{}^n$

$$|\mathbb{Z}_2{}^n| = 2^n$$

The set of Zero divisors of $\mathbb{Z}_2{}^n$ is denoted by $\mathbb{Z}(\mathbb{Z}_2{}^n)$ and is given by,

$\mathbb{Z}(\mathbb{Z}_2{}^n) = \{(b_1, b_2, b_3, ..., b_n) \mid b_is \in \{0, 1\}$, all $b_i \neq 0$, and all $b_i \neq 1, i = 1, 2, ..., n\}$

$$\therefore |\mathbb{Z}(\mathbb{Z}_2{}^n)| = 2^n - 2$$

The vertices $(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), (0, 0, 1, 0, ..., 0), ..., (0, 0, 0, .., 1)$ are adjacent to

$(0, b_2, b_3, ..., b_n), (b_1, 0, b_3, ..., b_n), ..., (b_1, b_2, b_3, ..., 0)$ respectively.

The vertex $(1, 0, 0, .., 0)$ is adjacent to $(0, b_2, b_3, .., b_n)$ and $\mathbb{Z}_2{}^n$ has two choices 0 or 1 but all $bi \neq 0, \forall i = 1, 2, ..., n$.

i.e the vertex $(1, 0, 0, ..., 0)$ is adjacent to $(0, b_2, b_3, ..., b_n)$ except $(0, 0, 0, ..., 0)$. i.e the degree of $(1, 0, 0, ..., 0)$ is $2^{n-1} - 1$ since $(1, 0, 0, ..., 0)$ is adjacent with $2^{n-1} - 1$ number of vertices. Similarly, the degree of

$(0, 1, 0, ..., 0), (0, 0, 1, 0, ..., 0), (0, 0, 0, 1, 0, ..., 0), ..., (0, 0, 0, ..., 1)$ is $2^{n-1} - 1$.

$\therefore$ These n elements have $2^{n-1} - 1$ degree.

$\therefore$ The n vertices in the Zero divisor graph of $\mathbb{Z}_2{}^n$, $\Gamma(\mathbb{Z}_2{}^n)$ have $2^{n-1} - 1$ degree.

3.1.1 Example

$B_3 = \mathbb{Z}_2^3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

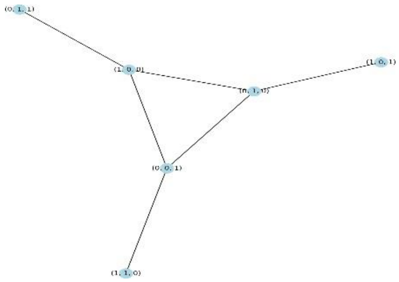


Fig 5: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

$\mathbb{Z}_2{}^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0), (1, 1, 1)\}$

$\mathbb{Z}\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0)\}$

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure above.

Here, $n = 3$. Number of vertices in the graph of $\mathbb{Z}\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 2^3 - 2 = 6$

Degree of vertex $(0, 0, 1) = 2^{3-1} - 1 = 2^2 - 1 = 3$

Degree of vertex $(0, 1, 0) = 2^{3-1} - 1 = 2^2 - 1 = 3$

Degree of vertex $(0, 0, 1) = 2^{3-1} - 1 = 2^2 - 1 = 3$

∴ 3 vertices have degree 3.

∴ The theorem holds.

3.1.2 ***Example***

$\mathbb{Z}_2{}^4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

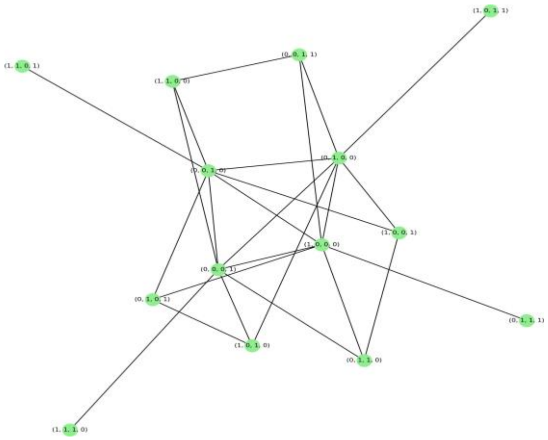


Fig 6: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is shown in the figure above. Here, $n = 4$. The number of vertices in the graph of Zero divisors of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is: $2^4 - 2 = 14$.

The degree of the vertices is computed as follows:

Degree of vertex $(0, 0, 0, 1) = 2^{4-1} - 1 = 2^2 - 1 = 7$

Degree of vertex $(0, 0, 1, 0) = 2^{4-1} - 1 = 2^2 - 1 = 7$

Degree of vertex (0, 1, 0, 0) = $2^{4-1} - 1 = 2^2 - 1 = 7$

Degree of vertex (1, 0, 0, 0) = $2^{4-1} - 1 = 2^2 - 1 = 7$

∴ 4 vertices have degree 7, and the theorem holds.

