Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

M/G/1 Queue with Two Stage Heterogeneous Service and Compulsory Deterministic Server Vacations

S. Vanitha[1]
[1]Assistant Professor, Department of Mathematics
Sri Sivasubramaniya Nadar College of Engineering, Tamilnadu, India
vanithas@ssn.edu.in

Article History:

Received: 16-11-2024

Revised: 15-12-2024

Accepted: 02-01-2025

Abstract:
We analyze a single server vacation queue with Poisson arrivals, two stages of heterogeneous service with different (arbitrary) service time distributions subject to deterministic vacation of constant duration $d(>0)$. After the first stage service, the server must provide the second stage service. We assume that after completion of second stage service, the server may decide to take a vacation of fixed length $d(>0)$. The time dependent probability generating functions using supplementary variable technique have been obtained in terms of their Laplace transforms and the corresponding steady state results are also obtained explicitly.

Keywords: Poisson arrivals, probability generating function, transient state, steady state, supplementary variable technique.

1. Introduction

In recent years queues with server vacations have emerged as an important area of queueing theory and have been studied extensively due to their various applications in production, inventory systems, communication systems, computer networks etc. Some of the few important monographs on queueing theory for the performance prediction of computer networks and communication systems include Kleinrock [11], Cohen ([2],[3]), Doshi ([4], [5], [6]), Lavenberg [13], Takagi [17] and several others.

The single server queue with two phases of service subject to vacation has paid attention recently by several researchers. Presently such type of models has been the subject matter of current research mainly due to its applications in computer and communication systems. One of the most important results that deals with such models is" Stochastic Decomposition Result" which was first established by Fuhrmann and Cooper [8] for M/G/1 type queues with generalized vacations. The queueing system with two phase service has been first studied by Krishna and Lee [12]. Doshi [7] has expanded two phase queueing system of Krishna and Lee into the case of batch and individual service times with general distributions. By using an embedded Markov chain approach, the distribution for the system size and sojourn time of a customer is derived.

Recently there have been several contributions considering queueing systems of M/G/1 type in which the server may provide a second phase of service. The steady state behaviour of a M/G/1 queue with two types of general heterogeneous service and optional repeated service subject to server's breakdowns and delayed repair is studied by Choudhury and Chandi Ram Kalita [1]. The joint distribution of state of the server and queue size by considering both elapsed and remaining time are derived explicitly for that model. A M/G/1 retrial queueing system with two phases of service in which the second phase is optional and the server operating under Bernoulli vacation schedule is investigated by Pavai Madheswari et. al [14]. In that model, the customer is allowed to balk upon arrival if he finds the server unavailable to serve his request immediately. The joint generating functions of orbit size and server status are derived using supplementary variable technique. A non-Markovian queueing model with setup time and Balking is studied by S. Shyamala and R. Vijayaraj [15]. In this model, the server provides two stages of service in which the service time for each customer follows general distribution. Upon completion of a service the server may go for a vacation with probability p or stay back in the system to serve a next customer with probability $1-p$ if any.

A new model of a queueing system which provides two stage first essential service followed by one of the two types of additional optional service is analysed by K. C Madan [9]. Moreover, the server has the option to take a vacation of constant length. A batch arrival single service queue with two stages of service (second stage is optional) and working breakdown is investigated by B. Somasundaram et. al [16]. When the system is in operation, it may breakdown at any time. During breakdown period, instead of terminating the service totally, it continues at a slower rate. K. C. Madan [10] has studied a M/G/1 single server vacation queue with two types of service and two types of vacation. Using supplementary variable technique, a steady state solution in terms of the generating function of the queue length as well as the steady state probabilities for all different states of the system are derived.

Many researchers have developed several models involving single vacation policy but only few models have been studied subject to compulsory vacation. Examples of this kind of situation may well be found in some transport systems in which a ferry driver or a locomotive driver may like to go to on a vacation after every round trip. Such a trip essentially involves a two stage service, that is trip to a particular destination and back to the starting point. There are probably many other situations such as some manufacturing processes which involves two stages of service and machines that need to be stopped for overhauling, etc., after each two stage service is complete.

