

Generalized q -Mittag-Leffler Matrix Function and its Properties

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Abstract: Motivated essentially by the success of the applications of the Mittag-Leffler functions and Matrix theory in Science and Engineering, we propose here a unification of certain q -extensions of generalizations of Mittag-Leffler function together with Saxena-Nishimoto's function, Bessel-Maitland function, Dotsenko function, Elliptic Function, etc. We obtain Mellin-Barnes contour integral representation, q -difference equation and eigen function property.

Keywords: Mittag-Leffler Matrix function, q -Mittag-Leffler function, q -difference equation, Eigen function

1. Preliminaries

Let the spectrum of a matrix in $C^{r \times r}$, denoted by $\sigma(A)$, be the set of all eigenvalues of A . Recall that a matrix $A \in C^{r \times r}$ is said to be positive stable when

$$\beta(A) = \min \{ \Re(z) / z \in \sigma(A) \} > 0 \quad (1.1)$$

For a positive stable matrix $A \in C^{r \times r}$, the q -gamma matrix function is defined by [10, Eq.(3.13)]

$$\Gamma_q(A) = \int_0^{\frac{1}{1-q}} t^{A-I} E_q^{-qt} dt \quad (1.2)$$

and the reciprocal q -gamma matrix function is defined as [10, Eq.(3.20)]

$$\Gamma_q^{-1}(A) = [A]_q [A+I]_q \dots [A+(n-1)I]_q \Gamma_q^{-1}(A+nI), \quad n \geq 1 \quad (1.3)$$

If A and B are positive stable matrices in $C^{r \times r}$ then the q -beta matrix function is defined as [10, Eq.(4.5)]

$$B_q(A, B) = \int_0^1 (tq; q)_\infty (tq^B; q)_\infty^{-1} t^{A-I} d_q t \tag{1.4}$$

Furthermore, if A, B and $A + B$ are positive stable matrices in $C^{r \times r}$ such that $AB = BA$ then the beta matrix function is defined as [10, Eq.(4.6)]

$$B_q(A, B) = \Gamma_q(A) \Gamma_q(B) \Gamma_q^{-1}(A + B) \tag{1.5}$$

2. Introduction

A generalized structure of the Mittag-Leffler matrix function is given by [11, Eq.(2.9)]:

$$E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r) = \sum_{n=0}^{\infty} [(A)_{\delta n}]^s \Gamma^{-1}(\alpha n I + B) \left[(C)_{\mu n} \right]^{-r} \frac{(\lambda z)^n}{n!} \tag{2.1}$$

where A, B, C are positive stable matrices in $C^{p \times p}$, $\alpha, \lambda, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\delta, \mu > 0$, $r \in \{-1, 0\} \cup \mathbb{N}$ and $s \in \{0\} \cup \mathbb{N}$.

Interestingly, the proposed function ((2.13) below) also enables us to define and include the q -analogues of

(i) Bessel-Maitland matrix function [11] :

$$J_{\nu I}^{\mu I}(z) = \sum_{n=0}^{\infty} (-1)^n \Gamma^{-1}(\nu I + n\mu I + I) \frac{z^n}{n!},$$

(ii) Dotsenko matrix function [11] :

$$\begin{aligned} & {}_2R_1(aI, bI; cI, \omega I; \nu; z) \\ &= \sum_{n=0}^{\infty} \Gamma(aI + nI) \Gamma^{-1}(aI) \Gamma(bI + n\frac{\omega}{\nu}I) \Gamma^{-1}(bI) \Gamma(cI) \Gamma^{-1}(cI + n\frac{\omega}{\nu}I) \frac{z^n}{n!}, \end{aligned}$$

(iii) Saxena and Nishimoto's matrix function [11]:

$$\begin{aligned} & E_{\gamma I, K}[(\alpha_j I, B_j)_{1,2}; z] \\ &= \sum_{n=0}^{\infty} (\gamma I)_{Kn} \Gamma^{-1}(\alpha_1 n I + B_1) \Gamma^{-1}(\alpha_2 n I + B_2) \frac{z^n}{n!}, \end{aligned}$$

where $z, \gamma, \alpha_j, \beta_j \in \mathbb{C}$, $\Re(\alpha_1 + \alpha_2) > \Re(K) - 1$, $\Re(K) > 0$, and

(iv) the Elliptic matrix function [11] :

$$K(I; k) = \frac{\pi}{2} {}_2F_1 \left(\begin{matrix} \frac{1}{2}I, & \frac{1}{2}I; & k^2 \\ I; \end{matrix} \right).$$

The following definitions and formulas will be used in this work.

For $A \in C^{r \times r}$ and $q \neq 1$, $q^A = e^{A \log q}$ then the q -shifted factorial matrix function is defined by [10]

$$(A; q)_n = \begin{cases} I & \text{if } n = 0 \\ (I - A)(I - Aq) \dots (I - Aq^{n-1}) & \text{if } n \in \mathbb{N}. \end{cases} \quad (2.2)$$

For any n ,

$$(A; q)_n = (q; q)_\infty (Aq^n; q)_\infty^{-1},$$

where

$$(A; q)_\infty = \prod_{k=0}^{\infty} (I - Aq^k), \quad |q| < 1.$$

$$(A; q)_\infty^{-1} = \prod_{k=0}^{\infty} (I - Aq^k)^{-1}, \quad \|A\| < 1, \quad |q| < 1.$$

A q -binomial coefficient is (cf. [2, Ex.(1.2), p.20] with $r = 1$):

$$\begin{bmatrix} n \\ m \end{bmatrix}_r = \frac{(q^r; q^r)_n}{(q^r; q^r)_{n-m} (q^r; q^r)_m}, \quad r \neq 0. \quad (2.3)$$

A q -gamma function is defined as [10, Eq.(3.18)]:

$$\Gamma_q(A) = (q; q)_\infty (q^A; q)_\infty^{-1} (1 - q)^{I - A}, \quad (2.4)$$

$q^{-n} \notin \sigma(q^A)$, $n = 0, 1, 2, \dots$

For $A \in C^{r \times r}$ the q -analogue of Legendre's duplication formula is of the form [10, Eq.(3.23)]

$$\Gamma_q(2A) \Gamma_{q^2} \left(\frac{1}{2} \right) = \Gamma_{q^2}(A) \Gamma_{q^2} \left(A + \frac{1}{2} I \right) (1 + q)^{2A - I} \quad (2.5)$$

where $q^{-n} \notin \sigma(q^{2A})$, $q^{-2n} \notin \sigma(q^{2A})$, $q^{-2n} \notin \sigma(q^{A+I/2})$, $n = 1, 2, \dots$

In view of [3, Eq.(6)], we can propose the the following theorem for matrix function.