3.1.3 ***Example***

Let $\mathbb{Z}^5_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ = {(0, 0, 0, 0, 0), (0, 0, 0, 0, 1), (0, 0, 0, 1, 0), (0, 0, 0, 1, 1), (0, 0, 1, 0, 0),

(0, 0, 1, 0, 1), (0, 0, 1, 1, 0), (0, 0, 1, 1, 1), (0, 1, 0, 0, 0), (0, 1, 0, 0, 1), (0, 1, 0, 1, 0), (0, 1, 0, 1, 1), (0, 1, 1, 0, 0),

(0, 1, 1, 0, 1), (0, 1, 1, 1, 0), (0, 1, 1, 1, 1), (1, 0, 0, 0, 0), (1, 0, 0, 0, 1), (1, 0, 0, 1, 0), (1, 0, 0, 1, 1), (1, 0, 1, 0, 0),

(1, 0, 1, 0, 1), (1, 0, 1, 1, 0), (1, 0, 1, 1, 1), (1, 1, 0, 0, 0), (1, 1, 0, 0, 1), (1, 1, 0, 1, 0), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0),

(1, 1, 1, 0, 1), (1, 1, 1, 1, 0), (1, 1, 1, 1, 1)}

$\mathbb{Z}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ = {(0, 0, 0, 0, 1), (0, 0, 0, 1, 0), (0, 0, 0, 1, 1), (0, 0, 1, 0, 0), (0, 0, 1, 0, 1), (0, 0, 1, 1, 0),

(0, 0, 1, 1, 1), (0, 1, 0, 0, 0), (0, 1, 0, 0, 1), (0, 1, 0, 1, 0), (0, 1, 0, 1, 1), (0, 1, 1, 0, 0), (0, 1, 1, 0, 1), (0, 1, 1, 1, 0),

(0, 1, 1, 1, 1), (1, 0, 0, 0, 0), (1, 0, 0, 0, 1), (1, 0, 0, 1, 0), (1, 0, 0, 1, 1), (1, 0, 1, 0, 0), (1, 0, 1, 0, 1), (1, 0, 1, 1, 0),

(1, 0, 1, 1, 1), (1, 1, 0, 0, 0), (1, 1, 0, 0, 1), (1, 1, 0, 1, 0), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0),

(1, 1, 1, 0, 1), (1, 1, 1, 1, 0)} The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure below,

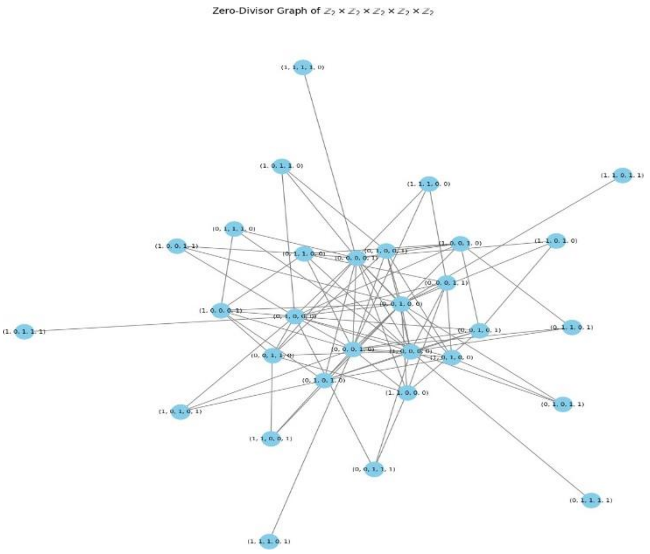


Fig 7: Γ ($\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$)

Here, n = 5 Number of vertices in $\mathbb{Z}$ ($\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$) = $2^5 - 2 = 30$

Degree of vertex $(0, 0, 0, 0, 1) = 2^{5-1} - 1 = 2^4 - 1 = 15$

Degree of vertex $(0, 0, 0, 1, 0) = 2^{5-1} - 1 = 2^4 - 1 = 15$

Degree of vertex $(0, 0, 1, 0, 0) = 2^{5-1} - 1 = 2^4 - 1 = 15$

Degree of vertex $(0, 1, 0, 0, 0) = 2^{5-1} - 1 = 2^4 - 1 = 15$

Degree of vertex $(1, 0, 0, 0, 0) = 2^{5-1} - 1 = 2^4 - 1 = 15$

∴ 5 vertices have degree 15.

∴ The theorem holds true.

3.2 ***Theorem***

Statement: Let $\mathbb{Z}_2^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ be a Boolean ring then n vertices of Zero divisor graph $\Gamma(\mathbb{Z}_2^n)$ have degree 1.