In the present work, we study a single server vacation queue with Poisson arrivals, two stages of heterogeneous service with different (arbitrary) service time distributions subject to deterministic vacation of constant duration $d(>0)$. After the first stage service, the server must provide the second stage service. Further, we assume that after completion of second stage service, the server may decide to take a vacation of fixed length $d(>0)$. This kind of problems in real life can be seen in production and manufacturing industries. The rest of the paper is organized as follows. The mathematical description of our model is in section 2. Definitions representing the model and equations governing the model are there in section 3. The time dependent solutions have been obtained in section 4 using supplementary variable technique and the corresponding steady state results have been derived explicitly in section 5.

2. Assumptions Underlying the Model

The following assumptions describe the mathematical model.

- Customers arrive at the system one by one in according to a Poisson stream with arrival rate $\lambda(>0)$.

- Each customer undergoes two stages of heterogeneous service provided by a single server on a first come first served basis. The service time of two stages follow different general (distributions) with distribution function $B_j(v)$ and the density function $b_j(v), j = 1,2$.
- Let $\mu_i(x)dx$ be the conditional probability of completion of the i^{th} stage of service during the interval $(x + dx]$ given that the elapsed service time is x, so that

$$\mu_i(\mathrm{x}) = \frac{b_i(x)}{1-B_i(x)}, i = 1,2. \qquad (1)$$

and therefore

$$b_i(v) = \mu_i(v)e^{-\int_0^{\infty} \mu_i(x)dx}. \qquad (2)$$

- As soon as the service of the second stage of a customer is complete, then the server goes for compulsory vacation.
- We assume that whenever the server takes a vacation, it is of constant duration $d(> 0)$.
- On returning from vacation the server instantly starts serving the customer at the head of the queue.
- The customers are served according to the first come, first served rule.
- Various stochastic processes involved in the system are independent of each other.

3. Definitions, Notations and Equations Governing the System

We define

- $P_n^{(1)}(x,t)$: Probability that at time t, there are $n \geq 0$ customers in the queue excluding one customer in first stage of service and the elapsed served time of this customer is x. Consequently $P_n^{(1)}(t) = \int_0^{\infty} P_n^{(1)}(x,t)dx$ denotes the probability that at time t, there are n customers in the queue excluding the one customer in first stage of service irrespective of the value of x .
- $P_n^{(2)}(x,t)$: Probability that at time t, there are $n \geq 0$ customers in the queue excluding one customer in second stage of service and the elapsed served time of this customer is x. Consequently $P_n^{(2)}(t) = \int_0^{\infty} P_n^{(2)}(x,t)dx$ denotes the probability that at time t, there are n customers in the queue excluding the one customer in second stage of service irrespective of the value of x .
- $V_n(t)$: Probability that at time t, there are $n \geq 0$ customers in the queue and the server is under vacation.
- $Q(t)$: Probability that at time t, there is no customer in the system and the server is idle but available in the system.
- we assume that k_r is the probability of r arrivals during a vacation period of duration d so that

$$k_r = \frac{e^{-\lambda d}(\lambda d)^r}{r!}, r = 0,1, \ldots \quad . \qquad (3)$$

The system is then governed by the following set of differential – difference equations

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)+\big(\lambda+\mu_1(x)\big)P_n^{(1)}(x,t)=\lambda P_{n-1}^{(1)}(x,t), n=1,2,..., \tag{4}$$

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t)+\frac{\partial}{\partial t}P_0^{(1)}(x,t)+\big(\lambda+\mu_1(x)\big)P_0^{(1)}(x,t)=0, \tag{5}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+\big(\lambda+\mu_2(x)\big)P_n^{(2)}(x,t)+\lambda P_{n-1}^{(1)}(x,t), n=1,2,..., \tag{6}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t)+\frac{\partial}{\partial t}P_0^{(2)}(x,t)+\big(\lambda+\mu_2(x)\big)P_0^{(2)}(x,t)=0, \tag{7}$$

$$\frac{d}{dt}V_0(t)=\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)dx+V_0(t)[-K_0-K_1-\cdots], \tag{8}$$