Theorem 2.1. *If $f(z) = \sum_{n=0}^{\infty} V_n z^n$ is an entire function then the order $\rho(f)$ of f is given by*

$$\rho(f) = \limsup_{n \rightarrow \infty} n \log n \log(\|V_n\|)^{-1}. \quad (2.6)$$

and the type of the function σ is given by [3]

$$e\rho\sigma = \limsup_{n \rightarrow \infty} \left(n \|V_n\|^{e/n} \right). \quad (2.7)$$

For every positive ϵ , the asymptotic estimate [3, Eq.(8)]

$$|f(z)| < \exp((\sigma + \epsilon) |z|^\rho), \quad |z| \geq r_0 > 0 \tag{2.8}$$

holds with ρ, σ as in (2.6), (2.7) for $|z| \geq r_0(\epsilon)$, $r_0(\epsilon)$ sufficiently large.

For any complex square matrix A and $|q| < 1$ the q -analogues of the exponential matrix functions are defined as [10, Eq.(3.11)]

$$e_q(A) = \sum_{n=0}^{\infty} \frac{A^n}{(q; q)_n} = ((1 - q)A; q)_{\infty}^{-1}, \tag{2.9}$$

and [10, Eq.(3.10)]

$$E_q(A) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{A^n}{(q; q)_n} = (-(1 - q)A; q)_{\infty}, \tag{2.10}$$

Let $A \in C^{r \times r}$ be a positive stable matrix then the q -beta matrix function is defined as [10, Eq.(4.5)]

$$B_q(A, B) = \int_0^1 (tq; q)_{\infty} (tq^B; q)_{\infty}^{-1} t^{A-I} d_q t, \tag{2.11}$$

The q -derivative of a function $f(x)$ is defined by [2, Ex.1.12, p.22]

$$D_q f(A) = [f(A) - f(Aq)][A(1 - q)]^{-1}. \tag{2.12}$$

In view of two q -analogues of generalized Mittag-Leffler function [3, 4], we define q -generalized Mittag-Leffler matrix functions in the form:

Definition 1. Let A, B, C be positive stable matrices in $C^{r \times r}$, $\alpha, \delta, \mu \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\delta, \mu > 0$, $r \in \{-1, 0\} \cup \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$ then

$$E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q) = \sum_{n=0}^{\infty} (-1)^{pnI} q^{pn(n-1)I/2} [\Gamma_q(\delta nI + A)]^s \times [\Gamma_q(\alpha nI + B)]^{-1} [\Gamma_q(\mu nI + C)]^{-r} \frac{(\lambda z)^n}{(q; q)_n}, \tag{2.13}$$

where $p = \alpha^2 + r\mu^2 - s\delta^2 + 1$ with $\Re(p) > 0$.

Definition 2. Let A, B, C be positive stable matrices in $C^{r \times r}$, $\alpha, \delta, \mu \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\delta, \mu > 0$, $r \in \{-1, 0\} \cup \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$ and $(\alpha^2 + r\mu^2 + 1)I = s\delta^2 I$ then

$$e_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q) = \sum_{n=0}^{\infty} [\Gamma_q(\delta nI + A)]^s [\Gamma_q(\alpha nI + B)]^{-1} \times [\Gamma_q(\mu nI + C)]^{-r} \frac{(\lambda z)^n}{(q; q)_n}. \tag{2.14}$$

Alternatively in view of the definition of q -Gamma function (2.4) these q -forms can also be put in the form:

$$E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q) = \sum_{n=0}^{\infty} (-1)^{pnI} q^{pn(n-1)I/2} (q^{\alpha nI + B}; q)_{\infty} \times [(q^{\mu nI + C}; q)_{\infty}]^r [(q^{\delta nI + A}; q)_{\infty}]^{-s} \frac{(\lambda z)^n}{(q; q)_n}, \tag{2.15}$$

and

$$e_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q) = \sum_{n=0}^{\infty} [(q^{\alpha n I + B}; q)_{\infty}] [(q^{\mu n I + C}; q)_{\infty}]^r \times [(q^{\delta n I + A}; q)_{\infty}]^{-s} \frac{(\lambda z)^n}{(q; q)_n}. \tag{2.16}$$

We shall refer to these functions as *q-gmlm*.

The objective of constructing this function is to

- (i) include certain existing generalizations of Mittag-Leffler matrix function [3, 11, 4, 7, 8, 9, 5, 6],
- (ii) also include the matrix functions such as Bessel Maitland function, Dotsenko function, Bessel function, generalized Bessel Maitland function, Lommel function etc. especially by means of parameters r, γ, λ (Table-1 below)
- (iii) obtain inverse inequality relations and some other inequalities by means of the integer 's'.

The *q*-analogues of the above stated Shukla and Prajapati's function (2.1) and those functions listed above from (i) through (iv) are all yielded by the *q-gmlm* (2.13) or (2.14).

They are tabulated below together with the indicated substitutions.