Proof: Let $\mathbb{Z}_2^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \ldots \mathbb{Z}_2$, be a Boolean ring $\mathbb{Z}_2^n$ has $(2^n - 2)$ Zero divisors and

$\mathbb{Z}_2^n$ is given by,

$\mathbb{Z}_2^n = \{(b_1, b_2, b_3, \ldots, b_n) \mid b_is \in \{0, 1\}, \forall i = 1, 2, \ldots, n\}$

Since, each b_is has two choices 0 and 1, so there are 2^n elements in $\mathbb{Z}_2^n$

$$|\mathbb{Z}_2^n| = 2^n$$

The set of Zero divisors of $\mathbb{Z}_2^n$ is denoted by $\mathbb{Z}(\mathbb{Z}_2^n)$ and is given by,

$\mathbb{Z}(\mathbb{Z}_2^n) = \{(b_1, b_2, b_3, \ldots, b_n) \mid b_is \in \{0, 1\}$, all $b_i \neq 0$, all $b_i \neq 1, i = 1, 2, \ldots, n\}$

$$|\mathbb{Z}(\mathbb{Z}_2^n)| = 2^n - 2$$

The vertices $(0, 1, 1, \ldots, 1), (1, 0, 1, 1, \ldots, 1), (1, 1, 0, 1, \ldots, 1), \ldots, (1, 1, 1, \ldots, 0)$ are adjacent to $(1, b_2, b_3, \ldots, b_n)$,

$(b_1, 1, b_3, \ldots, b_n), (b_1, b_2, 1, b_4, \ldots, b_n), \ldots, (b_1, b_2, b_3, \ldots, 1)$ respectively.

Clearly the vertex $(0, 1, 1, \ldots, 1)$ is adjacent to $(1, b_2, b_3, \ldots, b_n)$ and $b_2, b_3, \ldots, b_n$ have only one choice which is 0.

∴ The vertex $(0, 1, 1, \ldots, 1)$ is adjacent to $(1, 0, 0, \ldots, 0)$ only.

∴ The degree of $(0, 1, 1, \ldots, 1)$ is 1.

Similarly, the degree of $(1, 0, 1, 1, \ldots, 1), (1, 1, 0, 1, \ldots, 1), \ldots, (1, 1, 1, \ldots, 0)$ is 1.

∴ These n elements have degree 1.

∴ The n vertices of the Zero divisor graph of $\mathbb{Z}_2^n$ have degree 1

3.2.1 ***Example***

$\mathbb{Z}_2^2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$

$\mathbb{Z}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{(0, 1), (1, 0)\}$

Number of vertices in $\mathbb{Z}\ (\mathbb{Z}_2 \times \mathbb{Z}_2) = 2^2 - 2 = 2$

Degree of vertex (0,1) = 1

Degree of vertex (1,0) = 1

∴ 2 vertices have degree 1.

3.2.2 ***Example***

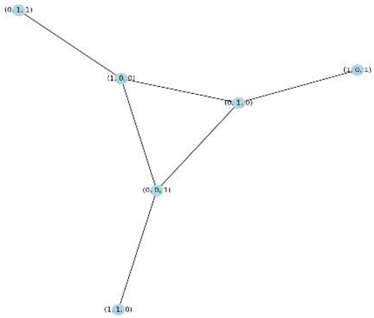


Fig 8: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

$\mathbb{Z}_2{}^3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0), (1, 1, 1)\}$

$\mathbb{Z}\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0)$

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure above. Number of vertices in the graph of $\mathbb{Z}\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 2^3 - 2 = 6$

Degree of vertex (0, 1, 1) = 1

Degree of vertex (1, 0, 1) = 1

Degree of vertex (1, 1, 0) = 1

∴ 3 vertices have degree 1.

3.2.3 ***Example***

$\mathbb{Z}_2{}^4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1), ((0, 1, 1, 0), (0, 1, 1, 1),$

$(1, 0, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1), (1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1)\}$

$\mathbb{Z}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = \{(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (0, 1, 1, 1),$

$(1, 0, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1), (1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0)\}$

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure below.

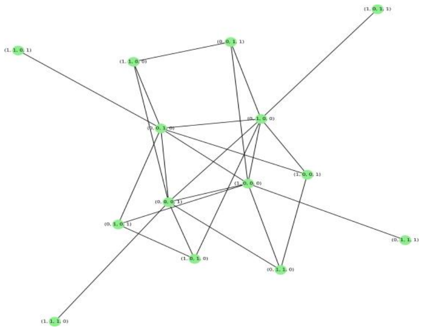


Fig 9: Γ ($\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$)

Here, $n = 4$. Number of vertices in the graph of $\mathbb{Z}$ ($\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$) = $2^4 - 2 = 14$

Degree of vertex (0, 1, 1, 1) = 1

Degree of vertex (1, 0, 1, 1) = 1

Degree of vertex (1, 1, 0, 1) = 1

Degree of vertex (1, 1, 1, 0) = 1

∴ 4 vertices have degree 1 and the theorem holds true.

