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Some Special Structures of δ_1 Near-Rings

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Abstract:

If, in a ring (N, +, ·) we ignore the commutativity of '+' and one of the distributive laws, (N, +, ·) becomes a Near-Ring. If we do not stipulate the left distributive law, (N, +, ·) is a right near-ring. This research aims to introduce the concept of δ_1 Near-Ring. For every x,y in N, xNy=Nx^2 y^2 is called δ_1Near-Ring. The element wise characterization for 〖δ〗_1 Near-Ring will be investigated and shall establish theorems and properties in this Near-Ring.

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1 Introduction

A right near-ring (N, +, ·) is an algebraic system with two binary operations such that (i) (N, +) is a group-not necessarily abelian-with 0 as its identity element, (ii) (N, ·) is a semigroup [we write xy for x.y for all x, y in N] and (iii) $(x + y)z = xz + yz$ for all x, y, z in N. Because of (iii) 0n =0 for all n in N. As we do not stipulate the left distributive law, "n0 = 0" need not hold good for all n in N. We say that N is zero-symmetric if $n0 = 0$ for all n in N. N is called an S-near-ring or an S'-near-ring according as $x \in Nx$ or $x \in xN$ for all $x \in N$. A subgroup M of N is called an Nsubgroup if $NM \subset M$ and an invariant N-subgroup if, in addition, $MN \subset M$.

An ideal I of N is called a semi prime ideal if for all ideals J of N. $J^2 \subset I \Rightarrow J \subset I$. If {0} is a semiprime ideal, then N is called a semi prime near-ring. An ideal I of N is called completely semi prime if x ∈ I whenever$x^2 \in I$. N is called a strictly prime near-ring if {0} is a strictly prime ideal i.e. if A and B are N-subgroups of N such that $AB = \{0\}$, then either $A = \{0\}$ or B = {0}. A near-ring N has property P_4 if for all ideals I of N, $xy \in I \Rightarrow yx \in I$. From p.289 of Pilz [3]

The concept of a mate function in N has been introduced in [4] with a view to handle the regularity structure in a near-ring with considerable ease. A map f from N into N is called a mate function for N, if $x = xf(x)x$ for all x in N. $f(x)$ is called a mate of x. A map f from N into N is called a P_3 mate function for N, if $x = xf(x)x$ and $xf(x) = f(x)x$ for all x in N.

Basic concepts and terms used but not defined in this paper can be found in Pilz [3]. Throughout this paper N stands for a near-ring – more precisely a right near-ring – with at least two elements.

As in p.249 Pilz [3], "if N is a near-field then either N is isomorphic to $M_c\ (Z_2)$ or N is zero-symmetric" (For the concept of $M_c\ (Z_2)$ one may refer to Example 1.4(a), p.8 and 1.15, p.12 of Pilz [3]. Obviously $M_c\ (Z_2)$ is a near-field of order 2 and is not zero-symmetric). All the near-fields in this paper are zero-symmetric.

2 Notations

(i) E denotes the set of all idempotent of N. [e in N is called an idempotent if $e^2 = e$]

(ii) L denotes the set of all nilpotent of N. [a in N is nilpotent if $a^k = 0$ for some positive integer k.]

(iii) $N_0 = \{n \in N \,/\, n0 = 0\}$ - zero-symmetric part of N.

(iv). $N_d = \{n \in N \,/\, n(x+y) = nx + ny$ for all x, y in $N\}$ – set of all distributive element of N.

(v) $C(N) = \{n \in N \,/\, nx = xn$ for all x in $N\}$ - center of N.

3. Preliminary results

We freely make use of the following results from [4], [3] and [2] and designate them as K (1), K (2) etc. (K for 'known results').

K (1): If N has a mate function m, then for every $x \in N$, $xf(x), f(x)x \in$ E and $Nx = Nf(x)x$ and $xN = xf(x)N$. (Lemma 3.2 of [4]).

K(2): If L ={0}and , N= N_0 then (i) $xy = 0 \Rightarrow yx = 0$ (for x, y in N) and (ii) N has "Insertion of Factors Property" – IFP for short – i.e. for x, y in N, $xy = 0 \Rightarrow xny = 0$ for all n in N. (In this paper we write that N has (*, IFP) if N has both (i) and (ii)) (Lemma 2.3 of [4]).

K (3): A zero-symmetric near-ring N has IFP if and only if (0: S) is an ideal, where S is any non-empty subset of N. (9.3, p.289 of [3]).

K (4): A near-ring N has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in N (Prob. 14, p.9 of [2]).