Table-1

q-Function of	r	s	α	B	A	δ	C	μ	Particular case of
Mittag-Leffler	0	1	α	1	1	1	-	-	(2.13)
Wiman	0	1	α	B	1	1	-	-	(2.13)
Prabhakar	0	1	α	B	γI	1	-	-	(2.13)
Shukla and Prajapati	0	1	α	B	γI	q	-	-	(2.13)
Bessel-Maitland	0	0	μ	(ν + 1)I	-	-	-	-	(2.13)
Dotsenko	-1	1	ω/ν	cI	a	I	bI	ω/ν	(2.14)
Saxena-Nishimoto	1	1	α ₁	B ₁	γI	K	B ₂	α ₂	(2.13)
Elliptic	-1	1	1	I	½I	1	½I	1	(2.14)

The explicit forms of the functions mentioned in this table are as stated below.

- q-Mittag-Leffler function:

$$E_{\alpha I}(\lambda z|q) = \sum_{n=0}^{\infty} [(-1)^n q^{n(n-1)/2}]^{\alpha^2 I} (q^{\alpha n I + I}; q)_{\infty} (\lambda z)^n.$$

- q-Analogue of Wiman's function:

$$E_{\alpha I}^B(\lambda z|q) = \sum_{n=0}^{\infty} [(-1)^n q^{n(n-1)/2}]^{\alpha^2 I} (q^{\alpha n I + B}; q)_{\infty} (\lambda z)^n.$$

- q -Analogue of Prabhakar's generalized ML-function:

$$E_{\alpha I}^{A,B}(\lambda z|q) = \sum_{n=0}^{\infty} [(-1)^n q^{n(n-1)/2}]^{\alpha^2 I} (q^{\alpha n I+B}; q)_{\infty} \\ \times [(q^{A+nI}; q)_{\infty}]^{-1} \frac{(\lambda z)^n}{(q; q)_n}.$$

- q -ML-function of Shukla and Prajapati (q is replaced by δ):

$$E_{\alpha I, \delta I}^{A,B}(\lambda z|q) = \sum_{n=0}^{\infty} [(-1)^n q^{n(n-1)/2}]^{(\alpha^2 - \delta^2 + 1)I} (q^{\alpha n I+B}; q)_{\infty} \\ \times (q^{A+\delta n I}; q)_{\infty}^{-1} \frac{(\lambda z)^n}{(q; q)_n}$$

- q -Bessel-Maitland function:

$$J_{\nu I}^{\mu} I(-z; q) = \sum_{n=0}^{\infty} \frac{[(-1)^n q^{n(n-1)/2}]^{(\mu^2+1)I} (q^{\mu n I+\nu I+I}; q)_{\infty}}{(q; q)_n} z^n.$$

- q -Dotsenko function:

$${}_2R_1(aI, bI; cI, \omega; \nu; z; q) = \sum_{n=0}^{\infty} (q^{cI+\frac{\omega}{\nu}nI}; q)_{\infty} [(q^{bI+\frac{\omega}{\nu}nI}; q)_{\infty}]^{-1} \\ \times [(q^{nI+aI}; q)_{\infty}]^{-1} \frac{z^n}{(q; q)_n}.$$

- q -Form of the particular case $m = 2$ of the function due to Saxena and Nishimoto

$$E_{\gamma I, K}[(\alpha_j I, B_j)_{1,2}; z|q] = \sum_{n=0}^{\infty} [(-1)^n q^{n(n-1)/2}]^{(\alpha_1^2 + \alpha_2^2 - K^2 + 1)I} \\ \times (q^{\alpha_1 n I+B_1}; q)_{\infty} (q^{\alpha_2 n I+B_2}; q)_{\infty} \\ \times [(q^{\gamma I+Kn}; q)_{\infty}]^{-1} \frac{z^n}{(q; q)_n}.$$

- q -Elliptic function:

$$K(\sqrt{z}|q) = \frac{\pi}{2} {}_2\phi_1 \left(\begin{matrix} \frac{1}{2}I, & \frac{1}{2}I; & z \end{matrix} \right).$$

We first show the convergence of series in (2.13) and (2.14); this is followed by Mellin-Barnes integral representation, q -difference equation and eigen function property.

3. Main Results

In this section, we prove the following results.

3.1 Convergence

Theorem 3.1. *Let A, B, C be positive stable matrices in $C^{p \times p}$, $\alpha, \delta, \mu \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\alpha^2) + r\mu^2 - s\delta^2 + 1 > 0$, $\delta, \mu > 0$, $r \in \{-1, 0\} \cup \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$ and $0 < q < 1$. Then $E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q)$ is an entire function of order zero.*

Proof. Put

$$V_n = (-1)^{pmI} q^{pn(n-1)I/2} [\Gamma_q(\delta nI + A)]^s \Gamma_q^{-1}(\alpha nI + B) [\Gamma_q^{-1}(\mu nI + C)]^r \times \frac{1}{(q; q)_n} \tag{3.1}$$

to get

$$E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q) = \sum_{n=0}^{\infty} V_n (\lambda z)^n.$$

Then in view of (2.4) and applying norm, we get after some simplification,

$$\begin{aligned} \sqrt[n]{\|V_n\|} &\sim \left\| (-1)^{pI} q^{p(n-1)I/2} \right. \\ &\quad \times (1-q)^{s(I-A)-r(I-C)-(I-B)/n} (1-q)^{(-s\delta+r\mu+\alpha)I} \\ &\quad \times \left[\prod_{h=0}^{\infty} (I - q^{\delta nI + A + hI}) \right]^s \left[\prod_{m=0}^{\infty} (I - q^{\alpha nI + B + mI}) \right] \\ &\quad \left. \times \left[\prod_{j=0}^{\infty} (I - q^{\mu nI + C + jI}) \right]^{-r} \frac{1}{(q; q)_n} \right\| \end{aligned}$$

Now by applying $\lim n \rightarrow \infty$, we get

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \sqrt[n]{\|V_n\|} \sim 0$$

when $|q| < 1$, $\Re(\alpha^2) + r\mu^2 - s\delta^2 + 1 > 0$. Thus, the function (2.13) is an entire function. Its order may be determined by using Theorem 2.1.

