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Premium Linear-Exponential Mixture of Poisson Distribution

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Abstract:
The proposed distribution is a discrete probability distribution with only one parameter and it has been derived by compounding the Poisson distribution with the Premium Linear-exponential distribution (PLED) of Sah (2022). So, it is named as Premium Linear-Exponential Mixture of Poisson Distribution (PLEMPD). The required statistical characteristics have been derived and explained to give the theoretical concept of this distribution. The main purpose of proposing this distribution is to provide a better goodness of fit for over-dispersed count data than the previously derived one parameter continuous mixtures of Poisson distribution of same nature by others. Taking some secondary over-dispersed count data and using the theoretical concept of this distribution, it has been observed that this distribution is more suitable for statistical modeling of over-dispersed count data than Poisson-Lindley distribution (PLD) of Sankaran (1970) and Poisson-Mishra distribution (PMD) of Sah (2017).

Keywords: Premium Linear-Exponential distribution, Probability distribution, Compounding, Moments, Estimation.

1. Introduction

This distribution is extracted by mixing the Poisson distribution with PLED. It is based on a single parameter and discrete in nature. The purpose of proposing this distribution is to give a better alternative to the one-parameter continuous mixture of Poisson distribution which have been published earlier such as PLD and PMD. It is natural that different characteristics of Poisson distribution as well as PLED will play a big role in the construction of different characteristics of this distribution. The probability density function(pdf) of PLED (Sah, 2022) was defined as

$$f_1(y;\alpha)=\frac{\alpha^2}{(1+\pi\alpha^2)}(\pi\alpha+y)e^{-\alpha y}\ ;y>0,\alpha>0 \qquad (1)$$

For modeling the count data, a distribution, PLD (Sankaran,1970), was introduced by Sankaran whose probability mass function (pmf) was given as

$$P_2(Y;\alpha)=\frac{\alpha^2(y+\alpha+2)}{(\alpha+1)^{y+3}}\quad ;y=0,1,2,...\quad ;\alpha>0 \qquad (2)$$

It has been obtained mixing Poisson distribution with Lindley distribution (Lindley, 1958) given by its pdf

$$f_3(y;\alpha)=\frac{\alpha^2}{(1+\alpha)}(1+y)e^{-\alpha y}\ ;y>0,\alpha>0 \qquad (3)$$

The expression (1) and (3) are continuous in nature and based on a single parameter. Hence, it is the appropriate reason to compare the proposed distribution with PLD. The pmf of PMD, (Sah, 2017), was given by

$$P_4(Y;\alpha)=\frac{\alpha^3[(1+\alpha)(y+\alpha+2)+(1+y)(2+y)]}{(\alpha^2+\alpha+1)(1+\alpha)^{y+3}} \quad ;y=0,1,2,... \quad ;\alpha>0 \tag{4}$$

Since PMD is also based on one parameter like PLEMPD, it seems useful to compare the results obtained from both distributions. When mixing Poisson distribution with Mishra distribution (Sah, 2015), it can be observed that the parameter of Poisson distribution follows Mishra distribution.

To give a systematic shape to work of this paper, it has been presented sequentially under different heading and sub-headings. The introduction part of this paper is placed under the first section. In the second section, the materials and methods required for this paper are placed. In the third section, the important part of this paper, which we have given the heading as results, is placed. For simplicity and clarity, the results extracted from this paper are placed under the following sub-headings.

- Premium Linear-Exponential Mixture of Poisson distribution (PLEMPD) and its related characteristics
- Statistical moments and related descriptive measures of PLEMPD
- Estimation of parameters of PLEMPD, and
- Applications and Goodness of fit of PLEMPD.

The conclusion of this paper is placed in the last section.

The information about the topics needed to explain the theoretical and application of this paper are mentioned in the sentences below. The probability mass function, probability generating function and moment generating function have been derived. To study the variation, shape and size of this distribution, the moments about origin as well as the mean have been well presented and described. The method of moments and the method of maximum likelihood have been used to estimate the parameter of this distribution. It may be observed from application section that this distribution would be a better alternative to the PLD and PMD for statistical modelling of over-dispersed count data which may be related to Biology, error distribution, ecology, thunderstorms and accident proneness.

2. Material and Methods

The materials required for this distribution are new theoretical concepts based on continuous mixtures of Poisson distribution and some over-dispersed count secondary data which have been used to check the validity of the theoretical work. We have previously established a new theory in its method and its validity has been confirmed by taking some data like the above mentioned to verify it.

3. Results

This section is an important part of the paper, so for convenience it has been divided into the following sub-headings.