3.2.4 ***Example***

$\mathbb{Z}_2^5 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ = {(0, 0, 0, 0, 0), (0, 0, 0, 0, 1), (0, 0, 0, 1, 0), (0, 0, 0, 1, 1), (0, 0, 1, 0, 0),

(0, 0, 1, 0, 1), (0, 0, 1, 1, 0), (0, 0, 1, 1, 1), (0, 1, 0, 0, 0), (0, 1, 0, 0, 1), (0, 1, 0, 1, 0), (0, 1, 0, 1, 1), (0, 1, 1, 0, 0),

(0, 1, 1, 0, 1), (0, 1, 1, 1, 0), (0, 1, 1, 1, 1), (1, 0, 0, 0, 0), (1, 0, 0, 0, 1), (1, 0, 0, 1, 0), (1, 0, 0, 1, 1), (1, 0, 1, 0, 0),

(1, 0, 1, 0, 1), (1, 0, 1, 1, 0), (1, 0, 1, 1, 1), (1, 1, 0, 0, 0), (1, 1, 0, 0, 1), (1, 1, 0, 1, 0), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0),

(1, 1, 1, 0, 1), (1, 1, 1, 1, 0), (1, 1, 1, 1, 1)}

$\mathbb{Z}(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2)$ = {(0, 0, 0, 0, 1), (0, 0, 0, 1, 0), (0, 0, 0, 1, 1) (0, 0, 1, 0, 0), (0, 0, 1, 0, 1), (0, 0, 1, 1, 0),

(0, 0, 1, 1, 1), (0, 1, 0, 0, 0), (0, 1, 0, 0, 1), (0, 1, 0, 1, 0), (0, 1, 0, 1, 1), (0, 1, 1, 0, 0), (0, 1, 1, 0, 1), (0, 1, 1, 1, 0),

(0, 1, 1, 1, 1), (1, 0, 0, 0, 0), (1, 0, 0, 0, 1), (1, 0, 0, 1, 0), (1, 0, 0, 1, 1), (1, 0, 1, 0, 0), (1, 0, 1, 0, 1), (1, 0, 1, 1, 0),

(1, 0, 1, 1, 1), (1, 1, 0, 0, 0), (1, 1, 0, 0, 1), (1, 1, 0, 1, 0), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0),

(1, 1, 1, 0, 1), (1, 1, 1, 1, 0)}

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure below,

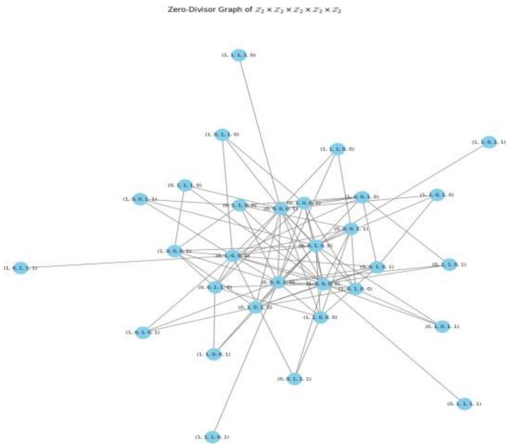


Fig 10: $\Gamma\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Here, $n = 5$

Number of vertices in $\mathbb{Z}\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 2^5 - 2 = 30$

Degree of vertex $(0, 1, 1, 1, 1) = 1$

Degree of vertex $(1, 0, 1, 1, 1) = 1$

Degree of vertex $(1, 1, 0, 1, 1) = 1$

Degree of vertex $(1, 1, 1, 0, 1) = 1$

Degree of vertex $(1, 1, 1, 1, 0) = 1$

∴ 5 vertices have degree 1.

∴ The theorem holds.

3.3 ***Corollary:***

All vertices of the Zero divisor graph $\Gamma(\mathbb{Z}_2{}^n)$ have odd degrees.

3.3.1 ***Example***

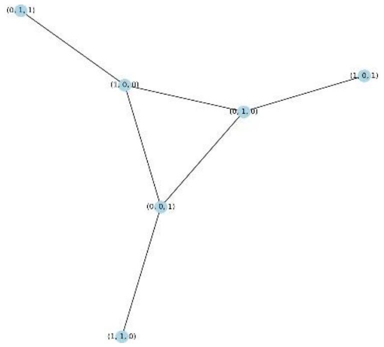


Fig 11: $\Gamma\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

$\mathbb{Z}_2{}^3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0), (1, 1, 1)\}$

$\mathbb{Z}\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0)$

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure above. Number of vertices in the graph of $\mathbb{Z}\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 2^3 - 2 = 6$

The degree of (0,1,1), (1,0,1), (1,1,0) is 1.