In fact, by choosing $f(z) = E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q)$ and $u_n = V_n$, Theorem 2.1 gets particularized to

$$\rho(E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q)) = \limsup_{n \rightarrow \infty} \frac{n \log n}{\log(\|V_n\|^{-1})},$$

where

$$\begin{aligned} \log(\|V_n\|^{-1}) &= \log \left(\left\| \Gamma_q(\alpha nI + B) [\Gamma_q(\mu nI + C)]^r \Gamma_q(n + 1) \right. \right. \\ &\quad \left. \left. \times q^{-n(n-1)(\alpha^2+r\mu^2-s\delta^2+1)I/2} [\Gamma_q(\delta nI + A)]^{-s} \right\| \right) \\ &= \log \|\Gamma_q(\alpha nI + B)\| + r \log \|\Gamma_q(\mu nI + C)\| \\ &\quad + \log |\Gamma_q(n + 1)| - \frac{1}{2} n(n - 1) [\Re(\alpha^2 + r\mu^2 - s\delta^2 + 1) \|I\|] \log q \\ &\quad - s \log \|\Gamma_q(\delta nI + A)\|. \end{aligned} \tag{3.2}$$

From the definition (2.4) of q -gamma function, one finds

$$\begin{aligned} \log \|\Gamma_q(\alpha n I + B)\| &= \log \left\| (q; q)_\infty [(q^{\alpha n I + B}; q)_\infty]^{-1} (1 - q)^{I - \alpha n I - B} \right\| \\ &= \log \left\| (q; q)_\infty [(q^{\alpha n I + B}; q)_\infty]^{-1} (1 - q)^{I - \alpha n I - B} \right\| \\ &= \log |(q; q)_\infty| + \|I - \alpha I - B\| \log |(1 - q)| \\ &\quad - \log \|(q^{\alpha n I + B}; q)_\infty\|; \end{aligned} \tag{3.3}$$

in which

$$\begin{aligned} \log \|(q^{\alpha n I + B}; q)_\infty\| &= \log \left(\prod_{k=0}^{\infty} \|I - q^{\alpha n I + B + k}\| \right) \\ &= \log \left(\lim_{m \rightarrow \infty} \prod_{k=0}^m \|I - q^{\alpha n I + B + k}\| \right) \\ &= \lim_{m \rightarrow \infty} \sum_{k=0}^m \log \|I - q^{\alpha n I + B + k}\| \\ &= \sum_{k=0}^{\infty} \log \|I - q^{\alpha n I + B + k}\|. \end{aligned}$$

Here it may be noted that [1, p.207]

$$\log \|I - q^{\alpha n I + B + k}\| \leq 1 - \|q^{\alpha n I + B}\| |q|^k$$

which leads us to

$$\sum_{k=0}^{\infty} \log \|I - q^{\alpha n I + B + k}\| \leq \|q^{\alpha n I + B}\| \sum_{k=0}^{\infty} |q|^k = \frac{\|q^{\alpha n I + B}\|}{1 - |q|}.$$

This implies that

$$\lim_{n \rightarrow \infty} \frac{\log \|(q^{\alpha n I + B}; q)_\infty\|}{n \log n} = 0.$$

Consequently from (3.3), it follows that

$$\lim_{n \rightarrow \infty} \frac{\log \|\Gamma_q(\alpha n I + B)\|}{n \log n} = 0.$$

This last limit and the trivial limit

$$\lim_{n \rightarrow \infty} \frac{n - 1}{\log n} = \infty$$

when used in (3.2), yields

$$\lim_{n \rightarrow \infty} \frac{\log (\|V_n\|)^{-1}}{n \log n} = \infty.$$

Thus,

$$\varrho(E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q)) = 0.$$

□

Theorem 3.2. *The function $e_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q)$ represents the series which converges absolutely for $|z| < |(1 - q)^{(s\delta - \alpha - r\mu - 1)}|$ and $|q| < 1$.*

Proof. Take

$$U_n = [\Gamma_q(\gamma + \delta n)]^s [\Gamma_q(\beta + \alpha n)]^{-1} [\Gamma_q(\lambda + \mu n)]^{-r} [\Gamma_q(n + 1)] \tag{3.4}$$

then

$$e_{\alpha, \beta, \lambda, \mu}^{\gamma, \delta}(z; s, r|q) = \sum_{n=0}^{\infty} U_n z^n.$$

Now in view of (2.4), we get

$$\sqrt[n]{\|U_n\|} \sim \|(1 - q)^{(s\delta - \Re(\alpha) - r\mu)I - 1}\|$$

whence

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \sqrt[n]{\|U_n\|} \sim |(1 - q)^{\alpha + r\mu - s\delta + 1}|.$$

Thus, the series in (2.14) converges absolutely if $|z| < R = (1 - q)^{s\delta - \Re(\alpha) - r\mu - 1}$. □

3.2 Contour integral

Theorem 3.3. *Let A, B, C be positive stable matrices in $C^{p \times p}$, $\alpha, \delta, \mu \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\alpha^2) + r\mu^2 - s\delta^2 + 1 > 0$, $\delta, \mu > 0$, $r \in \{-1, 0\} \cup \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$ and $0 < q < 1$. Then the function $E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q)$ is expressible as the Mellin - Barnes q -integral given by*

$$E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q) = \frac{1}{2\pi i} \int_L (-1)^{-pS} q^{-pS(-S-1)/2} \Gamma_q(S) [\Gamma_q(A - \delta IS)]^s \times [\Gamma_q(B - \alpha IS)] [\Gamma_q(C - \mu IS)]^{-r} (-z)^{-S} d_q S, \tag{3.5}$$

where $|\arg z| < \pi$. The contour L of integration begins from $-i\infty$ and proceeds towards $+i\infty$, and is indented to keep the poles of integrand at $S = -nI$ to the left; and the poles at $S = (A + nI)/\delta$ to the right of the path for all $n \in \mathbb{N} \cup \{0\}$.