- Premium Linear-Exponential Mixture of Poisson Distribution (PLEMPD) and related characteristics

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- Statistical moments and related measures of PLEMPD
- Estimation of parameters of PLEMPD, and
- Applications and Goodness of Fit of PLEMPD.

- ***Premium Linear-Exponential Mixture of Poisson Distribution (PLEMPD) and Its Related Characteristics:***

When we mix Poisson distribution with PLED, the parameter of Poisson distribution follows PLED and because of that we get PLEMPD and its probability mass function (pmf) can be obtained as

$$P(Y;\alpha)=\int_0^\infty\left[\left(\frac{e^{-\lambda}\lambda^y}{y!}\right)\left(\frac{\alpha^2}{1+\pi\alpha^2}\right)(\pi\alpha+\lambda)e^{-\alpha\lambda}\right]d\lambda\ \ ;y=0,1,2,\ ...;\lambda>0;\alpha>0$$

$$=\left(\frac{\alpha^2}{(1+\pi\alpha^2)}\frac{1}{y!}\right)\left[\pi\alpha\int_0^\infty\lambda^y e^{-(1+\alpha)\lambda}d\lambda+\int_0^\infty\lambda^{y+1}e^{-(1+\alpha)\lambda}d\lambda\right]$$

$$=\left(\frac{\alpha^2}{(1+\pi\alpha^2)}\right)\left[\frac{1+\pi\alpha(1+\alpha)+y}{(1+\alpha)^{y+2}}\right];y=0,1,2,...;\alpha>0 \qquad (5)$$

Graphical presentations of pmf of PLEMPD at varying values of parameter (α) are given as

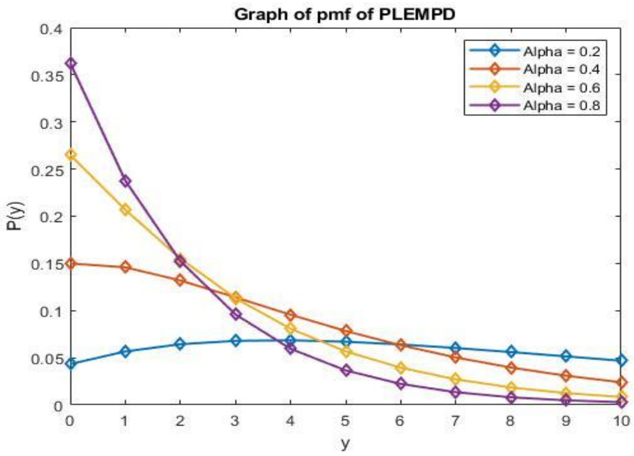

Fig.1 Graph of pmf of PLEMPD at $\alpha=0.2,0.4,0.6,0.8$

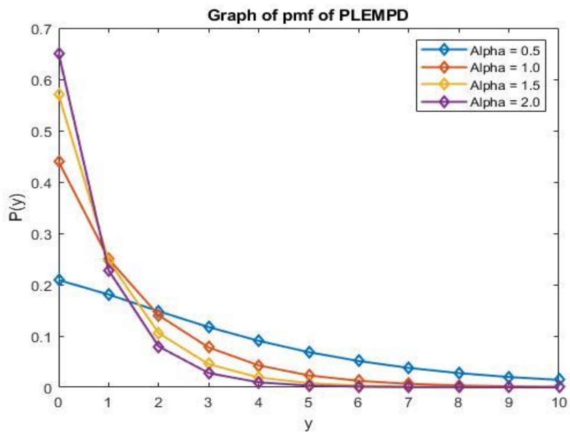

Fig.2 Graph of pmf of PLEMPD at $\alpha=0.5,1.0,1.5,2.0$

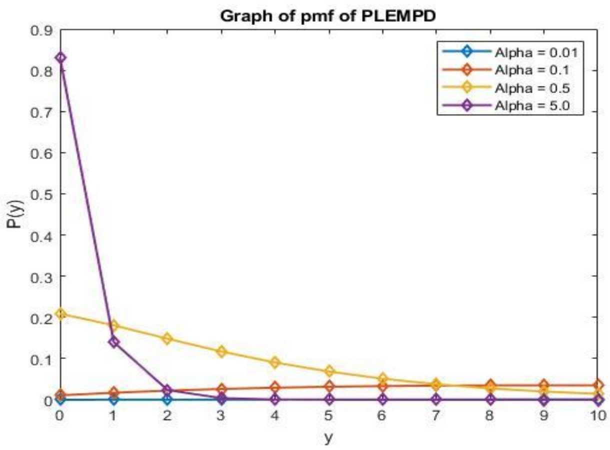

Fig.3 Graph of pmf of PLEMPD at $\alpha = 0.01, 0.1, 0.5, 5.0$

From the above figures, it has been observed that as the value of α increases, value of $P(y)$ also increases at $y = 0$. It can also be observed that the $P(y)$ curve touches X-axis first having greater value of α .

