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Application of Beal's Fuzzy Sets in Pattern Recognitions under Similarity Measures

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Abstract:
In this article, we established to basic collection of operations and construct the abstract properties that can be transmitted to the different models they are in joined with then to rank Beal's fuzzy set. we obtain the functions of score and accuracy, and formulate aggregate operators to be used with Beal's fuzzy sets. Alternatively, we develop the successful techniques "aggregative operators" to handle multi-criteria decision making problems in the Beal's fuzzy set environment. The proposed techniques has been illustrated and analysed through suitable example.

Keywords: Fuzzy set, Beal's fuzzy set, Score function, Accuracy function, Aggregate operators, Decision making.

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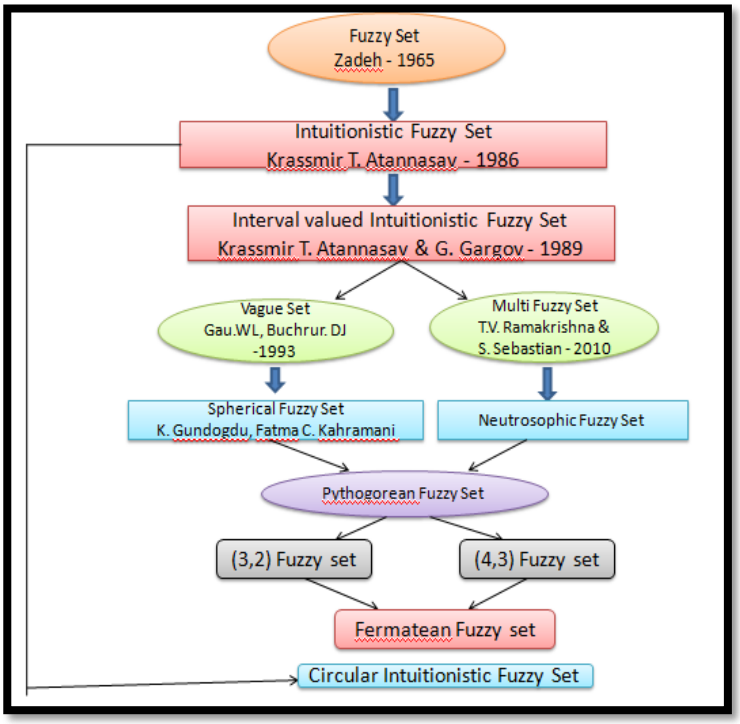
1. Introduction

The problems in the real world are too complicated since it includes ambiguity, uncertainty, or insufficient knowledge. So, decision-makers treat these problems using the methodology of fuzzification which is a vital method to address humanistic systems existing in real-world problems. L.A.Zadeh [19] in 1965, created fuzzy set (FSs) as a generalization of classical collection which are characterized by membership functions from the universe of discourse to the closed interval [0,1] . FS theory is applicable in various areas such as control theory, artificial intelligence, pattern recognition, database systems, and medical diagnosis. The fusion of technology and generalized forms of classical sets is very useful to solve many real-world complex problems that involve vague and uncertain information. A applicable set is defined by its members function from the universe of discourse to the two-point set {0, 1}. Classical set theory is insufficient to take the complex problems involving vague and fuzzy information. To handle the vagueness and uncertainty fuzzy sets (IFS). IFSs are widely used in many fields of mathematics, computer science, management and medical sciences. Szmidt [15] and Kacprzyk [16] and Wang and Xin [17] developed various distances and similarity measures between IFSs and studied applications of distance and of Atanassov [2] paper, several generalizations of IFSs have appeared in the literature. In 2020, a new notion called n-Pythagorean fuzzy sets (n-PFS) was created by Bryniarska [3] as a super class of FFSs and studied Yager's aggregation operations for n-PFSs. The distance and similarity measure on n-PFSs and their applications in MCDM problems were studied by Liu, Chen, and Peng [10] and Peng and Liu [11]

Ibrahim and his coworkers [6,7] initiated the study of (3,2)-Fuzzy sets was created by Alshami [1] and presented their applications to MCDM methods. Recently, Jun and his co-workers [8] created the class of Beal's fuzzy sets (BFSs) as a super class of n-PFSs. In this paper, we establish to basic set of operations and investigate the abstract properties that can be transmitted to the different models they are in connection with then to rank Beal's fuzzy sets, we obtain the mapping of score and accuracy, and formulate aggregate operators to be applied with Beal's fuzzy sets. Ultimately, we develop the successful techniques "aggregative operators" to handle multi-criteria decision making problems in the Beal's fuzzy sets environment. The proposed technique has been explained and analysed through suitable example.

1.1. Extension structure of fuzzy sets



2. Preliminaries

Throughout this article X be a universe of discourse N referred to the collection all natural numbers and $m, n \in N$.

Definition 2.1 [Fuzzy set] Let X be a nonempty set. Then a fuzzy set on x is defined by

$J_A : x \to [0,1]$. J_A is called the membership function. $J_A(x)$ is called the membership grade of X in J_A. we also write $J_A = \{(x, J_A(x))/x \in X\}$.

Example 2.2 Let $x = \{a, b, c, d\}$ and $J_A : x \to [0,1]$ defined by $J_A(a) = 0$, $J_A(b) = 0.6$, $J_A(c) = 0.3$, $J_A(d) = 1$.

Definition 2.3 [Intuitionistic fuzzy set] : An IFS is an extension of a fuzzy set introduced by K.Atanassov in 1983. An intuitionistic fuzzy set A in X is defined as $A = \{\langle x, J_A(x), K_A(x)\rangle / x \in X\}$, where $J_A : X \to [0,1]$ and $K_A : X \to [0,1]$ are respectively degree of membership and degree of non-membership for every $x \in X$ with $0 \leq J_A(x) + K_A(x) \leq 1$ and $\pi_A(x) = 1 - (J_A(x) + K_A(x))$ is the degree of indeterminacy of $x \in X$. Sometimes IFS is obviously called bifuzzy set.

Definition 2.4 A structure $A = \{\langle x, J_A(x), K_A(x)\rangle / x \in X\}$, where $J_A: X \to [0,1]$ and $K_A: X \to [0,1]$ are respectively degree of membership and degree of non-membership for every $x \in X$
to A is called

(i) Intuitionistic fuzzy set in X if $0 \leq J_A(x) + K_A(x) \leq 1$ for all $x \in X$.