$$\frac{d}{dt}V_n(t)=\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx+V_n(t)[-K_0-K_1-\cdots], n=1,2,..., \tag{9}$$

$$\frac{d}{dt}Q(t)=-\lambda Q(t)+V_0(t)K_0\,. \tag{10}$$

Equations $(4)-(10)$ are to be solved subject to following boundary conditions

$$P_0^{(1)}(0,t)=\lambda Q(t)+V_0(t)k_1+V_1(t)k_0 \tag{11}$$

$$P_n^{(1)}(0,t)=V_0(t)k_{n+1}+V_1(t)k_n+\cdots+V_n(t)k_1+V_{n+1}(t)k_0, n=1,2,..., \tag{12}$$

$$P_n^{(2)}(0,t)=\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx, n=0,1,... \tag{13}$$

We assume that initially there is no customer in the system and the server is idle so that initial conditions are

$$Q(0)=1, P_n^{(j)}(0)=0, j=1,2, V_n(0)=0, V_0(0)=0, n\geq 0. \tag{14}$$

4. Generating Functions of the Queue Length: The Time Dependent Solution

In this section we define the transient solution for the above set of differential-difference equations. We define probability generating functions

$$\left.\begin{aligned}P^{(1)}(x,z,t)&=\textstyle\sum_{n=0}^{\infty}z^n P_n^{(1)}(x,t),\\ P^{(1)}(z,t)&=\textstyle\sum_{n=0}^{\infty}z^n P_n^{(1)}(t)\\ P^{(2)}(x,z,t)&=\textstyle\sum_{n=0}^{\infty}z^n P_n^{(2)}(x,t),\\ P^{(2)}(z,t)&=\textstyle\sum_{n=0}^{\infty}z^n P_n^{(2)}(t)\\ V(z,t)&=\textstyle\sum_{n=0}^{\infty}z^n V(t).\end{aligned}\right\} \tag{15}$$

which are convergent inside the circle given by $|z|\leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\overline{f(s)}=\int_0^\infty e^{-st}f(t)dt, \Re(s)>0\,. \tag{16}$$

Taking Laplace transforms of equations (4) – (13) and using initial conditions (14), we obtain

$$\frac{\partial}{\partial x}\overline{P_n}^{(1)}(x,s)+\big(s+\lambda+\mu_1(x)\big)\overline{P_n}^{(1)}(x,s)=\lambda\overline{P}_{n-1}^{(1)}(x,s), n=1,2,..., \tag{17}$$

$$\frac{\partial}{\partial x}\overline{P_0}^{(1)}(x,s)+\big(s+\lambda+\mu_1(x)\big)\overline{P_0}^{(1)}(x,s)=0, \tag{18}$$

$$\frac{\partial}{\partial x}\overline{P_n}^{(2)}(x,s)+\big(s+\lambda+\mu_2(x)\big)\overline{P_n}^{(2)}(x,s)=\lambda\overline{P}_{n-1}^{(2)}(x,s), n=1,2,..., \tag{19}$$

$$\frac{\partial}{\partial x}\overline{P_0}^{(2)}(x,s)+\big(s+\lambda+\mu_2(x)\big)\overline{P_0}^{(2)}(x,s)=0, \tag{20}$$

$$s\overline{V}_0(s)=\overline{V}_0(s)[-k_0-k_1-\cdots]+\int_0^\infty\overline{P}_0^{(2)}(x,s)\mu_2(x)dx, \tag{21}$$

$$s\overline{V}_n(s)=\overline{V}_n(s)[-k_0-k_1-\cdots]+\int_0^\infty\overline{P}_n^{(2)}(x,s)\mu_2(x)dx, n=1,2,\ldots \tag{22}$$

$$(s+\lambda)\bar{Q}(s)=1+\overline{V}_0(s)k_0\,, \tag{23}$$

$$\overline{P}_0^{(1)}(0,s)=\lambda\overline{Q}(s)+\overline{V}_0(s)k_1+\overline{V}_1(s)k_0\,, \tag{24}$$

$$\overline{P}_n^{(1)}(0,s)=\overline{V}_0(s)k_{n+1}+\overline{V}_1(s)k_n+\cdots+\overline{V}_n(s)k_1+\overline{V}_{n+1}(s)k_0, n=1,2,\ldots, \tag{25}$$

$$\overline{P}_n^{(2)}(0,s)=\int_0^\infty\overline{P}_n^{(1)}(x,s)\mu_1(x)dx, n=0,1,\ldots \tag{26}$$