The degree of (0,0,1), (0,1,0), (1,0,0) is 3.

∴ All vertices of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ have odd degrees.

3.3.2 ***Example***

$\mathbb{Z}_2{}^4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2 = \{$(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1), ((0, 1, 1, 0), (0, 1, 1, 1),

(1, 0, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1), (1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1)$\}$

$\mathbb{Z}(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2) = \{$(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (0, 1, 1, 1),

(1, 0, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1), (1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0)$\}$

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure below.

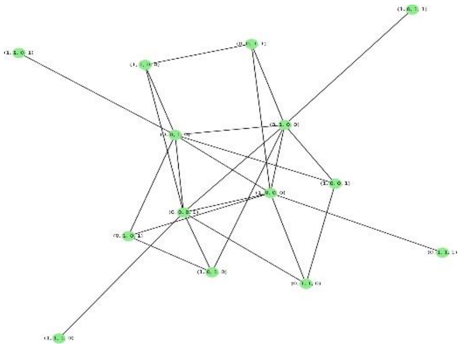


Fig 12: $\Gamma\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Here, $n = 4$.

Number of vertices in the graph of $\mathbb{Z}\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 2^4 - 2 = 14$

Degree of vertices (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0) is 7

Degree of vertices (0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0) is 3.

Degree of vertices (0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 0, 1), (1, 1, 1, 0) is 1.

∴ All vertices of Zero divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ have odd degree.

3.4 ***Theorem***

Statement: Let $\mathbb{Z}_2{}^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ be any Boolean ring then the Zero divisor graph of the Boolean ring $\Gamma(\mathbb{Z}_2{}^n)$ has a cycle of length n.

Proof: Let $\mathbb{Z}_2{}^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ be a Boolean ring $\mathbb{Z}_2{}^n$ has $(2^n - 2)$ Zero divisors and $\mathbb{Z}_2{}^n$ is given by,

$B_n = \{(b_1, b_2, b_3, ..., b_n) \mid b_i's \in \{0, 1\}, i = 1, 2, ..., n\}$

Since each b_is has two choices 0 and 1, there are 2^n elements in $\mathbb{Z}_2{}^n$.

$$|\mathbb{Z}_2{}^n| = 2^n$$

The set of Zero divisors of $\mathbb{Z}_2{}^n$ is denoted by $\mathbb{Z}(\mathbb{Z}_2{}^n)$ and is given by,

$\mathbb{Z}(\mathbb{Z}_2{}^n) = \{(b_1, b_2, b_3, ..., b_n) \mid b_is \in \{0, 1\},\text{all } b_i \neq 0, \text{ and all } b_i \neq 1, i = 1, 2, ..., n\}$

$$\therefore |\mathbb{Z}(\mathbb{Z}_2{}^n)| = 2^n - 2$$

The n vertices $(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), (0, 0, 1, 0, ..., 0), ..., (0, 0, 0, ..., 1)$ are adjacent to each other. But we consider the case where these vertices are only adjacent to its consecutive elements

i.e. $(1, 0, 0, ..., 0)$ is adjacent to $(0, 1, 0, ..., 0)$; $(0, 1, 0, ..., 0)$ is adjacent to $(0, 0, 1, 0, ..., 0)$; and so

on till $(0, 0, 0, ..., 1, 0)$ is adjacent to $(0, 0, 0, ..., 1)$ and $(0, 0, 0, ..., 1)$ is adjacent to $(1, 0, 0, ..., 0)$. So, we get these n vertices forming edges between them.

These n vertices form "n" edges between them. Hence, we get a closed loop formed by these n edges.

∴ The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has a cycle of length n.

3.4.1 ***Example***

$\mathbb{Z}_2{}^3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0), (1, 1, 1)\}$

$\mathbb{Z}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0)\}$

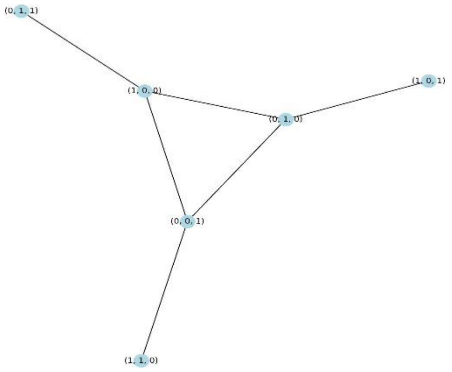


Fig 13: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure 10 above.

Number of vertices in the graph of $\mathbb{Z}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 2^3 - 2 = 6$

The vertices (0, 0, 1), (0, 1, 0) and (1, 0, 0) form a cycle.

∴ $\mathbb{Z}_2{}^3$ has a cycle of length 3.

∴ The theorem holds.