(ii) Pythagorean fuzzy set in X if $0 \leq J_A^2(x) + K_A^2(x) \leq 1$ for all $x \in X$.

(iii) (2, 1) - fuzzy set in X if $0 \leq J_A^2(x) + K_A(x) \leq 1$ for all $x \in X$.

(iv) (3, 2) - fuzzy set in X of $0 \leq J_A^3(x) + K_A^2(x) \leq 1$ for all $x \in X$.

(v) Fermatean fuzzy set in X if $0 \leq J_A^3(x) + K_A^3(x) \leq 1$ for all $x \in X$.

(vi) (4, 2) – fuzzy set in X if $0 \leq J_A^4(x) + K_A^2(x) \leq 1$ for all $x \in X$.

(vii) N – fuzzy set when $m, n \in N$ in X if $0 \leq J_A^m(x) + K_A^n(x) \leq 1$ for all $x \in X$ and $m \geq 4$ and $n \geq 4$.

(ix) q-runk fuzzy set when $m = n = q$ i.e., $0 \leq J_A^q(x) + K_A^q(x) \leq 1$ for all $x \in X$.

In what follows, we compare Beal's fuzzy set with the previous generalization of intuitionstic fuzzy sets.

Proposition 2.5 (i) Every bifuzzy set is a Beal's fuzzy set.

(ii) If $m \geq 2$ and $n \geq 2$, then a Pythagorean fuzzy set is a Beal's fuzzy set.

(iii) If $m \geq 3$ and $n \geq 3$, then a Fermatean fuzzy set is a Beal's fuzzy set.

(iv) If $m \geq q$ and $n \geq q$, then a q-runk ortho pair fuzzy set is a Beal's fuzzy set.

(v) If $m \geq 2$ and $n \geq 3$, then a (2, 3) – fuzzy set is a Beal's fuzzy set.

Proof: The proof is straight forward.

Remark: For all $a, b \in [0,1]$, we have $a + b \leq 1 \Rightarrow a^2 + b^2 \leq 1 \Rightarrow a^3 + b^3 \leq 1 \Rightarrow a^4 + b^4 \leq 1 \Rightarrow a^n + b^n \leq 1 \Rightarrow a^m + b^n \leq 1, m \geq 4 \text{ and } n \geq 4$ from Definition 2.1.

Definition 2.6 Let $A = (J_A, K_A) \in B_m^n(X)$ *and* $x \in X$. Then the expression

$$\prod_A(x) = \left(1 - J_A^m(x) - K_A^n(x)\right)^{\frac{2}{m+n}}$$ is said to be the degree of indeterminacy of x to A.

Remark: Clearly, $\prod_A^{\frac{(m+n)}{2}}(x) \quad for\ all\ x \in X$.

Definition 2.7 Let $A = (J_A, K_A) \in B_m^n(X)$. Then the complement of A, denoted by A^c, is defined as follows:

$$A^c = (J_A^c, K_A^c) = \left(K_A^{\frac{n}{m}}, J_A^{\frac{n}{m}}\right).$$

Theorem 2.8 Let $A = (J_A, K_A)$, $A_1 = (J_{A_1}, K_{A_1})$, $A_2 = (J_{A_2}, K_{A_2}) \in B_m^n(X)$. Then

$(i)\ A^c \in B_m^n(X)$

$(ii)(A^c)^c = A$

$(iii)(A_1 \cup A_2)^c = A_1^c \cap A_2^c$

$(iv)(A_1 \cap A_2)^c = A_1^c \cup A_2^c$.

Proof: (i) Since, $A^c = (J_A^c, K_A^c) = \left(K_A^{\frac{m}{n}}, J_A^{\frac{m}{n}}\right)$, we have

$$0 \le J_{A_c}^{\ m} + K_{A_c}^{\ n} = \left((K_A)^{\frac{n}{m}}\right)^m + \left((J_A)^{\frac{m}{n}}\right)^n$$
$$= K_A^n + J_A^m$$
$$= J_A^m + K_A^n \le 1$$

Therefore, $A^c \in B_m^n(X)$.

(ii) Easy and left to the reader.

(iii) We have $(A_1 \cup A_2)^c = (\max\{J_{A_1}, J_{A_2}), \min\{K_{A_1}, K_{A_2}\})^c$

$$= (\min\left\{K_{A_1}^{\frac{n}{m}}, K_{A_2}^{\frac{n}{m}}\right\}, \max\{J_{A_1}^{\frac{m}{n}}, J_{A_2}\left(\frac{m}{n}\right)\})$$
$$= \left(K_{A_1}^{\frac{n}{m}}, J_{A_1}^{\frac{m}{n}}\right) \cap (K_{A_2}^{\frac{n}{m}}, J_{A_2}\left(\frac{m}{n}\right))$$
$$= A_1^c \cap A_2^c.$$

(i) Follows by a similar process to (iii).

Definition 2.9 Let $A = (J_A, K_A) \in B_m^n(X)$. Then we need of terms and the possibility measure on A are called as follows:

(i) □A = $\left(J_A, (1 - J_A^m)^{\frac{1}{n}}\right)$.

(ii)◊A = $\left((1 - K_A^n)^{\frac{1}{m}}, K_A\right)$.

Example 2.10 Let $X = \{x\}$ and $A = \{\langle 0.75, 0.8\rangle\} \in B_6^5(x)$. Then

$\prod_A(x) = 0.6321432$, □A $=\{\{\langle 0.75, 0.8\rangle\}$, ◊A $= \langle x, 0.75932, 0.8\rangle\}$, $A^c = \{\langle x, 0.8, 0.723\rangle\}$.