Now multiplying equation (17) by $z^n$and summing over n from 1 to ∞, adding to equation (18) and using equation (15), we obtain

$$\frac{\partial}{\partial x}\overline{P}^{(1)}(x,z,s)+\big(s+\lambda-\lambda z+\mu_1(x)\big)\overline{P}^{(1)}(x,z,s)=0, \tag{27}$$

Performing similar operations on equations (19) – (22), we get

$$\frac{\partial}{\partial x}\overline{P}^{(2)}(x,z,s)+\big(s+\lambda-\lambda z+\mu_2(x)\big)\overline{P}^{(2)}(x,z,s)=0, \tag{28}$$

$$(s+1)\overline{V}(z,s)=\int_0^\infty\overline{P}^{(2)}(x,z,s)\mu_2(x)dx. \tag{29}$$

For boundary conditions, we multiply both sides of equation (24) by z, multiply both sides of equation (25) by z^{n+1}, sum over n from 1 to ∞, add two results and use equation (15) to get

$$z\overline{P}^{(1)}(0,z,s)=\lambda z\overline{Q}(s)+\overline{V}(z,s)e^{-\lambda d(1-z)}-\overline{V}_0(s)k_0 \tag{30}$$

Performing similar operations on equation (26) to obtain,

$$\overline{P}^{(2)}(0,z,s)=\int_0^\infty\overline{P}^{(1)}(x,z,s)\mu_1(x)dx. \tag{31}$$

Using equation (23), equation (30) becomes

$$z\overline{P}^{(1)}(0,z,s)=\lambda z\overline{Q}(s)+\overline{V}(z,s)e^{-\lambda d(1-z)}+1-(s+\lambda)\overline{Q}(s) \tag{32}$$

Integrating equation (27) from 0 to x yields

$$\overline{P}^{(1)}(x,z,s)=\overline{P}^{(1)}(0,z,s)e^{-(s+\lambda-\lambda z)x-\int_0^\infty\mu_1(t)dt}. \tag{33}$$

where $\overline{P}^{(1)}(0,z,s)$ is given by equation (32). Again, integrating equation (33) by parts with respect to x yields

$$\overline{P}^{(1)}(z,s)=\overline{P}^{(1)}(0,z,s)\left[\frac{1-\overline{B_1}(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \tag{34}$$

where

$$\overline{B_1}(s+\lambda-\lambda z)=\int_0^\infty e^{-(s+\lambda-\lambda z)x}\,\mathrm{dB}_1(\mathrm{x}) \tag{35}$$

is the Laplace - Steiltjes transform of the first stage of service time $B_I(x)$. Now multiplying both sides of equation of (33) by $\mu_I(x)$ and integrating over x, we obtain,

$$\int_0^\infty \overline{P}^{(1)}(x,z,s)\mu_1(x)dx = \overline{P}^{(1)}(0,z,s)\overline{B_1}(s+\lambda-\lambda z) \tag{36}$$

Similarly, on integrating equation (28) from 0 to x, we get

$$\overline{P}^{(2)}(x,z,s) = \overline{P}^{(2)}(0,z,s)e^{-(s+\lambda-\lambda z)x-\int_0^\infty \mu_2(t)dt}, \tag{37}$$

Where $\overline{P}^{(2)}(0,z,s)$ is given by equation (31). Again, integrating equation (37) by parts with respect to x yields

$$\overline{P}^{(2)}(z,s) = \overline{P}^{(2)}(0,z,s)\left[\frac{1-\overline{B_2}(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \tag{38}$$