Probability Generating Function (p.g.f.) of PLEMPD: It can be obtained as follows.

$$P_Y^{(t)} = \frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^\infty e^{-\lambda(1-t)}(\pi\alpha+\lambda)e^{-\alpha\lambda}d\lambda = \frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^\infty (\pi\alpha+\lambda)e^{-\lambda(1+\alpha-t)}d\lambda$$

$$= \frac{\alpha^2}{(1+\pi\alpha^2)}\left[\pi\alpha\int_0^\infty e^{-\lambda(1+\alpha-t)}d\lambda + \int_0^\infty \lambda e^{-\lambda(1+\alpha-t)}d\lambda\right] = \left(\frac{\alpha^2}{(1+\pi\alpha^2}\right)\left[\frac{1+\pi\alpha(1+\alpha-t)}{(1+\alpha-t)^2}\right]; \alpha > 0 \quad (6)$$

Moment Generating Function (M.G.F.) of PLEMPD: The M.G.F. ($M_Y^{(t)}$) of PLEMPD has been obtained as

$$M_Y^{(t)} = \frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^\infty (\pi\alpha+\lambda)e^{-\lambda(1+\alpha-e^t)}d\lambda = \frac{\alpha^2}{(1+\pi\alpha^2)}\left[\frac{\pi\alpha\Gamma 1}{(1+\alpha-e^t)^1} + \frac{\Gamma 2}{(1+\alpha-e^t)^2}\right]$$

$$= \left(\frac{\alpha^2}{(1+\pi\alpha^2}\right)\left[\frac{1+\pi\alpha(1+\alpha-e^t)}{(1+\alpha-e^t)^2}\right] \quad (7)$$

The expressions (5), (6), (7) are probability mass function, probability generating function and moment generating function of PLEMPD respectively. The first four moments about origin can also be obtained by using the following expression.

$$\mu_r' = \left[\frac{\partial^r[M_Y^{(t)}]}{\partial t^r}\right]_{t=0} \quad (8)$$

- ***Statistical Moments and Related Measures of PLEMPD:***

The moments about origin as well as the mean of PLEMPD have been explained and derived under this topic which are very useful to study about nature of this proposed distribution according to variation, shape and size. The r[st] moment about origin of PLEMPD can be obtained as

$$\mu_r' = E\left[E(Y^r / \lambda\right] = \frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^{\infty}\left(\sum_{y=0}^{\infty}\frac{y^r e^{-\lambda}\lambda^y}{y!}\right)(\pi\alpha+\lambda)e^{-\alpha\lambda}d\lambda \qquad (9)$$

Putting $r=1,2,3,4$ in the equation (9), we can obtain the 1st four moments about origin of PLEMPD as follows.

The 1st moment about origin of PLEPMD can be obtained as

$$\mu_1' = \frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^{\infty}\left(\sum_{y=0}^{\infty}\frac{y^1 e^{-\lambda}\lambda^y}{y!}\right)(\pi\alpha+\lambda)e^{-\alpha\lambda}d\lambda = \frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^{\infty}(\lambda)(\pi\alpha+\lambda)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^2}{(1+\pi\alpha^2)}\left[\int_0^{\infty}\pi\alpha\lambda e^{-\alpha\lambda}d\lambda + \int_0^{\infty}\lambda^2 e^{-\alpha\lambda}d\lambda\right] = \frac{\alpha^2}{(1+\pi\alpha^2)}\left[\frac{\pi\alpha}{\alpha^2}+\frac{2}{\alpha^3}\right] = \frac{(2+\pi\alpha^2)}{\alpha(1+\pi\alpha^2)} \qquad (10)$$

The expression (10) is the mean of PLEMPD. Graphical representation of the mean for varying values of alpha is given below.

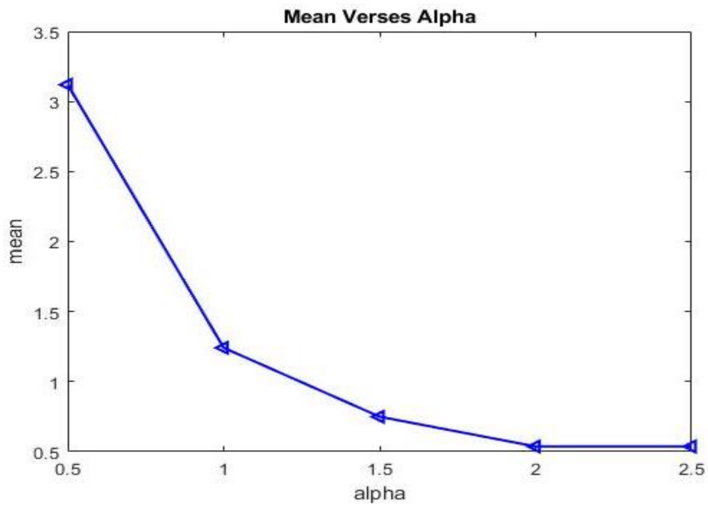

Fig.4 The mean of PLEMPD for $\alpha = 1.0, 1.5, 2.0, 2.5$

It can be observed that the mean of PLEMPD decreases as the value of the parameter of this distribution increases.