3.1 Definition and Examples

In the section, we introduce the notion of δ_1Near-Ring and furnish examples to illustrate it. To start with we have the following definition.

Definition 3.1.1

Let N be a right near-ring. If for every x, y in N, $xNy = Nx^2y^2$ then we say N is a δ_1Near-Ring.

Examples 3.1.2

(i) Let $(N, +)$ be the Klein's four group $\{0, a, b, c\}$. The near-ring $(N, +, \cdot)$ where '·' is defined as per scheme 12, p.408 of Pilz [33].

·	0	a	b	c

0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

is a δ_1Near-Ring.

(ii) Let $(N,+)$ be the Klein's four group $\{0,a,b,c\}$. The near-ring $(N,+,\cdot)$ where '$\cdot$' is defined as per scheme 11, p.408 of Pilz [33].

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	a
b	0	0	0	0
c	0	a	b	a

is a not δ_1Near-Ring. Since $aNb \neq Na^2b^2$

3.2 Properties of δ_1 near-ring

We shall obtain a complete characterization for $\delta_1$near-Ring and obtain structure theorem for such near-ring – under certain conditions.

Proposition 3.2.1

Let N be a $\delta_1$near-ring with identity.

(i) N is zero symmetric
(ii) If N has no non-zero nilpotent elements then N is an S-near-ring.
(iii) If N is an S'-near-ring then N has no non-zero nilpotent elements.

Proof.

(i) Let N be a $\delta_1$near ring.Then for all x,y in N, $xNy = Nx^2y^2$ …………….(1)
Putting y=1, we get $xN.1 = Nx^2.1$ for all x in N. $\Rightarrow xN = Nx^2$ for all x in N. When $x = 0, 0N = N0 = \{0\}$. It follows that N is zero symmetric.

(ii) Putting y=1 in equation (1), we get $xN = Nx^2$ for all x in N …………………….. (2).
Now $x^2 \in xN$ for all x in N $\Rightarrow x^2 \in Nx^2$ [By equation (2)]. We have $x^2 = zx^2$for some z in N. Therefore $(x - zx)x = 0$. By K (2), this implies that $x(x - zx) = 0$ and $zx(x - zx) = 0$. Consequently $(x - zx)^2 = 0$. By assumption $L = \{0\}$ therefore $x - zx = 0$ forcing $x = zx$. Thus $x \in Nx$ i.e. N is an S-near-ring.
(iii) If N is an S'-near-ring then $x \in xN$ and since $xN = Nx^2$ we get $x = nx^2$ for some $n \in N$. Therefore$x^2 = 0 \Rightarrow x = 0$. N has no non-zero nilpotent elements, from K (4).

Corollary3.2.2.

If N is a $\delta_I$near-ring without non-zero nilpotent elements, then from K (2), we see that N has (*, IFP). It is obvious that the property $\delta_I$is preserved by near-ring homomorphisms. Consequently, we have

Proposition 3.2.3

Any homomorphism image of a $\delta_I$near-ring is δ_I-near-ring.

Proof

Let f: N → N ′ be a near-ring epimorphism. Let $x^{'}, y' \in N'$. Then there exist $x, y \in N$ such that $f(x) = x', f(y) = y^{'}$. Also for $n' \in N'$ there exist $n \in N$ such that $f(n) = n'$.

Since N is $\delta_I{}^*$ near-ring, $xNy = Nx^2y^2$ ... ... (3)

Now, $x^{'}n^{'}y^{'} = f(x)f(n)f(y) = f(xny)$ [since f is a homomorphism]

$= f(nx^2y^2)$ [by Equation (3)] $=f(n)f(x^2)f(y^2)$

Therefore, $x^{'}n^{'}y^{'} \in N'x^{2}{}'y^{2}{}'$.

Consequently, $x^{'}N^{'y^{'}} \subseteq N^{'x^{2}{}'y^{2}{}'}$ ... ... (4)

Similarly, $N'x^{2}{}'y^{2}{}' \subseteq x^{'}N'y^{'}$…………… … (5)

Combining Equations (4) and (5), we get $x^{'}N'y^{'} \subseteq N'x^{2}{}'y^{2}{}'$

Hence N ′ is also δ_I near-ring and the desired result follows.

As an immediate consequence of Proposition 3.2.3we have the following theorem:

Theorem 3.2.4

Every $\delta_I$near-ring N is isomorphic to a subdirect product of subdirectly irreducible δ_I near-Ring.

Proof.