Theorem 2.11 $If\ A = (J_A, K_A) \in B_m^n(X)$, Then

(i) □A $\in B_m^n(X)$,

(ii)◊A $\in B_m^n(X)$,

Proof: (i) Follows on noting that:

$$J_{□A}^m + K_{□A}^m = J_A^m + \left((1 - J_A^m)^{\frac{1}{n}}\right)^n$$
$$= J_A^m + (1 - J_A^m)$$

$= 1.$

(ii)Follows on noting that

$$J_{\Diamond A}^{m} + K_{\Diamond A}^{n} = \left((1 - K_A^n)^{\frac{1}{m}}\right)^m + K_A^n$$
$$= (1 - K_A^n) + K_A^n$$
$$= 1.$$

Definition 2.12 Let $A = (J_A, K_A)$, $A_1 = (J_{A_1}, K_{A_1})$, $A_2 = (J_{A_2}, K_{A_2}) \in B_m^n(X)$ and $q \in N$. Then the operations $A_1 \oplus A_2$, $A_1 \otimes A_2$, qA, and A^q are defined as follows

(i) $A_1 \oplus A_2 = \left(J_{A_1}^{m} + J_{A_2}^{m} - J_{A_1}^{m} J_{A_2}^{m}, K_{A_1}^{n} K_{A_2}^{n}\right)$,

(ii) $A_1 \otimes A_2 = (J_{A_1}^{m} J_{A_2}^{m}, K_{A_1}^{n} + K_{A_2}^{n} - K_{A_1}^{n} K_{A_2}^{n})$

(iii) $qA = (1 - (1 - J_A^m)^q, K_A^n)$,

(iv) $A^q = \left(J_A^{mq}, 1 - (1 - K_A^n)^q\right)$.

Theorem 2.13 Let $A = (J_A, K_A)$, $A_1 = (J_{A_1}, K_{A_1})$, $A_2 = (J_{A_2}, K_{A_2}) \in B_m^n(X)$ and $q \in N$. Then

(i) $A_1 \oplus A_2 \in B_m^n(X)$,

(ii) $A_1 \otimes A_2 \in B_m^n(X)$,

(iii) $A_1 \cup A_2 \in B_m^n(X)$,

(iv) $A_1 \cap A_2 \in B_m^n(X)$,

(v) $qA \in B_m^n(X)$,

(vi) $A^q \in B_m^n(X)$.

Proof: (i) Since,

$$A_1 \oplus A_2 = \left(J_{A_1}^{m} + J_{A_2}^{m} - J_{A_1}^{m} J_{A_2}^{m}, K_{A_1}^{n} K_{A_2}^{n}\right),$$

we have,

$$J_{A_1 \oplus A_2}^{m} + K_{A_1 \oplus A_2}^{n} = \left(J_{A_1}^{m} + J_{A_2}^{m} - J_{A_1}^{m} J_{A_2}^{m}\right)^m + \left(K_{A_1}^{n} K_{A_2}^{n}\right)^n$$
$$= J_{A_1}^{m}\left(1 - J_{A_2}^{m}\right) + J_{A_2}^{m} + \left(K_{A_1}^{n} K_{A_2}^{n}\right)^n$$
$$\geq 0$$

and

$$J_{A_1 \oplus A_2}^{m} + K_{A_1 \oplus A_2}^{n} = \left(J_{A_1}^{m} + J_{A_2}^{m} - J_{A_1}^{m} J_{A_2}^{m}\right)^m + \left(K_{A_1}^{n} K_{A_2}^{n}\right)^n$$
$$\leq \left(\left(1 - K_{A_1}^{n}\right) + \left(1 - K_{A_2}^{n}\right) - \left(1 - K_{A_1}^{n}\right)\left(1 - K_{A_2}^{n}\right)\right)^m + \left(K_{A_1}^{n} K_{A_2}^{n}\right)^n$$
$$= \left(1 - K_{A_1}^{n} K_{A_1}^{n}\right)^m + \left(K_{A_1}^{n} K_{A_2}^{n}\right)^n$$
$$\leq 1$$

because $0 \leq K_{A_1}^{n} K_{A_2}^{n} \leq 1$ and $m, n \geq 1$.

Hence, $A_1 \oplus A_2 \in B_m^n(X)$,

(ii) Similar to (i).

(iii)Suppose max $\{K_{A_1}, K_{A_2}\} \leq K_{A_1}$, we have

$$0 \leq J^m_{A_1 \cup A_2} + K_{A_1 \cup A_2}$$
$$= (\max\{J_{A_1}, J_{A_2}\})^m + ((\min\{K_{A_1}, K_{A_2}\})^n$$
$$\leq J_{A_1}^m + K_{A_1}^n$$
$$\leq 1$$

Suppose now

max $\{J_{A_1}, J_{A_2}\} = J_{A_2}$. Since

$\min\{K_{A_1}, K_{A_2}\} \leq K_{A_2}$, we have

$$0 \leq J^m_{A_1 \cup A_2} + K^n{}_{A_1 \cup A_2},$$
$$= (\max\{J_{A_1}, J_{A_2}\})^m + ((\min\{K_{A_1}, K_{A_2}\})^n$$
$$\leq J_{A_1}^m + K_{A_1}^n$$
$$\leq 1$$

Thus, the proof of (iii) is completed.

(iv) Similar to (iii)

(v) Since $A \in B_m^n(X)$, we have $0 \leq J_A^m \leq 1$ and $0 \leq K_A^n \leq 1$.

$qA = (1 - (1 - J_A^m)^q, K_A^{nq})$, we have

$$\leq J_{qA}^m + K_{qA}^n$$
$$= (1 - (1 - J_A^m)^q)^m + (K_A^{nq})^n$$
$$\leq (1 - (1 - J_A^m)^q)m + (K_A^{nq})^n$$
$$\leq (1 - (K_A^{nq})^m) + (K_A^{nq})^n$$
$$\leq 1$$

Theorem 2.14 Let $A = (J_A, K_A), A_1 = (J_{A_1}, K_{A_1}),\ A_2 = (J_{A_2}, K_{A_2}) \in B_m^n(X)$ and $q, q_1, q_2 \in N$.

Then (i) $A_1 \oplus A_2 = A_2 \oplus A_1$

(ii) $A_1 \otimes A_2 = A_2 \otimes A_1$

(iii) $q(A_1 \oplus A_2) = qA_1 \oplus qA_2$

(iv) $(q_1 + q_2)A = q_1A \oplus q_2A$,

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$(v)(A_1 \otimes A_2)^q = A_1^q \otimes A_2^q$

$(vi)\ A^{q_1} \otimes A^{q_2} = A^{q_1+q_2}$.

Proof: The proof of above can easily understand.

Theorem 2.15 Let $A = (J_A, K_A) \in B_m^n(X)$ and $x \in X$. If $\prod_A(x) = 0$, then $\prod_{A^q}(x) = 0$ for all $q \in N$.