Putting $r=2$ in the expression (9), we get the second moment about origin of PLEMPD as follows

$$\mu_2' = \frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^{\infty}\left(\sum_{y=0}^{\infty}\frac{y^2 e^{-\lambda}\lambda^y}{y!}\right)(\pi\alpha+\lambda)e^{-\alpha\lambda}d\lambda = \frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^{\infty}\left(\lambda+\lambda^2\right)(\pi\alpha+\lambda)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^2}{(1+\pi\alpha^2)}\left[\frac{\pi\alpha}{\alpha^2}+\frac{2(1+\pi\alpha)}{\alpha^3}+\frac{6}{\alpha^4}\right] = \frac{[\pi\alpha^2(\alpha+2)+2(\alpha+3)]}{\alpha^2(1+\pi\alpha^2)} \qquad (11)$$

Putting $r=3$ in the expression (9), the third moment about origin can be obtained as

$$\mu_3' = \frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^{\infty}\left(\sum_{y=0}^{\infty}\frac{y^3 e^{-\lambda}\lambda^y}{y!}\right)(\pi\alpha+\lambda)e^{-\alpha\lambda}d\lambda = \frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^{\infty}\left(\lambda^3+3\lambda^2+\lambda\right)(\pi\alpha+\lambda)e^{-\alpha\lambda}d\lambda$$

$$=\frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^\infty\left(\pi\alpha\lambda+(1+3\pi\alpha)\lambda^2+(3+\pi\alpha)3\lambda^3+\lambda^4\right)e^{-\alpha\lambda}d\lambda$$

$$=\frac{1}{(1+\pi\alpha^2)}\left[\pi\alpha+\frac{2(1+3\pi\alpha)}{\alpha}+\frac{6(\pi\alpha+3)}{\alpha^2}+\frac{24}{\alpha^3}\right]=\frac{1}{(1+\pi\alpha^2)}\left[\left(\pi\alpha+\frac{2}{\alpha}\right)+\left(\frac{6\pi\alpha}{\alpha}+\frac{18}{\alpha^2}\right)+\left(\frac{6\pi\alpha}{\alpha^2}+\frac{24}{\alpha^3}\right)\right]$$

$$=\frac{(\pi\alpha^2+2)}{\alpha(1+\pi\alpha^2)}+\frac{6(\pi\alpha^2+3)}{\alpha^2(1+\pi\alpha^2)}+\frac{6(\pi\alpha^2+4)}{\alpha^3(1+\pi\alpha^2)} \qquad (12)$$

Putting $r=4$ in the expression (9), the fourth moment about origin can be obtained as

$$\mu_4'=\frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^\infty\left(\sum_{y=0}^\infty\frac{y^4e^{-\lambda}\lambda^y}{y!}\right)(\pi\alpha+\lambda)e^{-\alpha\lambda}d\lambda=\frac{\alpha^2}{(1+\pi\alpha^2)}\int_0^\infty\left(\lambda^4+6\lambda^3+7\lambda^2+\lambda\right)(\pi\alpha+\lambda)e^{-\alpha\lambda}d\lambda$$

$$=\frac{1}{(1+\pi\alpha^2)}\left[\pi\alpha+\frac{(2+14\pi\alpha)}{\alpha}+\frac{6(7+6\pi\alpha)}{\alpha^2}+\frac{24(\pi\alpha+6)}{\alpha^3}+\frac{120}{\alpha^4}\right]$$

$$=\frac{(\pi\alpha^2+2)}{\alpha(1+\pi\alpha^2)}+\frac{14(\pi\alpha^2+3)}{\alpha^2(1+\pi\alpha^2)}+\frac{36(\pi\alpha^2+4)}{\alpha^3(1+\pi\alpha^2)}+\frac{24(\pi\alpha^2+5)}{\alpha^4(1+\pi\alpha^2)} \qquad (13)$$

Central Moments of PLEMPD:

