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On the Structure of σ_1 Near - Rings

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Abstract:

If, in a ring (N, +, ·) we ignore the commutativity of '+' and one of the distributive laws, (N, +, ·) becomes a Near-Ring. If we do not stipulate the left distributive law, (N, +, ·) is a right near-ring. This research aims to introduce the concept of σ_1near-ring. N is called σ_1near - ring if N is a right near-ring and xy^2=yxyfor all x,y∈N .The element wise characterization for σ_1near-ring will be investigated and shall establish theorems and properties in this near - ring.

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1 Introduction

A right near-ring is a non-empty set N together with two binary operations "+"and "." such that (1) (N, +) is a group. (2) (N, ·) is a semi-group and (3) $(n_1 + n_2)n_3 = n_1n_3 + n_2n_3$ for all $n_1, n_2, n_3 \in N$.

Throughout this paper N stands for a right near-ring. $(N,+,.)$ with at least two elements 1 and $'0'$ denotes the identity element of the group $(N, +)$. Obviously, 0n = 0 for all n in N. N is said to be zero-symmetric if n0 = 0 for all n in N. As in [2], a subgroup of $(M,+)$ of $(N,+)$ is called an N-subgroup of N if $NM \subset M$ and an invariant N subgroup of N if, in addition, $MN \subset M$. In [6], N is defined to be pseudo commutative if $xyz = zyx$ for all x,y,z in N. The concept of a mate function in N has been introduced in [4] with a view to handling the regularity structure with considerable case. A map $'f'$ from N into N is called (i) a mate function for N if $x = xf(x)x$, (ii) a P_3 mate function, if, in addition, $xf(x) = f(x)x$ for all x in N. By identity 1 of N, we mean only the multiplicative identity of N.

Basic concepts and terms used but left undefined in this paper can be found in [2].

2 Notations

In this section, we furnish the notations which are used frequently throughout this paper.
(i) E denotes the set of all idempotent of N.

(e in N is called an idempotent if $e^2 = e$)

(ii) L denotes the set of all nilpotent of N.

(a in N is nilpotent if $a^k = 0$ for some positive integer k.) and N is said to be reduced if L=$\{0\}$.

(iii) $N_0 = \{n \in N / n0 = 0\}$ - zero-symmetric part of N and N is called zero symmetric if $N = N_0$.

(iv) $N_d = \{n \in N / n(x+y) = nx + ny$ for all x, y in $N\}$ – set of all distributive element of N and N is called distributive if $N = N_d$.

(v) $C(N) = \{n \in N / nx = xn$ for all x in $N\}$ - centre of N.

(vi) If A is any non – empty subset of N, then

i) $A^* = A - \{0\}$

ii) $C(A) = \{n \in N / na = an$ for all $a \in A\}$

iii) When $A = N, C(N) = \{na = an$ for all $a \in N\}$ −called the centre of N.

(vii) When $E \subseteq C(N)$, we say that the idempotent are central.

3 Preliminary Results

We freely make use of the following results and designate them as **R (1)**, **R (2)**....etc.

R (1) N is sub directly irreducible if and only if the intersection of any family of non-zero ideals of N is again non-zero (Theorem 1.60, p.25 of [2])

R (2) N has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in N (Problem 14, p.9 of [3]).

R (3) If f is a mate function for N, then for every x in $N, xf(x), f(x)x \in E$ and $Nx = Nf(x)x, xN = xf(x)N$ (Lemma 3.2 of [5]).

R (4) If $L = \{0\}$ and $N = N_0$, then (i) $xy = 0 \Rightarrow yx = 0$ for all x, y in N.(ii) N has Insertion of factors property- IFP for short- i.e for x, y in $N, xy = 0 \Rightarrow xny = 0$. for all n in N. If N satisfies (i) and (ii) then N is said to have ($*$, IFP) (Lemma 2.3 of [5])

R (5) Any pseudo commutative near-ring with a right identity is weak commutative ((i.e). $xyz = xyz$ for all x, y, z in N [2]) (Proposition 2.9 of [6])

R (6) A zero-symmetric near-ring N is a near- field if $N_d \neq \{0\}$ and for all $n \in N - \{0\}$, $N_n = N$ (Theorem 8.3, p.249 of [2]).