Proof: Since $\prod_A(x) = \left(1 - J_A^m(x) - K_A^n(x)\right)^{\frac{2}{m+n}}$

we have $\prod_A(x) = 0 \implies \left(1 - J_A^m(x) - K_A^n(x)\right)^{\frac{2}{m+n}}$

$$\implies J_A(x) + K_A^n(x) = 1$$

$$\implies J_A(x) = 1 - K_A^n(x).$$

By using this result, we have

$$A^q = \left(J_A^{mq}, 1 - (1 - K_A^n)^q\right)$$

$$= \left(J_A^{mq}, 1 - (1 - J_A^m)^q\right)$$

$$= (J_A^{mq}, 1 - J_A^{mq})$$

Hence,

$$\prod_{A^q}(x) = \left(1 - \left(J_A^{mq}(x)\right)^m - \left(1 - J_A^{mq}(x)\right)^n\right)^{\frac{2}{m+n}}$$

$$\implies \prod_{A^q}^{\frac{2}{m+n}}(x) = \left(1 - \left(J_A^{mq}(x)\right)^m - \left(1 - J_A^{mq}(x)\right)^n\right)$$

$$\implies \prod_{A^q}(x) = 0$$

Theorem 2.16 Let $A = (J_A, K_A) \in B_m^n(x)$ and $x \in X\ and\ q, q_1, q_2 \in N$. Then

$(i) q_1 \geq q_2 \implies A^{q_1} \subset A^{q_2}$

$(ii)\ q_1 \geq q_2 \implies q_2 A \subset q_1 A$

$$1 - J_{A_2}^{\ m} \leq 1 - J_{A_1}^{\ m}$$

$$\implies \left(1 - J_{A_2}^{\ m}\right)^q \leq \left(1 - J_{A_1}^{\ m}\right)^q$$

$$\implies \left(1 - \left(1 - J_{A_1}^{\ m}\right)^q\right) \leq \left(1 - \left(1 - J_{A_2}^{\ m}\right)^q\right)$$

$$\implies J_{qA_1} \leq J_{qA_2}$$

and

$$K_{A_1} \geq K_{A_2}$$

$$\implies K_{A_1}^{\ n} \geq K_{A_2}^{\ n}$$

$$\implies K_{A_1}^{\ nq} \geq K_{A_2}^{\ nq}$$

$$\Rightarrow K_{qA_1} \geq K_{qA_2}$$

Hence, $qA_1 \subset qA_2$.

(i)Similar to that of (i)

(ii)Follow since:

$$A_1 \cup A_2 = (\max\{J_{A_1}, J_{A_2}\}, \min\{K_{A_1}, K_{A_2}\})$$

and

$$(A_1 \cup A_2)^q = (J_{A_1 \cup A_2}, K_{A_1 \cup A_2})$$

$$= \left(\left(\max\{J_{A_1}, J_{A_2}\}\right)^{mq}, 1 - \left(1 - \left(\min\{K_{A_1}, K_{A_2}\}\right)^n\right)^q\right)$$

$$= (\max\{\{J_{A_1}^{mq}, J_{A_2}^{mq}\}, 1 - (1 - \min\{K_{A_1}^n, K_{A_2}^n\})^q)$$

$$= (\max\{\{J_{A_1}^{mq}, J_{A_2}^{mq}\}, 1 - \max\{1 - K_{A_1}^n, 1 - K_{A_2}^n\})^q)$$

$$= (\max\{\{J_{A_1}^{mq}, J_{A_2}^{mq}\}, 1 - \max\{1 - K_{A_1}^n, 1 - K_{A_2}^n\})^q)$$

$$= (\max\{\{J_{A_1}^{mq}, J_{A_2}^{mq}\}, 1 - (\max\{1 - K_{A_1}^n\}^q, (1 - K_{A_2}^n)^q\}) .$$

Proof: (i) Since

$$A^{q1} = \left(J_A^{mq1}, 1 - (1 - K_A^n)^{q1}\right)$$

$$A^{q2} = \left(J_A^{mq2}, 1 - (1 - K_A^n)^{q2}\right)$$

we have,

$$q_1 \geq q_2 \Rightarrow J_A^{q2} \geq J_A^{q1} \quad and \quad (1 - K_A^n)^{q1} \leq (1 - K_A^n)^{q2}$$

$$\Rightarrow J_A^{mq2} \geq Jm_A^{q1} \quad and \quad 1 - (1 - K_A^n)^{q2} \leq (1 - K_A^n)^{q1}$$

$$\Rightarrow J_A^{q2} \geq J_A^{q1} \quad and \quad K_A^{q2} \leq K_A^{q1}.$$

Hence $A^{q1} \subset A^{q2}$.

(ii) Similar to that of (i).

Theorem 2.17 Let $A_1 = (J_{A_1}, K_{A_1}),\ A_2 = (J_{A_2}, K_{A_2}) \in B_m^n(X)\ and\ q \in N$. Then

$(i)\ A_1 \subset A_2 \quad \Rightarrow qA_1 \subset qA_2$

$(ii)\ A_1 \subset A_2 \quad \Rightarrow A_1^q \subset A_2^q$

$(iii)(A_1 \cup A_2)^q \Rightarrow A_1^q \cup A_2^q$

$(iv)\ q(A_1 \cup A_2) \Rightarrow qA_1 \cup qA_2$

$(v)(A_1 \cap A_2) \Rightarrow A_1^q \cap A_2^q$

$(vi)q(A_1 \cap A_2) \Rightarrow qA_1 \cap qA_2$.