The 1st four central moments of PLEMPD have been obtained as

$$\mu_1=0$$

$$\mu_2=\mu_2'-\left(\mu_1'\right)^2=\frac{[\pi\alpha^2(\alpha+2)+2(\alpha+3)]}{\alpha^2(\pi\alpha^2+1)}-\left(\frac{(\pi\alpha^2+2)}{\alpha(\pi\alpha^2+1)}\right)^2$$

$$=\frac{[\pi^2\alpha^5+\pi^2\alpha^4+3\pi\alpha^3+4\pi\alpha^2+2\alpha+2]}{[\alpha(\pi\alpha^2+1)]^2} \qquad (14)$$

Theorem (1): *PLEMPD is an over-dispersed distribution.*

Proof:

A distribution is said to be over-dispersed if

Variance > Mean

$$\frac{[\pi^2\alpha^5+\pi^2\alpha^4+3\pi\alpha^3+4\pi\alpha^2+2\alpha+2]}{[\alpha(\pi\alpha^2+1)]^2}>\frac{(\pi\alpha^2+2)}{\alpha(\pi\alpha^2+1)}$$

$$(\pi^2\alpha^5+\pi^2\alpha^4+3\pi\alpha^3+4\pi\alpha^2+2\alpha+2)-(\pi^2\alpha^5+3\pi\alpha^3+2\alpha)>0$$

$$(\pi^2\alpha^4+4\pi\alpha^2+2)>0$$

which is true because $\alpha>0$ and $\pi=22/7$. Hence, PLEMPD is an over-dispersed distribution. Graphical representation of variance of PLEMPD with varying values of α is given below.

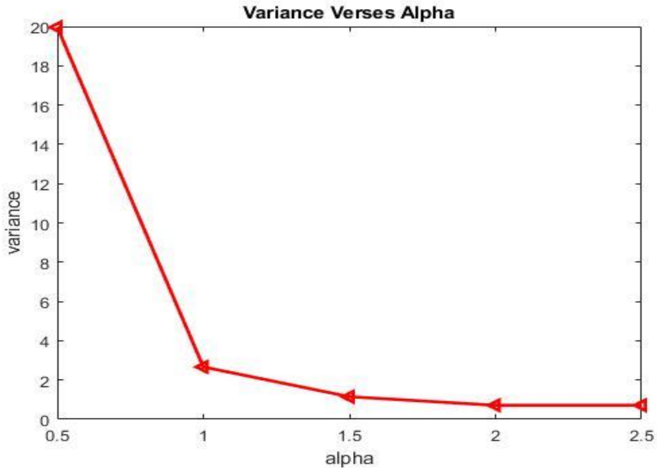

Fig.5 The Variance of PLEMPD for $\alpha = 1.0, 1.5, 2.0, 2.5$

From the figure (5), It can be observed that the variance of PLEMPD decreases as the value of the parameter of the PLEMPD increases.

The third central moment can be obtained as follows

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= \left[\frac{(\pi\alpha^2+2)}{\alpha(1+\pi\alpha^2)} + \frac{6(\pi\alpha^2+3)}{\alpha^2(1+\pi\alpha^2)} + \frac{6(\pi\alpha^2+4)}{\alpha^3(1+\pi\alpha^2)}\right] - 3\left[\frac{(\pi\alpha^2+2)}{\alpha(\pi\alpha^2+1)} + \frac{(\pi\alpha^2+6)}{\alpha^2(\pi\alpha^2+1)}\right]\left[\frac{(\pi\alpha^2+2)}{\alpha(\pi\alpha^2+1)}\right] + 2\left[\frac{(\pi\alpha^2+2)}{\alpha(\pi\alpha^2+1)}\right]^3$$

$$= \frac{[(\pi^3\alpha^6+2)(\alpha^2+3\alpha+2) + (\pi^2\alpha^4)(4\alpha^2+15\alpha+12) + (\pi\alpha^2)(5\alpha^2+18\alpha+12)]}{[\alpha(1+\pi\alpha^2)]^3} \quad (15)$$