3.4.2 ***Example***

$\mathbb{Z}_2^4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ = {(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1), ((0, 1, 1, 0), (0, 1, 1, 1),

(1, 0, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1), (1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1)}

$\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ = {(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (0, 1, 1, 1),

(1, 0, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1), (1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0)}

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure below.

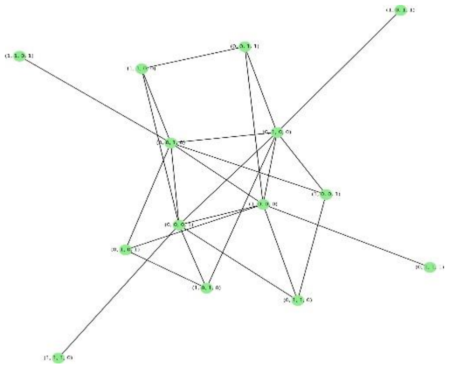


Fig 14: Γ $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Number of vertices in the graph of $\mathbb{Z}$ $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 2^4 - 2 = 14$

The vertices (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0) form a cycle of length 4.

The vertices (0, 1, 1, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 1) form a cycle of length 4.

The vertices (1, 0, 0, 0), (0, 1, 0, 1), (1, 0, 1, 0), (0, 1, 0, 0) form a cycle of length 4 and etc.

$\therefore \mathbb{Z}_2^n$ has a cycle of length 4 and the theorem holds.

3.4.3 ***Example:***

$B_5 = \mathbb{Z}_2^5 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ = {(0, 0, 0, 0, 0), (0, 0, 0, 0, 1), (0, 0, 0, 1, 0), (0, 0, 0, 1, 1), (0, 0, 1, 0, 0),

(0, 0, 1, 0, 1), (0, 0, 1, 1, 0), (0, 0, 1, 1, 1), (0, 1, 0, 0, 0), (0, 1, 0, 0, 1), (0, 1, 0, 1, 0), (0, 1, 0, 1, 1), (0, 1, 1, 0, 0),

(0, 1, 1, 0, 1), (0, 1, 1, 1, 0), (0, 1, 1, 1, 1), (1, 0, 0, 0, 0), (1, 0, 0, 0, 1), (1, 0, 0, 1, 0), (1, 0, 0, 1, 1), (1, 0, 1, 0, 0),

(1, 0, 1, 0, 1), (1, 0, 1, 1, 0), (1, 0, 1, 1, 1), (1, 1, 0, 0, 0), (1, 1, 0, 0, 1), (1, 1, 0, 1, 0), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0),

(1, 1, 1, 0, 1), (1, 1, 1, 1, 0), (1, 1, 1, 1, 1)}

$\mathbb{Z}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ = {(0, 0, 0, 0, 1), (0, 0, 0, 1, 0), (0, 0, 0, 1, 1) (0, 0, 1, 0, 0), (0, 0, 1, 0, 1), (0, 0, 1, 1, 0), (0, 0, 1, 1, 1), (0, 1, 0, 0, 0), (0, 1, 0, 0, 1), (0, 1, 0, 1, 0), (0, 1, 0, 1, 1), (0, 1, 1, 0, 0), (0, 1, 1, 0, 1), (0, 1, 1, 1, 0),

(0, 1, 1, 1, 1), (1, 0, 0, 0, 0), (1, 0, 0, 0, 1), (1, 0, 0, 1, 0), (1, 0, 0, 1, 1), (1, 0, 1, 0, 0), (1, 0, 1, 0, 1), (1, 0, 1, 1, 0),

(1, 0, 1, 1, 1), (1, 1, 0, 0, 0), (1, 1, 0, 0, 1), (1, 1, 0, 1, 0), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0),

(1, 1, 1, 0, 1), (1, 1, 1, 1, 0)}

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure below,

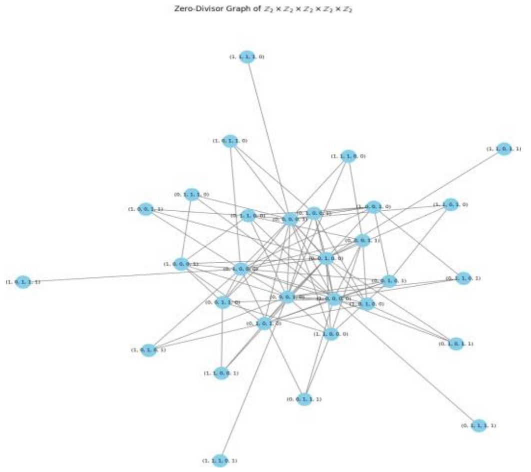


Fig 15: Γ ($\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$)

Here, n = 5

Number of vertices in $\mathbb{Z}$ ($\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$) = $2^5 - 2 = 30$

The vertices (0, 0, 0, 0, 1), (0, 0, 0, 1, 0), (0, 1, 0, 0, 0), (1, 0, 0, 0, 0), (0, 0, 1, 0, 0) form a cycle of length 5.