R (7) Let N be a right near-ring. If for every x, y in N, $xN\,y = Nxy$ then we say N is a β_1 near-ring. (Definition 3.1.1 of [4])

4. Definition and Examples of σ_1 Near – Rings

In this section we define $\sigma_1$near-ring and give certain examples of this new concept.

Definition 4.1

Let N be a right near- ring. Then N is said to be an $\sigma_I$near-ring if $xy^2 = yxy$ for all $x, y \in N$.

Example 4.2(a)

The near-ring $(N, +, .)$ defined on Klein's four group $(N, +)$ with $N = \{0, a, b, c\}$ where '·' is defined as per scheme 12, *P 408* of Pilz [2].

·	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

is a $\sigma_I$near-ring

b) Let $(N, +)$ be the Klein's four group as in (a) above. If multiplication is defined as per scheme 22, P.408 of pilz[2]

·	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	a

Then N is not a $\sigma_I$near-ring since $ab^2 \neq bab$.

5. Properties of $\sigma_I$near-ring

In this section we prove certain important properties of σ_I near-ring and give a complete characterization of such near-ring.

Proposition 5.1

Let $N = N_d$ be a $\sigma_I$near-ring with identity. Then N is commutative.

Proof

Let N be a $\sigma_I$near-ring.

Then for all $x, y \in N, xy^2 = yxy$………….... (1)

Replace the element y by $y + e.$ in (1)

$$x(y+e)^2 = (y+e)x(y+e)$$

$$x(y+e)(y+e) = (y+e)x(y+e)$$

$x[(y+e).y+(y+e).e] \;=\; (y+e)x(y+e)$

$x[y.y+y+y+e] \;=\; (yx+x)(y+e)$

$(xy)y+xy+xy+x \;=\; (yx)y+yx+xy+x$

$(xy)y+xy \;=\; (yx)y+yx$ (By right Cancellation law)

$x(y.y)+x.y \;=\; y(x.y)+y.x$ (By Associative Law)

$\Rightarrow xy^2+xy \;=\; xy^2+yx$ [By equation 1]

$xy = yx \;\forall\; x,y \in N$

Thus N is commutative.

Remark 5.2

A quasi weak commutative near-ring can become a $\sigma_I$near-ring

Proof

Let N be a quasi-weak commutative near-ring

Then $xyz = yxz$ for all $x,y,z \in N$.

If $y = z$, then $xyy = yxy$, (ie) $xy^2 = yxy$.

Consequently N becomes a $\sigma_I$near-ring.

Proposition 5.3

$\sigma_I$near-ring is always zero symmetric.

Proof

Suppose N is a $\sigma_I$near-ring Then $xy^2 = yxy\;\; for\ all\ x,y \in N$.

When $y = 0$, $x0 = 0\,x\,0 = 0$

It follows that N is zero symmetric.

Proposition 5.4

Let N be a $\sigma_I$near-ring. If N is weak commutative then $x^2y = y^2x$. for all $x,y \in N$

Proof

Let N be a weak Commutative near-ring

Then $xyz = xzy$ ………..… (1)

Let N be a $\sigma_I$near-ring

Then $xy^2 = yxy$ …………(2)

Now, $x^2y = xxy \;= xyx$ [By equation (1)] $= yx^2$ [By equation (2)]

Hence $x^2y = yx^2$for all $x,y \in N$.

Proposition 5.5

Homomorphic image of a $\sigma_I$near-ring is also a $\sigma_I$near-ring.

Proof

The proof is straight forward.