Where

$$\overline{B_2}(s+\lambda-\lambda z) = \int_0^\infty e^{-(s+\lambda-\lambda z)x}\,\mathrm{dB_2(x)}, \tag{39}$$

is the Laplace - Steiltjes transform of second stage of service time $B_2(x)$. We see that by virtue of equation (37), we have

$$\int_0^\infty \overline{P}^{(2)}(x,z,s)\mu_2(x)dx = \overline{P}^{(2)}(0,z,s)\overline{B_2}(s+\lambda-\lambda z). \tag{40}$$

Now by using equations (31) and (36), equation (40) becomes

$$\int_0^\infty \overline{P}^{(2)}(x,z,s)\mu_2(x)dx = \overline{P}^{(1)}(0,z,s)\overline{B_1}(s+\lambda-\lambda z)\,\overline{B_2}(s+\lambda-\lambda z). \tag{41}$$

From equation (32), we obtain

$$\overline{P}^{(1)}(0,z,s) = \left[\frac{\overline{V}(z,s)e^{-\lambda d(1-z)}+[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{z}\right]. \tag{42}$$

Substituting the value of $\overline{P}^{(1)}(0,z,s)$ into equation (34), we get

$$\overline{P}^{(1)}(z,s) = \left[\frac{1-\overline{B_1}(s+\lambda-\lambda z)}{(s+\lambda-\lambda z)}\right]\left[\frac{\overline{V}(z,s)e^{-\lambda d(1-z)}+[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{z}\right] \tag{43}$$

Now using equations (31), (32) and (36), equation (38) become

$$\overline{P}^{(2)}(z,s) = \left[\frac{\overline{V}(z,s)e^{-\lambda d(1-z)}+[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{z}\right]\overline{B_1}(s+\lambda-\lambda z)\left[\frac{1-\overline{B_2}(s+\lambda-\lambda z)}{(s+\lambda-\lambda z)}\right]. \tag{44}$$

From equation (29)

$$(s+1)\overline{V}(z,s) = \left[\frac{\overline{V}(z,s)e^{-\lambda d(1-z)}+[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{z}\right]\overline{B_1}(s+\lambda-\lambda z)\overline{B_2}(s+\lambda-\lambda z). \tag{45}$$

Which further reduces to

$$\overline{V}(z,s) = \left[\frac{[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{(s+1)[Z-\overline{B_1}(s+\lambda-\lambda z)\overline{B_2}(s+\lambda-\lambda z)+\overline{B_1}(s+\lambda-\lambda z)\overline{B_2}(s+\lambda-\lambda z)[1-e^{-\lambda d(1-z)}]]}\right]$$

$$\overline{B_1}(s+\lambda-\lambda z)\overline{B_2}(s+\lambda-\lambda z). \tag{46}$$

Let $\overline{P}(z,s)=\overline{P}^{(1)}(z,s)+\overline{P}^{(2)}(z,s)$ denote the probability generating function of the number in the queue irrespective of the type of service being provided. Then adding equations (43) and (44), we have

$$\overline{P}(z,s)=\left[\frac{\overline{V}(z,s)e^{-\lambda d(1-z)}+\left[1-s\overline{Q}(s)\right]+\lambda\overline{Q}(s)[z-1]}{z}\right]\left[\frac{1-\overline{B_1}(s+\lambda-\lambda z)\overline{B_2}(s+\lambda-\lambda z)}{(s+\lambda-\lambda z)}\right]. \tag{47}$$

Thus substituting the value of $\bar{V}(z,s)$ from equation (46) into equation (47), we get

$$\overline{P}(z,s)=\frac{\overline{N}(z,s)}{\overline{D}(z,s)} \tag{48}$$

$$\overline{N}(z,s)=\left(\begin{array}{c}(s+1)\left[\begin{array}{c}z-\overline{B_1}(s+\lambda-\lambda z)\overline{B_2}(s+\lambda-\lambda z)+\overline{B_1}(s+\lambda-\lambda z)\\ \overline{B_2}(s+\lambda-\lambda z)\left[1-e^{-\lambda d(1-z)}\right]\left[1-s\overline{Q}(s)+\lambda\overline{Q}(s)[z-1]\right]\end{array}\right]\\ +\overline{B_1}(s+\lambda-\lambda z)\overline{B_2}(s+\lambda-\lambda z)e^{-\lambda d(1-z)}\left[1-s\overline{Q}(s)+\lambda\overline{Q}(s)[z-1]\right]\end{array}\right)$$