Proof. The integral on the right hand side of (3.5) may be evaluated as the sum of the residues at the poles $S = 0, -1, -2, \dots$. In fact, in view of the definition of residue,

$$\begin{aligned} I &= \frac{1}{2\pi i} \int_L (-1)^{-pS} q^{-pS(-S-1)/2} \Gamma_q(S) [\Gamma_q(A - \delta IS)]^s (-z)^{-S} \\ &\quad \times [\Gamma_q(B - \alpha IS)] [\Gamma_q(C - \mu IS)]^{-r} d_q S \\ &= \sum_{n=0}^{\infty} S \underset{-n}{Res} \left[(-1)^{-pS} q^{-pS(-S-1)/2} \Gamma_q(S) (-z)^{-S} [\Gamma_q(B - \alpha IS)]^{-1} \right] \end{aligned}$$

$$\begin{aligned}
 & \times [\Gamma_q(C - \mu IS)]^{-r} [\Gamma_q(A - \delta IS)]^s \Big] \\
 = & \sum_{n=0}^{\infty} \lim_{S \rightarrow -n} \frac{\pi(S+n)}{\sin \pi S} (-1)^{-pS} q^{-pS(-S-1)/2} [\Gamma_q(A - \delta IS)]^s (-z)^{-S} \\
 & \times [\Gamma_q(B - \alpha IS)]^{-1} [\Gamma_q(C - \mu IS)]^{-r} [\Gamma_q(1 - S)]^{-1} \\
 = & \sum_{n=0}^{\infty} (-1)^{pn} q^{pn(n-1)/2} [\Gamma_q(A + \delta In)]^s [\Gamma_q(B + \alpha In)]^{-1} [\Gamma_q(C + \mu In)]^{-r} \\
 & \times \frac{z^n}{\Gamma_q(n+1)} \\
 = & E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q).
 \end{aligned}$$

□

Theorem 3.4. Let A, B, C be positive stable matrices in $C^{r \times r}$, $\alpha, \delta, \mu \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\delta, \mu > 0$, $r \in \{-1, 0\} \cup \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$ and $(\alpha^2 + r\mu^2 + 1)I = s\delta^2 I$, $0 < q < 1$. Then the function $e_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q)$ is expressible as the Mellin - Barnes q -integral given by

$$\begin{aligned}
 e_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q) &= \frac{1}{2\pi i} \int_L \frac{\Gamma_q(S) [\Gamma_q(A - \delta IS)]^s (-z)^{-S}}{[\Gamma_q(B - \alpha IS)]^{-1} [\Gamma_q(C - \mu IS)]^{-r} d_q S}, \quad (3.6)
 \end{aligned}$$

where $|\arg z| < \pi$; the contour L of integration begins from $-i\infty$ and proceeds towards $+i\infty$, and is indented to keep the poles of integrand at $S = -nI$ to the left; and the poles at $S = (A + nI)/\delta$ to the right of the path, for all $n \in \mathbb{N} \cup \{0\}$.

3.3 Difference equation

With the aid of the following operators, the difference equations of both q -analogues will be derived. Put

$$\Lambda_q f(A) = f(A) - f(Aq^{-1}), \quad \Theta f(A) = f(A) - f(Aq), \quad (3.7)$$

$$\begin{aligned}
 \mathcal{D}_q f(A) &= (1 - q) D_q f(A) := (1 - q) [f(A) - f(Aq)](A - Aq)^{-1} \\
 &= [f(A) - f(Aq)](A)^{-1}, \quad (3.8)
 \end{aligned}$$

$$\begin{aligned}
 & \left\{ \prod_{u=0}^{a-1} \prod_{v=0}^{a-1} [\Theta I + c^{-u} q^{I-(B+vI)/a} - I]^m \right\} \left\{ \prod_{u=0}^{a-1} \prod_{v=0}^{a-1} [c^{-u} q^{I-(B+vI)/a}]^m \right\}^{-1} \\
 = & \Phi_{u,v}^{(a, B, c; m)} \quad (3.9)
 \end{aligned}$$

and

$$\begin{aligned}
 & \left\{ \prod_{u=0}^{a-1} \prod_{v=0}^{a-1} [\Theta I + c^{-u} q^{(B+vI)/a} - I]^m \right\} \left\{ \prod_{u=0}^{a-1} \prod_{v=0}^{a-1} [c^{-u} q^{-(B+vI)/a}]^m \right\}^{-1} \\
 = & \Psi_{u,v}^{(a, B, c; m)}. \quad (3.10)
 \end{aligned}$$

In these notations, the q -difference equation satisfied by (2.13) is derived in the following theorem.

Theorem 3.5. Let $\alpha, \mu, \delta \in \mathbb{N}$, then $E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q)$ satisfies the equation

$$\begin{aligned} & \left[\Phi_{\ell, k}^{(\mu, C, \eta; r)} \Phi_{h, m}^{(\alpha, B, \sigma; 1)} \Theta \right] E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q) \\ & - \left[(-1)^p z \Psi_{j, i}^{(\delta, A, \zeta; s)} \right] E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z q^p; s, r|q) = 0 \end{aligned} \tag{3.11}$$

in which ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

Proof. In the first place, the coefficient of z^n in the series representation of $E_{\alpha I, \delta I, \mu I}^{A, B, C}(\lambda z; s, r|q)$ will be expressed in q -factorial notation with the help of the set of formulas [2, Appendix I]:

$$\begin{aligned} (A; q)_{kn} &= (A, Aq, \dots, Aq^{k-1}; q^k)_n, \\ (A^k; q^k)_n &= (A, A\omega_k, \dots, A\omega_k^{k-1}; q^k)_n; \omega_k = e^{(2\pi i)/k}, \\ (A; q^n)_{\nu k} &= (A^{1/n}; q)_{\nu k} (A^{1/n}\omega; q)_{\nu k} \dots (A^{1/n}\omega^{n-1}; q)_{\nu k}, \omega^n = 1, \end{aligned}$$

and

$$(q^A; q^\delta)_n = (q^{A/\delta}; q)_n (\varpi q^{A/\delta}; q)_n \dots (\varpi^{\delta-1} q^{A/\delta}; q)_n = \prod_{i=0}^{\delta-1} (\varpi^i q^{A/\delta}; q)_n, \varpi^\delta = 1.$$