Proposition 5.6

If I is an ideal of the $\sigma_I$near-ring N then N / I is also an $\sigma_I$near-ring

Proof

The function $\emptyset: N \to N/I$ defined by $\emptyset(x) = I + x$ is an epimorphism. The rest of the proof is taken care of by the above Proposition 5.5.

Proposition 5.7

Every $\sigma_I$near-ring N is isomorphic to a sub direct product of sub directly irreducible $\sigma_I$near-ring.

Proof

By theorem 1.62, P 26 of Pilz [2], N is isomorphic to a sub direct product of sub directly irreducible σ_I- near-ring.N_i's and each N_i is a homorphic image of N under the multiplication π_i. Now the desired result follows from the above Proposition 5.5.

Proposition 5.8

Let N be a $\sigma_I$near-ring with a mate function f.

Then we have

i) $L = \{0\}$

ii) N has $(*,$ IFP)

iii) $E \subseteq C(N)$

Proof

Let N be a $\sigma_I$near-ring

Then $xy^2 = yxy \; \forall \; x, y \in N$. ……………… (1)

Since f is a mate function for N then $x = xf(x)x \in xNx$ for all $x \in N$

$\therefore x = xnx$ for some n. ………………… (2)

i)For $n, x \in N$, $nx^2 = xnx$ [By equation (1)] $= x$ [By equation (2)]

Suppose $x^2 = 0$

Clearly then $x = 0$. [Since N is zero symmetric]. Then $R(2)$ guarantees that $L = \{0\}$

(ii) By i) $L = \{0\}$. Now R(4) guarantees that N has $(*,$ IFP)

(iii)Let $e \in E$. Since N is a $\sigma_I$near $-$ ring, $ne^2 = ene \Rightarrow ne = ene$for all n in N … …. . … … (3)

Also we have $(ene - en)e = 0$

This implies $e(ene - en) = 0$ and $en(ene - en) = 0$ $[by(ii)]$

Also $ene(ene - en) = en.0 = 0$ [Since N is zero symmetric]

Now $ene(ene - en) - en(ene - en) = 0$.

Consequently, $(ene - en)^2 = 0$ and (i) guarantees $ene - en = 0$.

Therefore $ene = en$ for all n in N ……………….….(4)

From Equations (3) and (4) we get $en = ne$ for all n in N. Thus $E \subseteq C(N)$

Proposition 5.9

Let N be a pseudo commutative near -ring with right identity. Then if N is a $\sigma_1$near-ring then for any a, b in N, $ab = 0$ implies $ba = 0$

Proof

Let N be a pseudo commutative near-ring. Then $xyz = zyx$ for all $x, y, z \in N$ …..…. (1)

Now R (5) guarantees that N is weak commutative.

$\therefore xyz = xzy$ for all $x, y, z \in N$ ………..(2)

Now,

$$\begin{aligned}
(xax)(yby) &= ax^2by^2 \\
xax\, yby &= axxbyy \\
xa(xyb)y &= a(xxb)yy \\
xa(byx)y &= a(bxx)yy \quad \text{[By Equation (1)]} \\
(xab)yxy &= ab\, xxy\, y \\
baxyxy &= ab(xxy)y \quad \text{[By Equation (1)]} \\
ba\, xy\, xy &= ab\, x\, yxy \quad \text{[By Equation (2)]}
\end{aligned}$$

$ba = ab$. Since $ab = 0$ it follows that $ba = 0$.