The vertices (1, 0, 1, 1, 0, 0), (0, 0, 1, 0, 0), (0, 1, 0, 0, 0), (0, 0, 0, 0, 1), (0, 0, 0, 1, 0) form a cycle of length 5, etc.

∴ $\mathbb{Z}_2^5$ has a cycle of length 5.

∴ The theorem holds.

*3.5 **Theorem***

Statement: Let $\mathbb{Z}_2^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 ... \mathbb{Z}_2$ be a Boolean ring. For $n \geq 3$, the girth of the Zero-divisor graph $\Gamma(\mathbb{Z}_2^n)$ is 3.

Proof: Let $\mathbb{Z}_2^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 ... \mathbb{Z}_2$ be a Boolean ring then $\mathbb{Z}_2^n$ has $(2^n - 2)$ Zero divisors

and $\mathbb{Z}_2^n$ is given by,

$\mathbb{Z}_2^n = \{(b_1, b_2, b_3, ..., b_n) \mid b_i's \in \{0, 1\}, i = 1, 2, ..., n\}$

Since each b_is has two choices 0 and 1, so there are 2^n elements in $\mathbb{Z}_2^n$.

$$|\mathbb{Z}_2^n| = 2^n$$

The set of Zero divisors of $\mathbb{Z}_2^n$ is denoted by $\mathbb{Z}(\mathbb{Z}_2^n)$ and is given by,

$\mathbb{Z}\,(\mathbb{Z}_2^n) = \{(b_1, b_2, b_3, ..., b_n) \mid b_i s \in \{0, 1\}$, all $b_i \neq 0$, and all $b_i \neq 1$, $i = 1, 2, ..., n\}$

$$\therefore |\mathbb{Z}(\mathbb{Z}_2^n)| = 2^n - 2$$

Consider the vertices $(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), (0, 0, 1, ..., 0), ..., (0, 0, 0, ..., 1)$, they all form their edges between them. By Theorem (2.1), the above elements have degree $2^{n-1} - 1$

For $n \geq 3$, the degree of the above elements would be greater than or equal to 3.

Any combination of 3 elements between these vertices will form a cycle, and the length of that cycle will be 3.

The cycle formed will be the shortest cycle of the graph.

We know, the girth of the graph is the length of the shortest cycle in the graph.

$\therefore$ The girth of the Zero-divisor graph $\Gamma(\mathbb{Z}_2^n)$ is 3. (For $n \geq 3$).

3.5.1 *Example*

$\mathbb{Z}_2^3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0), (1, 1, 1)\}$

$\mathbb{Z}\,(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0)$

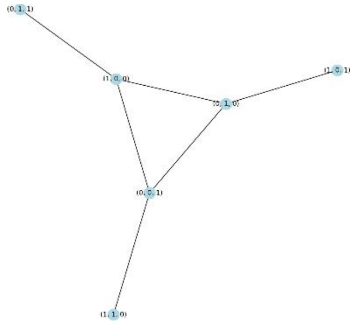


Fig 16: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure above.

Number of vertices in the graph of $\mathbb{Z}\,(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 2^3 - 2 = 6$

The vertices (0,0,1), (0,1,0) and (1,0,0) form a cycle.

$\therefore \mathbb{Z}_2^3$ has a cycle of length 3.

$\therefore$The girth of $\mathbb{Z}_2^3$ is 3.

$\therefore$ The theorem holds.

3.5.2 *Example*

$\mathbb{Z}_2^4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2 = \{$(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1), ((0, 1, 1, 0), (0, 1, 1, 1),

(1, 0, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1), (1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1)$\}$

$\mathbb{Z}(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2) = \{$(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (0, 1, 1, 1),

(1, 0, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1), (1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0)$\}$

The Zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the figure.

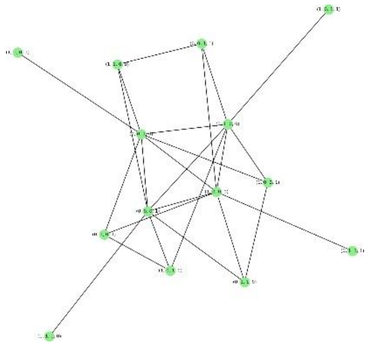


Fig 17: Γ $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Number of vertices in the graph of $\mathbb{Z}$ $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 2^4 - 2 = 14$

The vertices (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0) form a cycle of length 3.

The vertices (0, 1, 1, 0), (1, 0, 0, 0), (0, 0, 0, 1) form a cycle of length 3.

The vertices (1, 0, 0, 0), (0, 1, 0, 1), (0, 0, 1, 0) form a cycle of length 3 and etc.

$\therefore$ $\mathbb{Z}_2^4$ has the shortest cycle of order 3.