Then following the notation used in (3.1) for the coefficient of z^n , we get

$$\begin{aligned} V_n &= (-1)^{pn} q^{pn(n-1)/2} [(q^A; q)_{\delta n}]^s [(q^C; q)_{\mu n}]^{-r} [(q^B; q)_{\alpha n}]^{-1} \frac{1}{(q; q)_n} \\ &= (-1)^{pn} q^{pn(n-1)/2} [(q^A; q^\delta)_n]^s [(q^{A+I}; q^\delta)_n]^s \dots [(q^{A+\delta I-I}; q^\delta)_n]^s \\ &\quad \times [(q^C; q^\mu)_n]^{-r} [(q^{C+I}; q^\mu)_n]^{-r} \dots [(q^{C+\mu I-I}; q^\mu)_n]^{-r} \\ &\quad \times [(q^B; q^\alpha)_n]^{-1} [(q^{B+I}; q^\alpha)_n]^{-1} \dots [(q^{B+\alpha I-I}; q^\alpha)_n]^{-1} \frac{1}{(q; q)_n} \\ &= (-1)^{pn} q^{pn(n-1)/2} \left\{ \prod_{j=0}^{\delta-1} \prod_{i=0}^{\delta-1} [(\zeta^j q^{(A+iI)/\delta}; q)_n]^s \right\} \\ &\quad \times \left\{ \prod_{\ell=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(\eta^\ell q^{(C+kI)/\mu}; q)_n]^{-r} \right\} \\ &\quad \times \left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} [(\sigma^h q^{(B+mI)/\alpha}; q)_n]^{-1} \right\} \frac{1}{(q; q)_n}, \end{aligned} \tag{3.12}$$

where ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

Now take

$$\prod_{j=0}^{\delta-1} \prod_{i=0}^{\delta-1} [(\zeta^j q^{(A+iI)/\delta}; q)_n]^s = \mathcal{A}_n, \prod_{\ell=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(\eta^\ell q^{(C+kI)/\mu}; q)_n]^{-r} = \mathcal{B}_n, \tag{3.13}$$

and

$$\prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (\sigma^h q^{(B+mI)/\alpha}; q)_n = \mathcal{C}_n, (-1)^{pn} q^{pn(n-1)/2} = D_n \tag{3.14}$$

then

$$\sum_{n=0}^{\infty} V_n z^n = \sum_{n=0}^{\infty} \mathcal{A}_n D_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{z^n}{(q; q)_n} = W, \text{ say.}$$

Since the series in (2.13) converges, we have

$$\begin{aligned} \Theta W &= \sum_{n=0}^{\infty} \mathcal{A}_n D_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{1}{(q; q)_n} \Theta z^n \\ &= \sum_{n=0}^{\infty} \mathcal{A}_n D_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{1 - q^n}{(q; q)_n} z^n \\ &= \sum_{n=1}^{\infty} \mathcal{A}_n D_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{z^n}{(q; q)_{n-1}}. \end{aligned}$$

Next operating by $\Phi_{h,m}^{(\alpha,B,\sigma;1)}$, we get

$$\begin{aligned} \Phi_{h,m}^{(\alpha,B,\sigma;1)} \Theta I W &= \sum_{n=1}^{\infty} \mathcal{A}_n D_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{1}{(q; q)_{n-1}} \\ &\quad \times \left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (\Theta I + \sigma^{-h} q^{I-(B+mI)/\alpha} - I) \right\} \\ &\quad \times \left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (\sigma^{-h} q^{I-(B+mI)/\alpha}) \right\}^{-1} z^n \\ &= \sum_{n=1}^{\infty} \mathcal{A}_n D_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{1}{(q; q)_{n-1}} \\ &\quad \times \left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (I - \sigma^h q^{(n-1)I+(B+mI)/\alpha}) \right\} \\ &\quad \times \left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (\sigma^h q^{(B+mI)/\alpha}) \right\}^{-1} z^n \\ &= \sum_{n=1}^{\infty} \mathcal{A}_n D_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{1}{(q; q)_{n-1}} \mathcal{C}_n \mathcal{C}_{n-1}^{-1} z^n \\ &= \sum_{n=1}^{\infty} \mathcal{A}_n D_n \mathcal{B}_n^{-1} \mathcal{C}_{n-1}^{-1} \frac{z^n}{(q; q)_{n-1}} \end{aligned}$$

Finally,

$$\begin{aligned} &\Phi_{\ell,k}^{(\mu,C,\eta;r)} \Phi_{h,m}^{(\alpha,B,\sigma;1)} \Theta I W \\ &= \sum_{n=1}^{\infty} \mathcal{A}_n D_n \mathcal{C}_{n-1}^{-1} \mathcal{B}_n^{-1} \frac{1}{(q; q)_{n-1}} \left\{ \prod_{l=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(\Theta I + \eta^{-l} q^{I-(C+kI)/\mu} - I)]^r \right\} \\ &\quad \times \left\{ \prod_{l=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(\eta^{-l} q^{I-(C+kI)/\mu})]^r \right\}^{-1} z^n \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=1}^{\infty} \mathcal{A}_n D_n \mathcal{C}_{n-1}^{-1} \mathcal{B}_n^{-1} \frac{1}{(q; q)_{n-1}} \left\{ \prod_{l=0}^{\mu-1} \prod_{k=0}^{\mu-1} [-q^n + \eta^{-l} q^{I-(C+kI)/\mu}]^r \right\} \\
 &\quad \times \left\{ \prod_{l=0}^{\mu-1} \prod_{k=0}^{\mu-1} [\eta^{-l} q^{I-(C+kI)/\mu}]^r \right\}^{-1} z^n \\
 &= \sum_{n=1}^{\infty} \mathcal{A}_n D_n \mathcal{C}_{n-1}^{-1} \mathcal{B}_n^{-1} \frac{1}{(q; q)_{n-1}} \mathcal{B}_n \mathcal{B}_{n-1}^{-1} z^n \\
 &= \sum_{n=1}^{\infty} \mathcal{A}_n D_n \mathcal{B}_{n-1}^{-1} \mathcal{C}_{n-1}^{-1} \frac{z^n}{(q; q)_{n-1}}
 \end{aligned}$$