Proof: Since $A_1 \subset A_2$, we have

$$J_{A_1} \leq J_{A_2} \rightarrow J_{A_1}^{\ m} \leq J_{A_2}^{\ m}$$

$$= \max\{J_{A_2}^{\ mq}, J_{A_2}^{\ mq}\}, \min\{(1-(1-K_{A_1}^n)^q)\}$$

$$= A_1^q \cup A_2^q$$

(iv) Follows since

$$A_1 \cap A_2 = (\min\{J_{A_1}, J_{A_2}\}, \max\{K_{A_1}, K_{A_2}\}) \ and$$

$$(A_1 \cap A_2)^q = (J_{A_1 \cap A_2}, K_{A_1 \cap A_2})$$

$$= ((\min\{J_{A_1}, J_{A_2}\})^{mq}, 1-(1-\{\max\{K_{A_1}, K_{A_2}\})^n)^q\}$$

$$= ((\min\{J_{A_1}^{\ mq}, J_{A_2}^{\ mq}\}) \quad , 1-(1-\max\{K_{A_1}^{\ n}, K_{A_2}^n\})^k)$$

$$= (\min\{J_{A_1}^{mq}, J_{A_2}^{mq}\}, 1-(\min(1-K_{A_1}^n, 1-K_{A_2}^n)^q))$$

$$= (\min\{J_{A_1}^{mq}, J_{A_2}^{mq}\}, \max\{1-(1-KA_1^n)^q, (1-K_{A_2}^n)^q\})$$

$$= (min\{J_{A_1}^{mq}, J_{A_2}^{mq}\}, \max\{1-(1-K_{A_1}^n)^q (1-K_{A_2}^n)^q\})$$

$$= A_1^k \cap A_2^k$$

(v) The same that of result (iii)

(vi) Some way that $f(v)$

Definition 2.18 Let $A = (J_A, K_A) \in B_m^n(x)$ and $\alpha \in [0,1]$. Then the operator $G_\alpha(A)$ is expressed as follows:

$$G_\alpha(A) = \left(\left(J_A^m + \alpha \prod \frac{m+n}{2}\right)^{\frac{1}{m}}, J_A^n + ((1-\alpha) \prod \frac{m+n}{2}\right)^{\frac{1}{n}})$$

Theorem 2.19 Let $A = (J_A, K_A) \in B_m^n(x)$ and $\alpha, \beta \in [0,1]$. Then

$(i)\alpha \leq \beta \implies G_\alpha(A) \subset G_{\beta_1}(A)$

$(ii)G_0(A) = \square$

$(iii)G_1(A) = \lozenge$ A

Proof: (i) Immediate result to obtained

(ii) Since,

$$G_0(A) = \left(\left(J_A^m + \alpha \prod \frac{m+n}{2}\right)^{\frac{1}{m}} \left(K_A^n + (1-\alpha) \prod \frac{m+n}{2}\right)\left(\frac{1}{n}\right)\right)$$

$$= (J_A, \left(K_A^n + \prod \frac{m+n}{2}\right)^{\frac{1}{n}})$$

$$= \left(J_A, (K_A + 1 - J_A^m - K_A^n)^{\frac{1}{n}}\right)$$

$$= (J_A, (1 - J_A^m)^{\frac{1}{n}})$$

$= \square A$

The proof is completed.

It follows on nothing that,

$$G_1(A) = \left(\left(J_A^m + (1)\prod \frac{m+n}{2}\right)^{\frac{1}{m}}, K_A + (1-1)\prod \frac{m+n}{2}\right)^{\frac{1}{n}})$$

$$= ((J_A^m + \prod \frac{m+n}{2})^{\frac{1}{m}}, (K_A^n)^{\frac{1}{n}})$$

$$= \left(\left(J_A^m + (1 - J_A^m - K_A^n)^{\frac{1}{m}}, K_A\right)\right)$$

$$= \left((1 - K_A^n)^{\frac{1}{m}}, K_A\right)$$

$$= \lozenge A$$

Definition 2.20 Let $A = (J_A, K_A) \in B_m^n(x)$ and $\alpha, \beta \in [0,1]$ where $\alpha + \beta \leq 1$. we define the operator $H_{\alpha,\beta}(A)$ as

$$H_{\alpha,\beta}(A) = ((J_A^m + \alpha \prod \frac{m+n}{2})^{\frac{1}{m}}, \left(K_A^n + \beta \prod \frac{m+n}{2}\right)^{\frac{1}{n}})$$

Theorem 2.21 For any $A = (J_A, K_A) \in B_m^n(x)$ and $\alpha, \beta \in [0,1]$ where $\alpha + \beta \leq 1$. we have

$(i) H_{\alpha,\beta}(A) \in B_m^n(x)$

$(ii)\ 0 \leq t \leq \alpha \implies H_{\alpha,\beta}(A) \subset H_{\alpha,\beta}(A)$

$(iii)\ 0 \leq t \leq \beta \implies H_{\alpha,\beta}(A) \subset H_{\alpha,t}(A)$

$(iv) G_\alpha(A) = H_{\alpha, 1-\alpha}(A)$

$(v) \square A = H_{0,1}(A)$

$(vi) \lozenge A = H_{1,0}(A)$

$(vii) H_{\alpha,3}^c(M^c) = H_{3,\alpha}(A)$

Proof: (i) Follows since,

$$J_{1+\alpha,3}^m(A) + K_{1+\alpha,3}(A) = \left(\left(J_A^m + \alpha \prod \frac{m+n}{2}\right)^{\frac{1}{m}}\right)^m + \left(\left(K_A^n + \beta \prod \frac{m+n}{2}\right)^{\frac{1}{n}}\right)^n$$

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$$= G_\alpha(A)$$

(v) Follows since

$$H_{\alpha,1-\alpha}(x) = G_\alpha(A) \Longrightarrow H_{0,1}(A) = G_0(A).$$

$$\Longrightarrow H_{0,1}(A) = \square A$$

(vi) Follows on nothing that,

$$H_{\alpha,1-2}(A) = G_\alpha(A) \Longrightarrow H_{1,0}(A) = G_1(A)$$

$$\Longrightarrow H_{1,0}(A) = \lozenge A$$

(vii)Since $M^c = (K_A^{\frac{n}{m}}, J_n^{\frac{m}{n}})$ we have

$$H_{\alpha,\beta}(M^c) = \left(\left(\left(K_A^{\frac{n}{m}}\right)^m + \alpha \prod \frac{m+n}{2}\right)^{\frac{1}{m}}, \left(\left(J_A^{\frac{m}{n}}\right)^n + \beta \prod \frac{m+n}{2}\right)^{\frac{1}{n}}\right.$$

$$= \left(\left(K_A^n + \alpha \prod \frac{m+n}{2}\right)^{\frac{1}{m}} \left(J_A^m + \beta \prod \frac{m+n}{2}\right)^{\frac{1}{n}}\right)$$

$$d_{B_m^n}^H(A_1, A_2) = \frac{1}{2}\sum_{i=1}^{n}\{|J_{A_1}(x_i) - J_{A_2}(x_i)| + \left|K_{A_1}(x_i) - K_{A_2}(x_i)\right| + |\prod_{A_1}(x_i) - \prod_{A_2}(x_i)|\}$$