Proposition 5.10

N is a σ_1 near-ring if and only if every x in N can be written as $xy^2 = u + v$ where $u \epsilon N_o$ and $v \epsilon N_c$ and $u = y_0(ny + m) - y_0\, x\, y_c, v = y_0 x\, y_c + y_c,\ y = y_0 + y_c \epsilon N_0 \oplus N_c$ where $y_0, n \in N_0, y_c, m \in N_c$. Further more $u \in N_0, v \in N_c.$

Proof

For the ‘only if’ part, let $y \in N$. Since N is σ_1 there exist x in N such that $xy^2 = yxy$. By using pierce decomposition we can write $x = n + m$ and $y = y_0 + y_c$ where $x \in N, n, y_0 \in N_0$ and $m, y_c \in N_c$

Now $xy^2 = (y_0 + y_c)(n + m)y$

$$= (y_0 + y_c)(ny + my)$$

$$= (y_0 + y_c)(ny + m) \quad [since\ m\ \in N_c]$$

$$= y_0(ny + m) + y_c(ny + m)$$

$$= y_0(ny + m) + y_c \qquad [since\ y_c \in N_c]$$

$$= y_0(ny + m) - y_0\, x\, y_c + y_0\, x\, y_c + y_c$$

$= u + v$ where $u = y_0(ny + m) - y_0\, x\, y_c$ and $v = y_0\, x\, y_c + y_c$

Now, $u.o = [y_0(ny + m) - y_0\, x\, y_c].0 = y_0(ny + m)0 - y_0\, x\, y_c\, .0$

$$= y_0(ny0 + m0) - y_0\, x\, y_c.\, 0$$

$$= y_0(ny0 + m) - y_0\, x\, y_c \qquad [since\ \ m, y_c \in N_c]$$

$$= y_0(ny_c + my_c) - y_0\, x\, y_c \quad [since\ y0 = y_c\ and\ m\ \in N_c]$$

$$= 0.$$

Also, $v.0 = [y_0\, x\, y_c + y_c].0 = y_0\, x\, y_c 0 + y_c\ 0 = y_0\, x\, y_c + y_c\ \ [since\ y_c \in N_c]\ = v$

Thus $xy^2 = u + v$ where $u \in N_0$ and $v \in N_c$.

For the "if part" Assume for every y in N with $xy^2 = u + v$ where $u \in N_0$ and $v \in N_c$ with $u = y_0(ny + m) - y_0 x\, y_c$, $v = y_0\, x\, y_c + y_c$ where $y = y_0 + y_{c,}\, y_0$, $n \in N_c$ and

$y_c, m \in N_c$

We shall show that N is a $\sigma_1$near – ring.

Now, $xy^2 = u + v$

$$= y_0(ny + m) - y_0 x\, y\,_c + y_0\, x\, y_c + y_c$$

$$= y_0(ny + my) + y_c \quad [since\ m\ \in N_c]$$

$$= y_0(n + m)y + y_c$$

$$= y_0\, xy + y_c\, xy \qquad [since\ y_{c,} \in N_c]$$

$$= (y_0 + y_c)xy$$

$$= yxy$$

Thus for every $y \in N, xy^2 = yxy$ for all x in N.

Hence N is a $\sigma_1$near – ring.

Theorem 5.11

Let N be a zero symmetric weak commutative σ_1 near – ring then right cancellative near rings are integral.

Proof

Let $a \neq o$ and $ab = 0$. Then $ba = 0$ [By Proposition 5.10] $= 0a$ and therefore by right cancellative law we get $b = 0$. And if $b \neq 0$ and $ab = 0$ then $ab = 0b$. This implies $a = 0$, by right cancellative law. Hence $ab = 0 \Longrightarrow$ either $a = 0$ or $b = 0$ and the result follows.

Theorem 5.12

Let N be a σ_1 near –ring with mate function f. If N is regular and I is a proper ideal of N, then every element of I is a zero divisor.

Proof

Let N be an $\sigma_1$near – ring

Then $xy^2 = yxy \ \forall x, y \in N$ ...……………….. (1)

Since f is a mate function for N, then $x = xf(x)x \in xNx$

$\therefore x = xnx = nx^2$ for some n [By equation 1]

Put $x^2 = 0 \Longrightarrow n\, 0 = x \Longrightarrow x = 0 \therefore L = \{0\}$.