By Theorem 1.62, p.26 of Pilz [3], N is isomorphic to a sub direct product of sub directly irreducible near-ring N_i's, say, and each N_i is a homomorphic image of N under the projection map π_i. The desired result now follows from Proposition 3.2.3.

We shall now discuss the behavior of N-subgroups and ideals of δ_I nearRing. To start with we have the following:

Proposition 3.2.5.

Let N be a $\delta_I$near-ring with identity, If N is left bipotent, and then every N-subgroup of N is invariant.

Proof.

Let N be a δ_I near ring with identity. Then for all x,y in N, $xN = Nx^2$ …………….(6). Let A be any N-subgroup of N, Then $A = \sum_{x \in A} Nx$ …………… (7). Now, $NxN = NNx^2 1 = Nx^2$ [By equation (6)] $\subseteq Nx$. [Since N is left bipotent] (i.e.) $NxN \subseteq Nx$………… (8). Therefore, AN =$(\sum_{x \in A} Nx)\ N$[By equation (7)] $\subseteq \sum_{x \in A} NxN \subseteq \sum_{x \in A} Nx$[By equation (8)]= A (ie.)$AN \subseteq A$. Consequently, A is invariant N-subgroup.

Proposition 3.2.6.

Let N be a $\delta_I$near-ring. Then every left ideal of N is an ideal.

Proof.

Let A be a left ideal of N. Since N is zero-symmetric, $NA \subseteq A$ i.e. A is an N-subgroup of N. Proceeding as in Proposition 3.2.5 we get $AN \subseteq A$. Hence A becomes an ideal.

It is easy to observe the following:

Corollary 3.2.7

Every left ideal (and therefore every ideal) of a δ_I near-ring N is an invariant N-subgroup of N.

Proposition 3.2.8.

If N is a $\delta_I$near-ring with identity, then N has strong IFP.

Proof.

Let N be a $\delta_I$near ring with identity. Then for all x,y in N, $xN = Nx^2$ ……………….(9)

In view of Proposition 9.2 Pilz [3], we need only to establish that for all ideals I of N and for all a, b, n in N, $ab \in I \Rightarrow anb \in I$. Since I is an ideal, $IN \subseteq I$ and since N is zero-symmetric, I is an N-subgroup of N. i.e. $NI \subseteq I$. Now $an \in aN = Na^2$ [By equation (9)] ⇒ $an = n^{'}a^2$ for some $n' \in N \Rightarrow anb = (n^{'}a^2)b = (n'a)(ab) \in NI \Rightarrow anb \in I$.

Notation 3.2.9

If a $\delta_I$near-ring N is an S (or S')-near-ring then we write that N is an S - $\delta_I$near-ring (or S'- $\delta_I$near-ring).

Remark 3.2.10

For an S- δ_I near-ring, we see that for all x in N, $x \in Nx = x^2 N \Rightarrow x = x^2 n$ for some $n \in N$. Hence $x^2 = 0 \Rightarrow x = 0$ and K (4) demands that $L = \{0\}$.

Proposition 3.2.11.

In a δ_I near-ring, $E \subseteq C(N)$.

Proof.

Let e ∈ E. Since N is $\delta_I$near-ring ,$eNe = Ne^2e^2 = Ne$. Therefore forsome n in N, $ene = ue$ and $ne = eve$ for some u, v in N. Now,$ene = e(ue)$ and $e(ne) = eve$. Thus $ene = ne$ for all n in N. …………..... (10).Also we have $(ene - en)e = 0$. This implies $e(ene - en) = 0$ and $en(ene - en) = 0$. Also $ene(ene - en) = en.0 = 0$ [since N is zero-symmetric]. Now, $ene(ene - en) - en(ene - en) = 0$.

Consequently, $(ene - en)^2 = 0$ and K (4) guarantees $ene - en = 0$.Therefore,$ene = en$ for all n in N…………….. (11).From Equations (10) and (11) we get $en = ne$ for all n in N. Thus $E \subseteq C(N)$.

Remark 3.2.12.

It is worth noting that we do not stipulate that N admits mate functions for the validity of the above results.

Proposition 3.2.13.

Let N be a $\delta_I$near-ring with identity. Then N has a mate function if and only if N is an S'-near-ring.

Proof.

When N has a mate function `m' for all $x \in N, x = xf(x)x \in xN$ and obviously N is an S'-near-ring. Conversely let N be an S' near-ring. $x \in xN = Nx^2 \Rightarrow x = nx^2$ for some $n \in N \Rightarrow x^2 = xnx^2 \Rightarrow (x - xnx)\,x = 0$ .Using Proposition 3.2.1(iii) and Corollary 3.2.2 we get $x(x - xnx) = 0$ and $xnx(x - xnx) = 0$ and consequently $(x - xnx)^2 = 0$. Since $L = \{0\}$ we get $x - xnx = 0$i.e. $x = xf(x)x$ where we get $f(x) = n$. This guarantees that $f : N \to N$ is a mate function for N.