The normalize Hamming distance is explained as

$$d_{B_m^n}^H(A_1, A_2) = \frac{1}{2r}\sum_{i=1}^{n}\{|J_{A_1}(x_i) - J_{A_2}(x_i)| + \left|K_{A_1}(x_i) - K_{A_2}(x_i)\right| + |\prod_{A_1}(x_i) - \prod_{A_2}(x_i)|$$

The Euclidean distance is defined as

$$d_{B_m^n}^H(A_1, A_2) = \sqrt{\frac{1}{2}\sum_{i=0}^{r}\left\{\left(J_{A_1}(x_i) - J_{A_2}(x_2)\right)^2 + \left(K_{A_1}(x_i) - K_{A_2}(x_i)\right)^2 + (\prod_{A_1}(x_i) - \prod_{A_2}(x_i))^2\right\}}$$

The normalized Euclidean distance is defined as

$$d_{B_m^n}^H(A_1, A_2) = \sqrt{\frac{1}{2r}\sum_{i=0}^{r}\left\{\left(J_{A_1}(x_i) - J_{A_2}(x_i)\right)^2 + \left(K_{A_1}(x_i) - K_{A_2}(x_i)\right)^2 + (\prod_{A_1}(x_i) - \prod_{A_2}(x_i))^2\right\}}$$

and so,

$$(H_{\alpha,3}(M^c))^c = \left(\left((J_A^m + \beta \prod \frac{m+n}{2}\right)^{\frac{1}{n}}\right)^{\frac{m}{n}} + \left(\left(K_A^n + \alpha \prod \frac{m+n}{2}\right)^{\frac{1}{m}}\right)^{\frac{m}{n}})$$

$$= \left(\left(J_a^m + \beta \prod \frac{m+n}{2}\right)^{\frac{1}{m}}, \left(K_A^n + \alpha \prod \frac{m+n}{2}\right)^{\frac{1}{n}}\right)$$

$$= H_{\beta,\alpha}(A).$$

Hence the proof.

3. Various types of distance function over Beal's fuzzy set

Definition 3.1 Let $X = \{ x_1, x_2, x_3 \dots, x_r\}$ be a whole of discourse and $A_1 = (J_{A_1}, K_{A_1})$, $A_2 = (J_{A_2}, K_{A_2})$, $A_3 = (J_{A_3}, K_{A_3}) \in B_m^n(X)$, the distance function $d: B_m^n(X) \times B_m^n(X) \to [0,1]$ is defined as

$(i) 0 \leq d(A_1, A_2) \leq 1$ (Boundness)

$(ii) d(A_1, A_2) = 0 \implies A_1 = A_2$ (Separable)

$(iii) d(A_1, A_2) = d(A_2, A_1)$ (Symmetric)

$(iv) d(A_1, A_3) + d(A_2, A_3) \geq d(A_1, A_2)$ (Triangle Inequality)

Definition 3.2 Let $X = (x_1, x_2\}$ be a whole of discourse and $A_1, A_2, A_3 \in B_7^5(X)$, where

$$A_1 = \{\langle x_1, 0.5, 0.6\rangle, \langle x_2, 0.4, 0.5\rangle\},$$

$$A_2 = \{\langle x_1, 0.8, 0.3\rangle, \langle x_2, 0.9, 0.5\rangle\},$$

$$A_3 = \{\langle x_1, 0.6, 0.8\rangle, \langle x_2, 0.8, 0.5\rangle\},$$

Find the following: (1) Hamming Distance (2) Normalized Hamming Distance (3) Euclidean Distance (4) Normalized Euclidean Distance

$dB_7^5(A_1, A_2) = 0.6$ $\quad$ $dB_7^5 (A_3, A_1) = 0.5$

$d^E B_7^5 (A_1, A_2) = 0.3$ $\quad$ $dB_7^5 (A_3, A_1) = 0.2$

$d^n B_{m^n}^5 (A_1, A_2) = 0.6$ $\quad$ $dB_7^5 (A_3, A_1) = 0.5$

$d^H B_7^5 (A_2, A_3) = 0.5$ $\quad$ $d^n B_7^5 (A_3, A_1) = 0.3$

$d^H B_7^5 (A_2, A_3) = 0.2$ $\quad$ $d^E B_7^5 = 0.5$

$d^n B_7^5 = 0.4$

Example 3.3 Let $X = \{x_1, x_2, x_3, x_4\}$ and $A_1, A_2, A_3 \in B_8^6(X)$ where

$$A_1 = \{\langle x_1, 0.2, 0.7\rangle, \langle x_2, 0.9, 0.7\rangle\},$$

$$A_2 = \{\langle x_1, 0.9, 0.3\rangle, \langle x_2, 0.5, 0.5\rangle\},$$

$$A_3 = \{\langle x_1, 0.6, 0.7\rangle, \langle x_2, 0.8, 0.5\rangle\},$$

$dB_8^5(A_1, A_2) = 0.9$ $\quad$ $d^H B_8^5(A_1, A_2) = 0.3$

$d^n B_8^5(A_1, A_2) = 0.6$ $\quad$ $d^E B_8^5(A_1, A_2) = 0.8$

Definition 3.4 Let $X = \{ x_1, x_2, x_3 \dots, x_r\}$ be a whole of discourse and $A_1 = (J_{A_1}, K_{A_1})$, $A_2 = (J_{A_2}, K_{A_2})$, $A_3 = (J_{A_3}, K_{A_3}) \in B_m^n(X)$. The similarity measure $S: B_m^n(X) \times B_m^n(X) \to [0,1]$ is expressed as

$(i) 0 \leq S(A_1, A_2) \leq 1$ (Boundedness)

$(ii) S(A_1, A_2) = 1 \Leftrightarrow A_1 = A_2$ (Separability)

$(iii) S(A_1, A_2) = S(A_2, A_1)$ (Symmetric)

$(iv) S(A_1, A_3) + S(A_2, A_1) \geq S(A_1, A_2)$ (Triangle Inequality)

Using distance functions and similarity measures, the below of result proved.

Theorem 3.5 Let $A_1 = (J_{A_1}, K_{A_1})$, $A_2 = (J_{A_2}, K_{A_2}) \in B_m^n(X)$. If $d(A_1, A_2)$ is a distance measure between Beal's fuzzy sets A_1 and A_2, then

$$S(A_1, A_2) = 1 - d(A_1, A_2)$$

is a similarity measure of A_1 and A_2.