$$\left[1-\overline{B_1}(s+\lambda-\lambda z)\overline{B_2}(s+\lambda-\lambda z)\right] \tag{49}$$

$$\overline{D}(z,s)=(s+\lambda-\lambda z)\left[(s+1)z\left[\begin{array}{c}z-\overline{B_1}(s+\lambda-\lambda z)\overline{B_2}(s+\lambda-\lambda z)\\ +\overline{B_1}(s+\lambda-\lambda z)\overline{B_2}(s+\lambda-\lambda z)\left[1-e^{-\lambda d(1-z)}\right]\end{array}\right]\right] \tag{50}$$

If we let $z=1$ in equation (48), we can easily verify that

$$\overline{Q}(s)+\bar{V}(z,s)+\overline{P}(z,s)=\frac{1}{s}, \tag{51}$$

as it should be.

Thus $\overline{P}^{(1)}(z,s)$, $\overline{P}^{(2)}(z,s)$ and $\bar{V}(z,s)$, are completely determined from equations (43), (44) and (46) respectively.

5. The Steady State Results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0} s\,\overline{f}(s)=\lim_{t\to\infty} f(t), \tag{52}$$

In order to determine $\overline{P}^{(1)}(z,s)$, $\overline{P}^{(2)}(z,s)$ and $\overline{V}(z,s)$ completely, we have yet to determine the unknown Q. For that purpose, we shall use the normalizing condition.

$$P^{(1)}(1)+P^{(2)}(1)+V(1)=1. \tag{53}$$

Thus, multiplying both sides of equations (43), (44) and (46) by s, taking limit as $s\to 0$, applying property (51) and simplifying we have

$$P^{(1)}(z)=\left[\frac{V(z)e^{-\lambda d(1-z)}+\lambda Q(z-1)}{z}\right]\left[\frac{1-B_1(\lambda-\lambda z)}{(\lambda-\lambda z)}\right], \tag{54}$$

$$P^{(2)}(z) = \left[\frac{V(z)e^{-\lambda d(1-z)} + \lambda Q(z-1)}{z}\right] B_1(\lambda - \lambda z) \left[\frac{1-B_2(\lambda-\lambda z)}{(\lambda-\lambda z)}\right], \tag{55}$$

$$V(z) = \frac{B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)Q(z-1)}{z-B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)+B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)[1-e^{-\lambda d(1-z)}]} \tag{56}$$

Where

$$P(z) = P^{(1)}(z) + P^{(2)}(z) \tag{57}$$

$$= \left[\frac{V(z)e^{-\lambda d(1-z)} + \lambda Q(z-1)}{z-B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)}\right] \left[\frac{1-B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)}{(\lambda-\lambda z)}\right] \tag{58}$$

Now using equation (56) into equation (58), we get

$$P(z) = \left[\frac{\{B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)-1\}Q}{z-B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)+B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)[1-e^{-\lambda d(1-z)}]}\right] \tag{59}$$

We see that for $z = 1$, the right side of both equations (54) and (55) are indeterminate of the form $\frac{0}{0}$. Therefore applying LHopital's rule, we obtain

$$V(1) = \frac{\lambda Q \mu_1 \mu_2}{\mu_1\mu_2 - \lambda(\mu_1+\mu_2) - \lambda\mu_1\mu_2 d}, \tag{60}$$

$$P(1) = \frac{\lambda(\mu_1+\mu_2)Q}{q\mu_1\mu_2 - \lambda(\mu_1+\mu_2) - \lambda\mu_1\mu_2 d}. \tag{61}$$

Wherein we have used facts that $\overline{B_1}(0) = \overline{B_2}(0) = 1, -\overline{B_1}'(0) = \frac{1}{\mu_1}$ and $-\overline{B_2}'(0) = \frac{1}{\mu_2}$.