Proposition 3.2.14.

Let N be an S'-$\delta_I$near-ring. Then N has a P_3 mate function.

Proof.

When N is an S'- $\delta_I$near-ring it admits a mate function 'f'. From Proposition 3.2.12 we have $x = f(x)\,x^2 \Rightarrow (xf(x) - f(x)x)\,x = 0 \Rightarrow (xf(x) - f(x)x)^2 = 0$. [since N has (*, IFP)] $xf(x) - f(x)x = 0 \Rightarrow xf(x) = f(x)x$ i.e. $f(x) \in C(x)$ i.e. 'f' is a P_3 mate function.

Proposition 3.2.15.

If N has property $\delta_I$and a mate function 'f' then $L = \{0\}$ and N has (*, IFP).

We now give a complete characterization of δ_I when they admit mate functions.

Theorem 3.2.16.

Let N be a near-ring with a zero symmetric mate function 'f' and left bipotent. Then the following statements are equivalent:

(i) N is δ_I

(ii) $E \subseteq C(N)$

Proof. (ii) ⇒ (i) Let $E \subseteq C(N)$. Now $Nx^2y^2 = Nxy^2$ [N is left bipotent] $= (Nf(x)x)y^2$ [By K (1)]$=$ $(f(x)xN)y^2$[Since E ⊆ C (N)] $= xf(x)Ny$ [N is left bipotent]$=$ xNy [By K (1)].i.e. $Nx^2y^2 = xNy$

Proof of '(i) ⇒(ii)' is similar.

Remark 3.2.17

Let N admit a mate function '$f(x)$' and let E ⊆ C (N). It is easy to observe that for every x in N,$x = x\,f(x)x \Rightarrow x = f(x)x^2$. Consequently Proposition 3.2.16 guarantees that m is a P_3 mate function.

Theorem 3.2.18

Every N-subgroup of N is an ideal in an S'- $\delta_I$near-ring.

Proof:

Since N is an S'-$\delta_I$near-ring, it admits a mate function 'f(x)' [from Proposition 3.2.13] and L {0} [from Proposition 3.2.3(iii)]. It is clear from K (2) that N has (*, IFP). Again for any non-empty $S \subseteq N$,(0 : S) is an ideal of N [by K(3)]. If M is any N-subgroup of N, then $= \sum_{x \in M} Nx$. We first show that each Nx is an ideal. Let $S = (0 : Nx)$. We claim that $Nx = (0 : S)$. Clearly$Nx \subset (0 : S)$………….. (12) Now if $y \in (0 : S)$ then $yS = \{0\}$. Also $(y - yf(x)x)f(x)x = 0$ ……….. (13) $\Rightarrow (y - yf(x)x)Nf(x)x = \{0\} \Rightarrow (y - yf(x)x)Nx = \{0\}$ [using K(1)] $\Rightarrow (y - yf(x)x) \in (0 : Nx) = S$. Since $yS = \{0\}, y(y - yf(x)x) = 0$………..... (14) .Using the fact that N has (*, IFP), it is easy to get from equations (13) and (14),$(y - yf(x)x)^2 = 0.$ Since $L = \{0\}$ we get $(y - yf(x)x) = 0 \Rightarrow y = yf(x)x \Rightarrow\ y \in Nx$. Therefore $(0 : S) \subset Nx$………… (15) From equations (12) and (15) we get $Nx =\ (0 : S)$ and hence Nx is an ideal. The desired result now follows.

Remarks 3.2.19.

(a) It is worth noting that in a $\delta_1$near-ring with mate functions the concepts of N-subgroups, left ideals, right ideals and ideals are equivalent.

(b) Recall that the nil radical of N is the greatest nil ideal of N. Since L = {0}, for an S'-$\delta_1$near-ring N, it follows that the nil radical of $N =\ \{0\}$.

Proposition 3.2.20

Let N be an S'-$\delta_1$near-ring. Then any N-subgroup of N is a completely semi prime ideal.

Proof.

Suppose I is an N-subgroup of N. From Theorem 3.2.18 it follows that I is an ideal.

Let $x^2\ \in\ I$. Since N has strong IFP, $xf(x)x \in I$ i.e. $x\ \in$I. Hence I is a completely semi prime ideal.