Theorem 3.6 Let $A_1 = (J_{A_1}, K_{A_1})$, $A_2 = (J_{A_2}, K_{A_2}) \in B_m^n(X)$. Suppose $A_1 \subset A_2 \subset A_3$, then

$(i) d(A_1, A_2) \geq d(A_1, A_2) and$

$(ii) d(A_1, A_3) \geq d(A_2, A_3)$

$(iii) S(A_1, A_3) \leq S(A_1, A_2) and$

$(iv) S(M_1, M_3) \leq S(M_2, M_3)$.

Theorem 3.7 Let $A_1 = (J_{A_1}, K_{A_1}), A_2 = (J_{A_2}, K_{A_2}) \in B_m^n(X)$. Then

$(i) d(A_1, A_2) = d(A_1^c, A_2^2)$

$(ii) S(A_1, A_2) = S(A_1^c, A_2^c)$.

Note. The two similarity measures are defined are defined as follows

$S_1 = 1 - d_{B_m^n}^n H(A_1, A_2)$,

$S_2 = 1 - d_{B_m^n}^n E(A_1, A_2)$

Note: $A_1, A_2 \in B_m^n(X), X = \{x_1, x_2\}$,

$S_1 \in (A_1, A_2)$, $S_2 \in (A_1, A_2)$, $A_1, A_2, A_3 \in B_m^n(X)$

$S_1(A_1, A_2) = 0.6$, $S_2(A_1, A_2) = 0.8$, $S_1(A_1, A_3) = 0.6$, $S_2(A_1, A_3) = 0.8$.

In order to rank Beal's fuzzy sets, we give the score mapping and accuracy, mapping of the Beal's fuzzy set.

Definition 3.8 The score mapping function of a Beal's fuzzy set $A = (J_A, K_A)$ can be given as

$S(A) = J_A^n - K_A^m$.

The accuracy function of a Beal's fuzzy set $A = (J_A, K_A)$ can be given as

$\overline{\alpha}$ (A) = $J_A^n + K_A^m$.

Example 3.9 Consider the Beal's fuzzy set $A = (0.62, 0.54)$ then the score function is

$$S(A) = \begin{cases} -0.14 \ \ if\ n = 10, m = 3 \\ -0.00 \ if\ n = 8, m = 6 \\ -0.01 \ if\ n = 7, m = 5 \\ 0.06 \ \ if\ n = 5, m = 6 \end{cases}$$

and the accuracy function is

$$\overline{\alpha}(A) = \begin{cases} 0.16 \ if\ n = 10, m = 3 \\ 0.05 \ if\ n = 8, m = 6 \\ 0.08 \ if\ n = 7, m = 5 \\ 0.11 \ \ if\ n = 5, m = 6 \end{cases}$$

Theorem 3.10 Let $A = (J_A, K_A)$ be a Beal's fuzzy set. Then the suggested score fuzzy mapping $S(A) \in [-1,1]$.

Proof: Since for any Beal's fuzzy set A. we have $J_A^n + K_A^n \leq 1$

Hence,

$$J_A^n - K_A^n \leq J_A^n \leq 1 \ and\ J_A^n - K_A^n \geq -K_A^m \geq -1$$

Thus $-1 \leq J_A^n - K_A^n \leq 1$, namely $S(A) \in [-1,1]$.

In such way that, if $A = (0,1)$ then $S(A) = -1$ and if $A = (0,1)$ then $S(A) = 1$.

Remark: For any Beal's fuzzy set $A_1 = (J_{A_1}, K_{A_1})$ and $A_2 = \left(J_{A_2}, K_{A_2}\right)$, the comparison technique is supposed as

$(i) If\ S(A_1) < S(A_2)\ then\ A_1 < A_2$

$(ii) If\ S(A_1) > S(A_2)\ then\ A_1 > A_2$

$(iii) If\ S(A_1) = S(A_2)\ then\ A_1 = A_2$

$(iv)\ \overline{\alpha}\,(A_1) < \overline{\alpha}\,(A_2)\ then\ A_1 < A_2$

$(v) If\ \overline{\alpha}\,(A_1) > \overline{\alpha}\,(A_2)\ then\ A_1 > A_2$

$(vi) If\ \overline{\alpha}\,(A_1) = \overline{\alpha}\,(A_2)\ then\ A_1 = A_2$.

Beal's fuzzy weighted power average

In this section, we study the various operation on Beal's fuzzy sets, and some intellectual structure are indicated in detail.

Definition 4.1. Let $A_i = \left(J_{A_i}, K_{A_i}\right)$ $(i = 1,2, \dots . k)$ be a value of Beal's fuzzy sets and $\in = (\in_1, \in_2, \dots \in_k)^T$ be a weighted vector of A_i with $\in_i > 0, \sum_{i=1}^{k} \in_i = 1$. Then a Beal's fuzzy weighted power mean operator is a function

$$BFWPA\colon\ A^k \to A$$

where

$BFWPA(A_1, A_2, \dots A_k) = \left(\left(\sum_{i=1}^{k} \in_i J_{A_i}^{n}\right)^{\frac{1}{n}}, \left(\sum_{i=1}^{k} \in_i K_{A_i}^{m}\right)^{\frac{1}{m}} \right).$

Example 4.2. Consider $A_1 = (0.8, 0.4), A_2 = (0.3, 0.9), A_3 = (0.6, 0.7), A_4 = (0.8, 0.7)$ *and* $A_5 = (0.9, 0.5)$ as five Beal's fuzzy sets and let $\in = (0.18, 0.27, 0.25, 0.16, 0.14)^T$ be a weighted vector of $A_i (i = 1,2,3,4,5)$. Then BFWPA $(A_1, A_2, \dots . A_5)$

$= ((0.8^n \times 0.18 + 0.3^n \times 0.27 + 0.6^n \times 0.25 + 0.8^n \times 0.16 + 0.9^n \times 0.14)^{\frac{1}{n}},$

$(0.4^n \times 0.18 + 0.9^n \times 0.27 + 0.7^n \times 0.25 + 0.7^n \times 0.16 + 0.5^n \times 0.14)^{\frac{1}{m}})$

$$= \begin{cases} 0.667, 0.818 \ \text{if } n = 2, m = 3 \\ 0.696, 0.832 \ \text{if } n = 3, m = 5 \\ 0.717, 0.862 \ \text{if } n = 4, m = 8 \\ 0.783, 0.851 \ \text{if } n = 10, m = 14 \\ 0.824, 0.89 \ \text{if } n = 20, m = 30 \\ 0.875, 0.84 \ \text{if } n = 70, m = 50. \end{cases}$$

Theorem 4.3. Let $A_i = (J_{A_i}, K_{A_i}) (i = 1,2, \dots k)$ be a family of Beal's fuzzy sets and let $\in = (\in_1, \in_2 .. \in_k)^T$ be a weighted vector of A_i with $\in_i > 0$ *and* $\sum_{i=1}^{k} \in_i = 1$. Then Beal's fuzzy weighted power mean $(A_1, A_2, \dots A_k)$ is a Beal's fuzzy set.