Thus,

$$\Phi_{\ell,k}^{(\mu,C,\eta;r)} \Phi_{h,m}^{(\alpha,B,\sigma;1)} \Theta I W = \sum_{n=0}^{\infty} \mathcal{A}_{n+1} D_{n+1} \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{z^{n+1}}{(q; q)_n} \tag{3.15}$$

On the other hand,

$$\begin{aligned}
 &\Psi_{j,i}^{(\delta,A,\zeta;s)} E_{\alpha I, \delta I, \mu I}^{A,B,C}(zq^p; s, r|q) \\
 &= \sum_{n=0}^{\infty} \mathcal{A}_n D_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{1}{(q; q)_n} \left\{ \prod_{j=0}^{\delta-1} \prod_{i=0}^{\delta-1} [(\Theta I + \zeta^{-j} q^{-(A+iI)/\delta} - I)]^s \right\} \\
 &\quad \times \left\{ \prod_{j=0}^{\delta-1} \prod_{i=0}^{\delta-1} [(\zeta^{-j} q^{-(A+iI)/\delta})]^s \right\}^{-1} z^n \\
 &= \sum_{n=0}^{\infty} \mathcal{A}_n D_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{1}{(q; q)_n} \left\{ \prod_{j=0}^{\delta-1} \prod_{i=0}^{\delta-1} [(I - \zeta^j q^{nI+(A+iI)/\delta})]^s \right\} z^n \\
 &= \sum_{n=0}^{\infty} \mathcal{A}_n D_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{1}{(q; q)_n} \mathcal{A}_{n+1} \mathcal{A}_n^{-1} z^n \\
 &= \sum_{n=0}^{\infty} \mathcal{A}_{n+1} D_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{z^n}{(q; q)_n}
 \end{aligned}$$

that is,

$$z (-1)^p \Psi_{j,i}^{(\delta,A,\zeta;s)} E_{\alpha I, \delta I, \mu I}^{A,B,C}(zq^p; s, r|q) = \sum_{n=0}^{\infty} \mathcal{A}_{n+1} D_{n+1} \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{z^{n+1}}{(q; q)_n}. \tag{3.16}$$

On comparing (3.15) and (3.16), the equation (3.11) is obtained. □

The q -difference equation satisfied by the function (2.14) is given in following theorem whose proof follows line-to-line just dropping the factor $q^{n(n-1)/2}$ that is, dropping D_n in (3.14).

Theorem 3.6. *Let $\alpha, \mu, \delta \in \mathbb{N}$, then $Y = e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z; s, r|q)$ satisfies the equation*

$$\left[\Phi_{\ell,k}^{(\mu,\lambda,\eta;r)} \Phi_{h,m}^{(\alpha,\beta,\sigma;1)} \Theta - z \Psi_{j,i}^{(\delta,\gamma,\zeta;s)} \right] Y = 0, \tag{3.17}$$

where ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

3.4 Eigen function property

Take

$$\begin{aligned} & \left\{ \prod_{u=0}^{a-1} \prod_{v=0}^{a-1} [(\Lambda_q + c^{-u} q^{I-(B+vI)/a} - I)]^m \right\} \left\{ \prod_{u=0}^{a-1} \prod_{v=0}^{a-1} [c^{-u} q^{I-(B+vI)/a}]^m \right\}^{-1} \\ &= \Omega_{u,v}^{(a,B,c;m)}, \end{aligned} \tag{3.18}$$

and

$$\Delta_q = \mathcal{D}_q \Omega_{j,i}^{(\delta,A,\zeta;-s)} \Phi_{\ell,k}^{(\mu,C,\eta;r)} \Phi_{h,m}^{(\alpha,B,\sigma;1)}. \tag{3.19}$$

Here the operators $\Omega_{j,i}^{(\delta,\gamma,\zeta;-s)}$, $\Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}$, $\Phi_{h,m}^{(\alpha,\beta,\sigma;1)}$ in (3.19) are not commutative with the operator \mathcal{D}_q .

This property does not hold for the function $E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z; s, r|q)$, but for the other version: $e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z; s, r|q)$, it is established in

Theorem 3.7. *Let $\alpha, \mu, \delta \in \mathbb{N}$ and q -difference operator Θ be defined by (3.7) then $e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z; s, r|q)$ is an eigen function with respect to the operator Δ_q defined by (3.19). That is, for any non zero c ,*

$$\Delta_q e_{\alpha I, \delta I, \mu I}^{A,B,C}(cz; s, r|q) = c e_{\alpha I, \delta I, \mu I}^{A,B,C}(z; s, r|q). \tag{3.20}$$

$$\tag{3.21}$$

Proof. With \mathcal{A}_n , \mathcal{B}_n and \mathcal{C}_n as in (3.13) and in (3.14),

$$e_{\alpha I, \delta I, \mu I}^{A,B,C}(cz; s, r|q) = \sum_{n=0}^{\infty} \mathcal{A}_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{z^n}{(q; q)_n}$$