Let $a \in I$. Then Na is an N – subgroup. Since N is regular $a = axa$ for some $x \in N$.

Let $n\, a \in Na$ for any $n \in N. \Longrightarrow na = n(axa) = naxa \in NaNa$

And if $m \in NaNa$ then for $u, v \in N$

$m = uava = (uav)a \in Ia\ (NIN \subseteq I) \Longrightarrow m \in Na$

Consequently, $NaNa = Na$

Let $na = uava$

Then $na - uava = 0 \Longrightarrow (n - uav)a = 0$………………(2)

If a is not a zero divisor, then $n - uav = 0$.

i.e.)$n = uav \in NIN \subseteq I$

$\Longrightarrow N = I$, which is a contradiction to I is a proper ideal of N.

Hence a is a right zero divisor.

Now equation (2) $\Longrightarrow a(n - uav) = 0$. [by Proposition 5.9].

This leads to the result that a is a left divisor too. Thus the result follows.

Corollary 5.12

If N is a σ_1 near – ring with no non- zero divisors then N contains no proper ideals of N.

Lemma 5.13

If N is σ_1 – near ring with mate function f then for $a, b \in N, ab = b^2$ and $ba = a^2$ imply $a = b$.

Proof:

Let N be a $\sigma_1$near-ring with a mate function f.

Then we have $L = \{0\}$……………………(1) [By Proposition 5.8]

Let $u_1 = a - b$ $u_2 = au_1$ and $u_3 = bu_1$

We can easily obtain the following equations.

$u_1 a = (a-b)a = a^2 - ba = 0$……………… (2)

$u_1 b = (a-b)b = ab - b^2 = 0$………………. (3)

Also. $u_1{}^2 = (a-b)u_1 = au_1 - bu_1 = u_2 - u_3$…………… (4)

As N is zero symmetric.

$u_2{}^2 = (au_1)(au_1) = a(u_1 a)u_1 = a.0u_1 = 0$ [By equation (2)] ………. (5)

Also, $u_3{}^2 = (bu_1)(bu_1) = b(u_1 b)u_1 = b(0)u_1 = 0$ [By equation (3)] ………. (6)

Equations (5) & (6) imply $u_2 = 0$ and $u_3 = 0$ respectively. [By equation (1)]

Making use of these in equation (4), we get $u_1{}^2 = 0$.

It follows that $u_1 = 0$, [By equation (1)] i.e.) $a - b = 0$. Thus $a = b$

References:

[1] J.R.Clay, The near-rings on groups of low order, Math Z. 104 (1968), 364-371.

[2] Gunter Pilz, Near - Ring,North Holland, Amsterdam,1983.

[3] N.H.McCoy, The theory of Rings, Macmillan & Co,1970.

[4] Sugantha G and Balakrishnan R , β1 Near-Rings, International Journal of Algebra vol.4 2010, no. 2, 71-79

[5] S.Suryanarayanan and N.Ganesan, Stable and Pseudo Stable Near-Rings, Indian J. Pure and Appl.Math 19 (12) December, 1988, 1206-1216.

[6] S.Uma, Pseudo Commutative Near-Rings, Scientia Magna, An international Journal. Vol.6 (2010) No 2, 75-85.

[7] Dheena P. On strongly regular near–rings, Journal of the Indian Math.Soc.,49 (1985),201-206.

[8] Dheena P., A generalization of strongly regular near–rings, Indian J. Pure Appl. Math., 20 (1): 1989,58-63.

[9] Von Neumann J., On regular rings, Proc. Nat. Acad. USA, 22 (1936),707-713.

[10] S.Uma, R.Balakrishnan, T.Tamilzchelvan α_1 Near-Rings ,International Journal of Algebra vol.4 2010, No.2, 71-79.

[11] Volety V.S. Ramachadran, Commutativity of semi Near rings – Journal of Science and Arts, 2011, No.4 (17) 367 - 368.