Theorem 4.4. Let $A_i = (J_{A_i}, K_{A_i}) (i = 1,2, ..k)$ be a value of Beal's fuzzy sets $A = (J_A, K_A)$ be a Beal's fuzzy set and $\in = (\in_1, \in_2, \dots \in_k)^T$ be a weighted vector of A_i with $\sum_{i=1}^{k} \in_i = 1$. Then

$BFWPA(A_1 \oplus A_2 \oplus \dots . A_k \oplus A) \geq BFWPA(A_1 \otimes A_2 \otimes \dots A_k \otimes A).$

Theorem 4.5. Let $A_i = (J_{A_i}, K_{A_i}) (i = 1,2, \dots, k)$ be a value of Beal's fuzzy sets, $A = (J_A, K_A)$ be a Beal's fuzzy set, and $\in = (\in_1, \in_2, \dots \in_k)^T$ be a weighted vector of A_i with $\sum_{i=1}^{k} \in_i = 1$. Then

$(i) BFWPA\ (A_1 \oplus A_2 \oplus \dots . A_k \oplus A) \geq BFWPA(A_1, A_2, \dots A_K) \otimes A.$

$(ii) BFWPA\ (A_1, A_2, \dots A_k) \oplus A \geq BFWPA(A_1, A_2, \dots, A_k) \otimes A.$

Theorem 4.6. Let $A_i = (J_{A_i}, K_{A_i})$ and $B_i = (J_{B_i}, K_{B_i}) (i = 1,2, \dots, k)$ be two values of Beal's fuzzy sets and $\in = (\in_1, \in_2, \dots \in_k)^T$ be a weighted vector of them with $\sum_{i=1}^{k} \in_i = 1$. Then

$(i) BFWPA(A_1 \oplus B_1, A_2 \oplus B_2, \dots, A_k \oplus B_k) \geq (A_1 \otimes B_1, A_2 \otimes B_2, \dots, A_k \otimes B_k).$

$(ii) BFWPA\ (A_1, A_2, \dots, A_k) \oplus BFWPA\ (B_1, B_2, \dots, B_k)$

$(iii) BFWPA\ (A_1, A_2, \dots, A_k) \otimes BFWPA\ (B_1, B_2, \dots, B_k)$

$$= ((\sum_{i=1}^{k} \in_i J_{A_i}^{n})^{\frac{1}{n}} \ (\sum_{i=1}^{k} \in_i J_{B_i}^{n})^{\frac{1}{n}} \sqrt{\sum_{i=1}^{k} \in_i K_{A_i}^{m} + \sum_{i=1}^{k} \in_i K_{A_i}^{m} - \sum_{i=1}^{k} \in_i K_{A_i}^{m} \sum_{i=1}^{n} \in_i K_{AB_i}^{m}})$$

Application of Beal's fuzzy set in pattern recognition

In this section, we propose similarity measures that can be dignity in commander's chosen in military office, medical diagnosis of disease. Plant leaf diseases classifications, construction material selections and other multi-attribute decision-making problems.

The following contribution showed to accept on unknown pattern using the new idea of similarity theorem.

Example 4.7 Let us taken three unknown patterns $P_i(i = 1,2,3)$ which are represented by the Beal's fuzzy set $(m = 5, n = 5)P_i(i = 1,2,3)$ in the characteristic as: $R = \{r_1, r_2, r_3\}$.

$$P_1 = \langle r_1, 0.5, 0.6\rangle, \langle r_2, 0.1, 0.7\rangle, \langle r_3, 0.4, 0.7\rangle$$

$$P_2 = \langle r_1, 0.1, 0.8\rangle, \langle r_2, 0.8, 0.8\rangle, \langle r_3, 0.7, 0.6\rangle$$

$$P_3 = \langle r_1, 0.5, 0.7\rangle, \langle r_2, 0.6, 0.6\rangle, \langle r_3, 0.7, 0.5\rangle$$

Consider, as unknown pattern $P \in B_4^5\ (R)$ that will be recognized, where:

$$P = \langle r_1, 0.9, 0.7\rangle, \langle r_2, 0.8, 0.7\rangle, \langle r_1, 0.6, 0.8\rangle.$$

Then the proposed similarity techniques S_1 *and* S_2 which have been evaluated from P to $P_i(i = 1,2,3)$ are given in the following table.

From the number point of given presented in the previous data, we know the similarity measures between P_2 and P are the biggest or highest one.

Similarity measure between $P_i(i = 1,2,3)$ and P

Similarity Measure	(P_1, P)	(P_2, P)	(P_3, P)
$S_1(P_i, P)$	0.7180	0.8487	0.7589
$S_2(P_i, P)$	0.7679	0.8382	0.7625

Among the calculated measures, (P_2, P) is the largest one.

5. Conclusion

In the Beal's fuzzy set, the complement, necessity, possibility and various arithmetic operators are explained and various results related to properties of these operations have been demonstrated in this article. Furthermore, some distance and similarity measures over Beal's fuzzy set as developed and their validity are verified by suitable examples. An example of the application of similarity measures exposed in pattern recognition is investigated.

6. Future Work

1. Researchers may study the various hybrid structure of Beal's fuzzy sets with soft sets and rough sets. These can be identified and their uses in different fields of mathematics can be applied.

2. Also, the dice, cosine and cotangent similarity measures and weighted aggregation operators of Beal's fuzzy set can be studied and their applications in multicriterion decision making and multi attribute decision making problems can be survived.

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