Now if $e_{\alpha I, \delta I, \mu I}^{A,B,C}(cz; s, r|q) = Y_c$ then in the notation (3.9),

$$\begin{aligned} \Phi_{h,m}^{(\alpha,B,\sigma;1)} Y_c &= \sum_{n=0}^{\infty} c^n \mathcal{A}_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{1}{(q; q)_n} \\ &\quad \times \left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (\Theta I + \sigma^{-h} q^{I-(B+mI)/\alpha} - I) \right\} \\ &\quad \times \left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (\sigma^{-h} q^{I-(B+mI)/\alpha}) \right\}^{-1} z^n \\ &= \sum_{n=0}^{\infty} c^n \mathcal{A}_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (I - \sigma^h q^{(n-1)I+(B+mI)/\alpha}) \right\} \\ &\quad \times \left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (\sigma^h q^{(B+mI)/\alpha}) \right\}^{-1} \frac{z^n}{(q; q)_n} \\ &= \sum_{n=0}^{\infty} c^n \mathcal{A}_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{1}{(q; q)_n} \mathcal{C}_n \mathcal{C}_{n-1}^{-1} z^n \\ &= \sum_{n=0}^{\infty} c^n \mathcal{A}_n \mathcal{B}_n^{-1} \mathcal{C}_{n-1}^{-1} \frac{z^n}{(q; q)_n} \end{aligned}$$

Next

$$\begin{aligned}
 \Phi_{\ell,k}^{(\mu,C,\eta;r)} \Phi_{h,m}^{(\alpha,B,\sigma;1)} Y_c &= \sum_{n=0}^{\infty} c^n \mathcal{A}_n \mathcal{C}_{n-1}^{-1} \mathcal{B}_n^{-1} \frac{1}{(q; q)_n} \\
 &\times \left\{ \prod_{l=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(\Theta I + \eta^{-l} q^{I-(C+kI)/\mu} - I)]^r \right\} \\
 &\times \left\{ \prod_{l=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(\eta^{-l} q^{I-(C+kI)/\mu})]^r \right\}^{-1} z^n \\
 &= \sum_{n=0}^{\infty} c^n \mathcal{A}_n \mathcal{C}_{n-1}^{-1} \mathcal{B}_n^{-1} \frac{1}{(q; q)_n} \\
 &\times \left\{ \prod_{l=0}^{\mu-1} \prod_{k=0}^{\mu-1} [-q^n + \eta^{-l} q^{I-(C+kI)/\mu}]^r \right\} \\
 &\times \left\{ \prod_{l=0}^{\mu-1} \prod_{k=0}^{\mu-1} [\eta^{-l} q^{I-(C+kI)/\mu}]^r \right\}^{-1} z^n \\
 &= \sum_{n=0}^{\infty} c^n \mathcal{A}_n \mathcal{C}_{n-1}^{-1} \mathcal{B}_n^{-1} \frac{1}{(q; q)_n} \mathcal{B}_n \mathcal{B}_{n-1}^{-1} z^n \\
 &= \sum_{n=0}^{\infty} c^n \mathcal{A}_n \mathcal{B}_{n-1}^{-1} \mathcal{C}_{n-1}^{-1} \frac{z^n}{(q; q)_n}
 \end{aligned}$$

Further using (3.18),

$$\begin{aligned}
 \Omega_{j,i}^{(\delta,A,\zeta;-s)} \Phi_{\ell,k}^{(\mu,C,\eta;r)} \Phi_{h,m}^{(\alpha,B,\sigma;1)} Y_c &= \sum_{n=0}^{\infty} c^n \mathcal{A}_n \mathcal{B}_{n-1}^{-1} \mathcal{C}_{n-1}^{-1} \frac{1}{(q; q)_n} \left\{ \prod_{j=0}^{\delta-1} \prod_{i=0}^{\delta-1} [(\Delta_q I + \zeta^{-j} q^{-(A+iI)/\delta} - I)]^{-s} \right\} \\
 &\times \left\{ \prod_{j=0}^{\delta-1} \prod_{i=0}^{\delta-1} [(\zeta^{-j} q^{-(A+iI)/\delta})]^{-s} \right\}^{-1} z^n \\
 &= \sum_{n=0}^{\infty} c^n \mathcal{A}_n \mathcal{B}_{n-1}^{-1} \mathcal{C}_{n-1}^{-1} \frac{1}{(q; q)_n} \left\{ \prod_{j=0}^{\delta-1} \prod_{i=0}^{\delta-1} [(I - \zeta^j q^{nI+(A+iI)/\delta})]^s \right\} z^n \\
 &= \sum_{n=0}^{\infty} c^n \mathcal{A}_n \mathcal{B}_{n-1}^{-1} \mathcal{C}_{n-1}^{-1} \frac{1}{(q; q)_n} \mathcal{A}_{n-1} \mathcal{A}_n^{-1} z^n \\
 &= \sum_{n=0}^{\infty} c^n \mathcal{A}_{n-1} \mathcal{B}_{n-1}^{-1} \mathcal{C}_{n-1}^{-1} \frac{z^n}{(q; q)_n}
 \end{aligned}$$

Finally,

$$\begin{aligned} \Delta_q Y_c &= \mathcal{D}_q \Omega_{j,i}^{(\delta,A,\zeta;-s)} \Phi_{\ell,k}^{(\mu,C,\eta;r)} \Phi_{h,m}^{(\alpha,B,\sigma;1)} Y_c \\ &= \sum_{n=1}^{\infty} c^n \mathcal{A}_{n-1} \mathcal{B}_{n-1}^{-1} \mathcal{C}_{n-1}^{-1} \frac{z^{n-1}}{(q; q)_{n-1}} \\ &= \sum_{n=0}^{\infty} c^{n+1} \mathcal{A}_n \mathcal{B}_n^{-1} \mathcal{C}_n^{-1} \frac{z^n}{(q; q)_n} \\ &= c e_{\alpha I, \delta I, \mu I}^{A, B, C}(cz; s, r|q). \end{aligned}$$

□

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