Proposition 3.2.21

An S'-$\delta_1$near-ring has property P_4

Proof.

Let I be an ideal of N and let $xy\ \in I$.

Now $(yx)^2 = (yx)(yx) = y(xy)x \in NIN \subset I$ [using Remark 3.2.19 (a)] $\Rightarrow (yx)^2 \in I$ .Using Proposition 3.2.20 we get $yx \in I.$ i.e.). N has property P_4.

3.3 In this section we obtain a structure theorem for $\boldsymbol{\delta_1}$ near-Ring.

Throughout this section N denotes an S'-$\delta_1$near-ring and m is a mate function for N.

Theorem 3.3.1. N is sub directly irreducible if and only if N is a near-field.

Proof.

Suppose N is sub directly irreducible. First we claim that no non-zero idempotent of N is a zero-divisor. Let J be the set of all non-zero idempotent which are zero-divisors and let $\neq \emptyset$. Let $I = \bigcap_{e \in J}(0 : e)$. Since N is subdirectly irreducible,$I \neq \emptyset$. Let $a\ \in\ I - \{0\}$. Thus $ae = 0$ for all e in J …………. (16) This $\Rightarrow f(a)ae = 0\ \ \Rightarrow ef(a)a = 0$ [using K (2)] $\Rightarrow f(a)\ a \in\ \ J$. From equation (16) we get $af(a)a =\ 0 \Rightarrow\ a =\ 0.\dots\dots\dots\dots(17)$. This contradiction implies that no non-zero idempotent of N is a zero-divisor. We shall now prove that N has no non-trivial N-Subgroups. Let M be any N-subgroup of N such that $M \neq\ \{0\}$ and let $x(\neq 0) \in\ M.$ Let N be a $\delta_1$near ring. Then for all x,y in N, $xNy = Nx^2y^2$ .Putting $x = 1$, We get $xN.1 = Nx^2.1$ for all x in N. $\Rightarrow Ny = Ny^2$ for all y in N. For any $n \in N$, there exists n_1 in N such that $ny = n_1y^2\ \ \Rightarrow (n - n_1y)y = 0 \Rightarrow (n - n_1y)f(y)y = 0 \Rightarrow n - n_1y = 0$[by equation 17]$\Rightarrow n = n_1y \in NM \subseteq M.$ Therefore $N \subseteq M$ i.e. $M = N$. Thus N has no non-trivial N-subgroups. Clearly for $n \in\ N -\ \{0\},$ Nn is an N-subgroup of N. Consequently $Nn = N$ for all $n \in\ N -\ \{0\}$……..…. (18) .Also, it is clear that $N_d \neq \{0\}$ [since E $\subseteq$ C(N)]$\subseteq N_d$]. This and equation (18) guarantee that N is a near-field. [Theorem 8.3, Pilz [3])]

Converse is obvious.

As an immediate consequence of Theorem 3.1, we have the following:

Corollary 3.3.2. N has no non-zero zero-divisors if and only if N is a near-field.

We are now in a position to give a structure theorem for N.

Theorem 3.3.3. N is isomorphic to a sub direct product of near-fields.

Proof. From Theorem 3.2.4, N is isomorphic to a sub direct product of sub directly irreducible $\delta_I$near-ring, N_i's, say. Obviously the existence of a mate function is preserved under homomorphisms. Hence each N_i admits a mate function. Appealing to Theorem 3.3.1 we get N is isomorphic to a sub direct product of near-fields.

Remark 3.3.4. From 8.11 of [3], the additive group of a near-field is abelian. It follows that for any δ_I near-ring N with mate functions, (N, +) is abelian.

Proposition 3.3.5. Let N be a Boolean near-ring. Then N is $\delta_I$if and only if it is a commutative ring.

Proof. We observe that identity function is a mate function for N. Appealing to Theorem 3.2.16 and Remark 3.3.4 we see that when N is a $\delta_I$near-ring, $N = E \subset C(N)$ and $(N, +)$ is abelian and hence N is a commutative ring. Conversely, N is Boolean and a commutative ring. Then for all x in N,$xn = nx$ for all n in N⇒ $xny = nxy$ for all y in N,⇒ $xNy = Nx^2y^2$Hence the result.

Proposition 3.3.6. If N is distributively generated and has no non-zero zero-divisors then N is a division ring.

Proof. Corollary 3.3.2 guarantees that N is a near-field. Also (N, +) is abelian [by Remark 3.3.4)] Since N is distributively generated, we see that N is a ring [from Theorem 6.6(c) of Pilz [3]] and hence the result.

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