The fourth central moment of PLEMPD can be obtained as

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 + 3(\mu_1')^4$$

$$= \left[\frac{(\pi\alpha^2+2)}{\alpha(1+\pi\alpha^2)} + \frac{14(\pi\alpha^2+3)}{\alpha^2(1+\pi\alpha^2)} + \frac{36(\pi\alpha^2+4)}{\alpha^3(1+\pi\alpha^2)} + \frac{24(\pi\alpha^2+5)}{\alpha^4(1+\pi\alpha^2)}\right]$$
$$-4\left[\frac{(\pi\alpha^2+2)}{\alpha(1+\pi\alpha^2)} + \frac{6(\pi\alpha^2+3)}{\alpha^2(1+\pi\alpha^2)} + \frac{6(\pi\alpha^2+4)}{\alpha^3(1+\pi\alpha^2)}\right]\left[\frac{(\pi\alpha^2+2)}{\alpha(\pi\alpha^2+1)}\right]$$
$$+6\left[\frac{(\pi\alpha^2+2)}{\alpha(\pi\alpha^2+1)} + \frac{(\pi\alpha^2+6)}{\alpha^2(\pi\alpha^2+1)}\right]\left[\frac{(\pi\alpha^2+2)}{\alpha(\pi\alpha^2+1)}\right]^2 - 3\left[\frac{(\pi\alpha^2+2)}{\alpha(\pi\alpha^2+1)}\right]^4$$

$$= \frac{\begin{array}{l}[(24+48\alpha+26\alpha^2+2\alpha^3) + (\pi\alpha^2)(96+180\alpha+92\alpha^2+7\alpha^3) \\ +(\pi\alpha^2)^2(132+240\alpha+116\alpha^2+9\alpha^3) \\ +(\pi\alpha^2)^3(72+126\alpha+60\alpha^2+5\alpha^3) + (\pi\alpha^2)^4(9+18\alpha+10\alpha^2+\alpha^3)]\end{array}}{[\alpha(1+\pi\alpha^2)]^4} \quad (16)$$

The nature of PMPLED according to shape and size can be analysed by obtaining the co-efficient of skewness and kurtosis based on moments as follows.

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{3/2}}$$

$$=\frac{[(\pi^3\alpha^6+2)(\alpha^2+3\alpha+2)+(\pi^2\alpha^4)(4\alpha^2+15\alpha+12)+(\pi\alpha^2)(5\alpha^2+18\alpha+12)]}{[\pi^2\alpha^5+\pi^2\alpha^4+3\pi\alpha^3+4\pi\alpha^2+2\alpha+2]^{3/2}} \tag{17}$$

The expression (17) co-efficient of skewness of PLEMPD based on moments. From this expression, it has been observed that $(\sqrt{2})<\gamma_1<\infty$.Graphical representation of γ_1 with different values of the parameter α is given below.

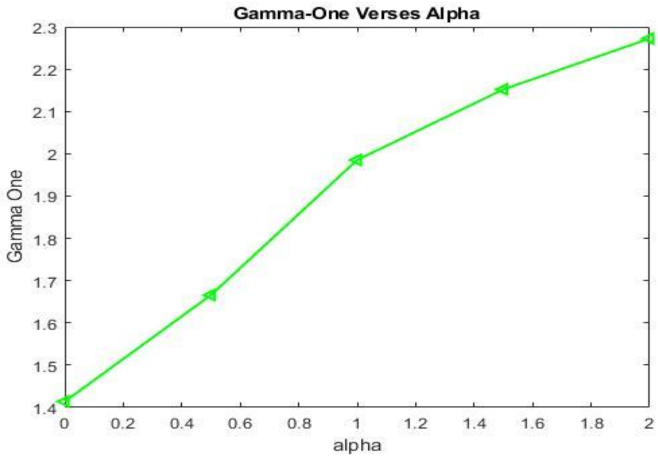

Fig.6: Skewness of PLEMPD for $\alpha=1.0,1.5,2.0,2.5$

$$\beta_2=\frac{\mu_4}{(\mu_2)^2}$$

$$=\frac{\begin{array}{l}[(24+48\alpha+26\alpha^2+2\alpha^3)+(\pi\alpha^2)(96+180\alpha+92\alpha^2+7\alpha^3)\\+(\pi\alpha^2)^2(132+240\alpha+116\alpha^2+9\alpha^3)\\+(\pi\alpha^2)^3(72+126\alpha+60\alpha^2+5\alpha^3)+(\pi\alpha^2)^4(9+18\alpha+10\alpha^2+\alpha^3)]\end{array}}{[\pi^2\alpha^5+\pi^2\alpha^4+3\pi\alpha^3+4\pi\alpha^2+2\alpha+2]^2} \tag{18}$$

The expression (18) is the co-efficient of kurtosis based on moments and it can be noted that $6<\beta_2<\infty$. Hence, this distribution is leptokurtic by size. The graphical representation of β_2 for different values of α is given below.