$\therefore$The girth of $\mathbb{Z}_2^3$ is 3 and the theorem holds.

4 Conclusion

An attempt has been made to generalize the Zero divisor graphs of Boolean rings. In this project, we have tried to prove the results regarding the degree of vertices, length of cycles, and girth of the graph for the Zero divisor graphs of these Boolean rings. By studying the graphs for different values of n, we could develop these results and hence form the proofs of the theorems stated in the project before. The ultimate goal of this project is to understand the relationship between different aspects of the Zero-divisor graphs of Boolean rings and their various properties.

5 Sage

Sage is free, open-source mathematics software that can be used alternatively to Mathematica or Matlab. Sage Math (previously Sage or SAGE," System for Algebra and Geometry Experimentation") is a computer algebra system (CAS) with features covering many aspects of mathematics, including algebra, combinatorics, graph theory, numerical analysis, number theory, calculus, and statistics. We have been using this tool to form graphs.

Commands that that been used to form a zero-divisor graph of $\mathbb{Z}_2^3$ are:

We have defined the function G1=Graph()

```
G1.add_vertex('(0, 0, 1)')
G1.add_edge('(0, 1, 0)',' (1, 0, 0)')
G1.add_vertices(['(1, 1, 0)',' (1, 0, 1)',' (0, 1, 1)'])
G1.add_ edges([('(0, 1, 0)',' (0, 0, 1)'), ('(0, 0, 1)',' (1, 0, 0)'), ('(1, 1, 0)',' (0, 0, 1)'),
('(1, 0, 1)',' (0, 1, 0)'), ('(0, 1, 1)',' (1, 0, 0)')])
G1. Show ()
```

Similarly, we have used the commands for generating graphs of $\mathbb{Z}_2^4$ and $\mathbb{Z}_2^5$.

6 Acknowledgement

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References

[1] Euler L. Leonhard Euler and the Königsberg bridges. Scientific American. 1953 Jul 1;189(1):66-72.

[2] M. BehZad and A. Chartrand, Introduction to the Theory of Graphs, Allyn and Becon Inc., Boston, 1971.

[3] I. Beck (1988). Coloring of commutative rings, J. Algebra, 116, 208–226.

[4] D.F. Anderson, A. FraZier, A. Lauve, P. Livingston, The Zero-Divisor Graph of a Commutative Ring, II, Lecture Notes in Pure and Applied Mathematics, Vol. 220, Marcel Dekker, New York, Basel, 2001, pp. 61–72.

[5] Mulay, S. B. (2002). Cycles and symmetries of Zero-divisors, Comm. Algebra, 30, 3533–3558.

[6] D. F. Anderson, R. Levy, and J. Shapiro, “Zero-divisor graphs, von Neumann regular rings, and Boolean algebras”, J. Pure Appl. Algebra 180:3 (2003), 221–241.

[7] Anderson, David F., Ron Levy, and Jay Shapiro. “Zero-divisor graphs, von Neumann regular rings, and Boolean algebras.” Journal of Pure and Applied Algebra 180.3 (2003): 221-241.

[8] Redmond S. P. (2003). A deal-based Zero-divisor graph of a commutative ring, Comm. Alge- bra, 31, 4425–4443.

[9] Akbari, S., and A. Mohammadian. “On the Zero-divisor graph of a commutative ring.” Journal of algebra 274.2 (2004): 847-855.

[10] J. D. LaGrange, “Complemented Zero-divisor graphs and Boolean rings”, J. Algebra 315:2 (2007), 600–611. MR 2008h:13012 Zbl 1133.13005

[11] Mohammadian, Ali. “On Zero-divisor graphs of Boolean rings.” Pacific journal of mathematics 251.2 (2011): 375-383.

[12] Anderson, David F., and John D. LaGrange. “Commutative Boolean monoids, reduced rings, and the compressed Zero-divisor graph.” Journal of Pure and Applied Algebra 216.7 (2012): 1626-1636.

[13] LaGrange, John D. “Boolean rings and reciprocal eigenvalue properties.” Linear algebra and its applications 436.7 (2012): 1863-1871.

[14] Muneshwar, R. A., S. S. Agrawal, and S. G. Jakkewad. “Some Properties of Graph of a Finite Group.” 2278-5728, Volume 10, (Nov - Dec. 2014), PP 55-58).

[15] Deepa Sinha and Baleen Kaur, On Beck’s Zero-divisor graph, Vol. 25, 2019, No. 4, 150–157.

[16] Anderson, David F., and Grace McClurkin. “Generalizations of the Zero-divisor graph.” International Electronic Journal of Algebra 27.27 (2020): 237-262.

[17] Wu, Tongsuo. “Boolean graphs-A survey.” Ring Theory 2019: Proceedings of the Eighth China–Japan–Korea International Symposium on Ring Theory 2021.