Now to determine the only unknown Q. We see (60) and (61) in the normalizing condition

$Q + P(1) + V(1) = 1$ and have

$$Q = \frac{\mu_1\mu_2 - \lambda(\mu_1+\mu_2) - \lambda\mu_1\mu_2 d}{\mu_1\mu_2(1-\lambda d) + \lambda\mu_1\mu_2} \tag{62}$$

$$= 1 - \frac{2\lambda\mu_1\mu_2}{\mu_1\mu_2(1-\lambda d) + \lambda\mu_1\mu_2} \tag{63}$$

where $\lambda < \mu_1\mu_2[1-\lambda d]$.

Equation (62) gives the steady state probability that there is no customer in the system and the server is idle.

Also, from equation (63), we obtain ρ, the utilization factor of the system as

$$\rho = 1 - Q = \frac{2\lambda\mu_1\mu_2}{\mu_1\mu_2(1-\lambda d) + \lambda\mu_1\mu_2} < 1 \tag{64}$$

Acknowledgement

The author thanks the management of Sri Sivasubramaniya Nadar College of Engineering for providing the necessary requirements during the preparation of this paper.

References

[1] Choudhury, G., Chandi Ram Kalita. (2018), An M/G/1 queue with two types of general heterogeneous service and optional repeated service subject to server's breakdown and delayed repair, Quality Technology and Quantitative Management, 15(5), 662 – 654.

[2] Cohen, J, W. (1982), The single server queue. 2 nd edn. North – Holland, Amsterdam.

[3] Cohen, J, W. (1988), Queueing and Its Applications, North – Holland, Amsterdam.

[4] Doshi, B.T. (1985), A note on stochastic decomposition in a GI/G/1 queue with vacation or setup times, Journal of Applied Probability, 22, 419 – 428.

[5] Doshi, B.T. (1986), Queueing system with vacations - A survey, Queueing Systems, 1, 29–66.

[6] Doshi, B.T. (1990): Single server queues with vacations, Stochastic Analysis of Computer and Communication Systems, Takagi, H. (Ed.), Elsevier, Amsterdam.

[7] Doshi, B.T. (1991), Analysis of a two phase queueing system with general service times, Operation Research Letters, 10, 265 – 272.

[8] Fuhrmann., S.W. and Cooper, R.B. (1985), Stochastic Decompositions in the M/G/1 Queue with Generalized Vacations, Operation Research, 83, 1117–1129.

[9] Kailash C. Madan. (2021), On A single server queue with two stage first essential service followed by one of the two types of additional optional service and optional deterministic server Vacations, International Journal of Circuits Systems and Signal Processing, 15, 1730-1736.

[10] Kailash C. Madan. (2024), On a single server vacation queue with two types of service and two types of vacation, WSEAS Transactions on Systems, 23, 98 – 104.

[11] Kleinrock, L. (1976), Queueing Systems Vol. I, Theory, John Wiley and Sons, New York.

[12] Krishna, C.M. and Lee, Y.H. (1990), A study of a two phase service, Operation Research Letters, 9, 91–97.

[13] Lavenberg, G. (1988), A perspective on queueing models of computer performance in queueing theory and its applications, Liber Amicorium, for J.W. Cohen; CWI Monograph 7.

[14] Pavai Madheswari,S., Krishna Kumar, B., Poomalai Suganthi, (2019), Analysis of M/G/1 retrial queues with second optional service and customer balking under two types of Bernoulli vacation schedule, RAIRO-Operation Research, 53(2), 415 – 443.

[15] Shyamala, S. and Vijayaraj, R. (2020), Time dependent solution of two stages $M^X/G/1$ queue model server vacation random setup time and balking with Bernoulli schedule. AIP Conference Proceedings, 2261, 1-10.

[16] Somasundaram, B., Karpagam, S., Lokesh, R., Kavin Sagana Mary, A. (2023), An $M^{[X]}/G/1$ queue with optional service and working breakdown, Ural mathematical journal,9,162 – 175.

[17] Takagi., H. (1991), Queueing Analysis: Vacation and Priority Systems, Vol. 1. North Holland, Amsterdam.