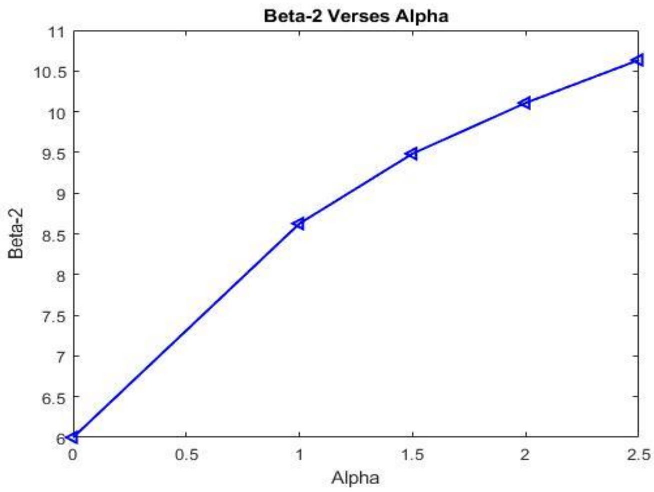

Fig.7: Kurtosis of PLEMPD for $\alpha=1.0,1.5,2.0,2.5$

From figure (6), it can also be noted that $(\sqrt{2}) < \gamma_1 < \infty$. It can also be observed that value of γ_1 increases as the value of α increases. From figure (7), it can also be noted that $6 < \beta_2 < \infty$. It can also be observed that value of β_2 increases as the value of α increases, while the values of mean and variance are inversely proportional to the value of α.

Remarks:

PLEMPD is always over-dispersed.
It is always positively skewed.

It is always leptokurtic by size.

- **Estimation of Parameters of PLEMPD:**

Under this sub-section, the value of the parameter of this distribution has been estimated using the following two methods (a) Method of moments and (b) Maximum likelihood method.

(a) ***Method of moments*****:**

Since there is only one parameter with this distribution, the estimator of the parameter has been obtained using the first moment about the origin of PLEMPD.

$$\mu_1' = \frac{(\pi\alpha^2 + 2)}{\alpha(\pi\alpha^2 + 1)}$$

Or, $\mu_1'(\alpha + \pi\alpha^3) - (\pi\alpha^2 + 2) = 0$ (19)

The expression (19) is a polynomial equation of third degree in α.

(b) ***Method of maximum likelihood:***

A sample of size n is taken from the PLEPMD population to get an estimator of the parameter (α) from this method as follows

$y_i \; : y_1 \;\; y_2 \;\; y_3 \; \ldots \; y_k$
$f_i \; : f_1 \;\; f_2 \;\; f_3 \; \ldots \; f_k$

The maximum likelihood equation has been obtained as

$$L = \left(\frac{\alpha^2}{1+\pi\alpha^2}\right)^n (1+\alpha)^{-\sum_{i=1}^{k}(y_i+2)f_i} \prod_{i=1}^{k}(1+y_i+\pi\alpha(1+\alpha))^{f_i} \quad (20)$$

Or, $log(L) = 2n\log a - n\log(1+\pi\alpha^2) - \left(\sum_{i=1}^{k}(y_i+2)f_i\right)\left(\log(1+\alpha)\right) + \sum_{i=1}^{k} f_i \log\left((1+y_i+\pi\alpha(1+\alpha))\right)$

Or, $\frac{\partial(log(L))}{\partial\alpha} = \frac{2n}{\alpha} - \frac{2n\pi\alpha}{(1+\pi\alpha^2)} - \frac{\left(\sum_{i=1}^{k}(y_i+2)f_i\right)}{(1+\alpha)} + \sum_{i=1}^{k}\frac{(\pi+2\pi\alpha)f_i}{(1+y_i+\pi\alpha(1+\alpha))}$ (21)

An estimate of α can also be obtained by solving the expression (21).

- **Applications of PLEMPD:**

Applications and goodness of fit have been conducted to this distribution with the support of secondary data used by other researchers. To test goodness of fit the following data are used.

Example (1): Distribution of mistakes in copying groups of random digits, Kemp and Kemp (1965).

Number of errors per group	0	1	2	3	4^{+}
Observed Frequency	35	11	8	4	2

Example (2): Distribution of Pyrausta nablilalis in 1937, Beall (1940).

Number of insects per leaf	0	1	2	3	4	5
Observed Frequency	33	12	6	3	1	1

Example (3): Distribution of mammalian cytogenic dosimetry lesions in rabbit lymphoblast included by Streptonigrin [NSC-45383], Exposure-70($\mu g / kg$).

Class / Exposure ($\mu g / kg$)	0	1	2	3	4	5	6
Observed Frequency	200	57	30	7	4	0	2

Example (4): Distribution of number of red mites on apple leaves, reported by Garman (1923).

Number of red mites per leaf	0	1	2	3	4	5	6	7
Observed Frequency	38	17	10	9	3	2	1	

Table I: Observed Verses Expected Frequency of Example (1)

Number of errors per group	Observed frequency	Expected frequency		
		PLD	PMD	PLEMPD
0	35	33.0	32.9	33.6
1	11	15.3	15.3	14.9
2	8	6.8	6.8	6.5
3	4	2.9	3.6	2.9
4	2	2.0	1.4	2.1
Total	60	60.0	60.0	60.0
μ'_1	0.78333333			
μ'_2	1.8500			
$\hat{\alpha}$		1.7434	2.1654758	1.445385607
d.f.		2	2	2
χ^2		1.78	1.72	1.62
P-value		0.61	0.625	0.64

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Table II: Observed Verses Expected Frequency of Example (2)

Number of insects per leaf	Observed frequency	Expected frequency		
		PLD	PMD	PLEMPD
0	33	31.5	31.4	31.9
1	12	14.2	14.2	13.8
2	6	6.1	6.2	5.9
3	3	2.5	2.6	2.5
4	1	1.0	1.0	1.1
5	1	0.7	0.6	0.8
Total	56.0	56.0	56.0	56.0
μ'_1	0.75			
μ'_2	1.8571			
$\hat{\alpha}$		1.8081	2.234	1.498812719
d.f.		2	2	2
χ^2		0.53	0.47	0.32
P-value		0.83	0.85	0.88

Table III: Observed Verses Expected Frequency of Example (3)

Class per Exposure ($\mu g / kg$)	Observed frequency	Expected frequency		
		PLD	PMD	PLEMPD
0	200	191.8	191.6	193.0
1	57	70.3	70.2	68.9
2	30	24.9	25.1	24.6
3	7	8.6	8.7	8.7
4	4	2.9	2.9	3.1
5	0	1.0	1.0	1.1
6	2	0.5	0.5	0.6
Total	300	300	300	300.0
μ'_1	0.553333333			
μ'_2	1.253333333			
$\hat{\alpha}$		2.353339	2.784976722	1.94719828
d.f.		3	3	3
χ^2		3.91	3.81	3.51
P-value		0.43	0.45	0.485

Table IV: Observed Verses Expected Frequency of Example (4)

Number of red mites per leaf	Observed frequency	Expected frequency	
		PLD	PLEMPD
0	38	35.8	36.8
1	17	20.7	20.1
2	10	11.4	10.8
3	9	6.0	5.8
4	3	3.1	3.1
5	2	1.6	1.7
6	1	0.8	0.9
7^+	0	0.6	0.8
Total	80	80.0	80.0
μ_1'	1.15		
μ_2'	3.4		
$\hat{\alpha}$		1.255891	1.061191
d.f.		3	3
χ^2		2.47	1.16
P-value		0.48	0.82

The first example is a secondary data related to error per page. This is an over-dispersed data count data and it has been collected by Kemp and Kemp (Kemp and Kemp, 1965). The second example is related to the number of insects per leaf and it is due to Beall (Beall, 1940). The third example a secondary over-dispersed count data and it is due to Catcheside at al (Catcheside at al, 1946) and the fourth example is due to Garman (Garman ,1923). In each table, along with the observed frequency, the expected frequencies due to PLD and PMD have also been kept, and the expected frequency obtained using PLEMPD has also been also kept, so it helps to compare the distributions. The first two examples have been used by Sah (Sah,2013) in the doctoral thesis for the study of generalisation of Poisson-Lindley distribution.

4. Discussion

In the table-V, degrees of freedom, value of Chi-Square and P-value due to PLD, PMD and PLEMPD obtained in the above all tables have been included, which provides an opportunity and to mark a concluding remark for the proposed distribution.

Table- V

(PLD and PMD) Verses PLEMPD

Table	PLD			PMD			PLEMPD		
	$d.f.$	$\chi^2_{d.f.}$	$P-Value$	$d.f.$	$\chi^2_{d.f.}$	$P-Value$	$d.f.$	$\chi^2_{d.f.}$	$P-Value$
I	2	1.78	0.61	2	1.72	0.625	2	1.62	0.64
II	2	0.53	0.83	2	0.47	0.85	2	0.32	0.88
III	3	3.91	0.43	3	3.81	0.45	3	3.51	0.485
IV	3	2.47	0.48	3	-	-	3	1.23	0.88

Hence, we have identified the following concluding points for PLEMPD.

- In most of the over-dispersed data-sets, it is better alternative of PLD (Sankaran, 1970) and PMD (Sah,2017) for statistical modeling because P-value obtained by using PLEMPD is greater than the P-value obtained by using PLD as well as PMD.
- PLEMPD is over-dispersed, positively skewed by shape because $(\sqrt{2}) < \gamma_1 < \infty$ and Leptokurtic by size because $6 < \beta_2 < \infty$.

Conflict of Interest

The authors of this paper have no any conflict of interest.

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