Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

Subclass of univalent functions involving Raducanu-Orhan differential operator connected with Pascal distribution series

[1]Saravanan K, [2]Thirucheran M, [3]Bhuvaneswari Raja, [4]Stalin T, [5]Britto Manoj A

[1]Department of Mathematics, Dr Ambedkar Government Arts College, Chennai, 600 039, India. saravanandagac@gmail.com

[2]Department of Mathematics, L N Government College, Ponneri, Chennai, 601 204, India.
drthirucheran@gmail.com

[3]Department of Mathematics, Aalim Muhammed Salegh College of Engineering, Chennai, 600 055, India. rbrs1947@gmail.com

[4]Department of Mathematics, Vel Tech Rangarajan Dr Sagunthala R &D Institute of Science and Technology, Chennai, 600 062, India.
drstalint@veltech.edu.in, https://orcid.org/0000-0002-8735-3567

[5]Department of Advanced Computer Science and Engineering, Vignan's Foundation for Science, Technology & Research, Guntur-522213, Andhra Pradesh, India. brittomanoj@gmail.com

Article History:

Received: 13-11-2024

Revised: 17-12-2024

Accepted: 20-01-2025

Abstract:

Recent years have shown us how fascinating the univalent function is many new publications have been written in this field. Currently, operators of normalized analytic functions, differential and integral operators are highly sought after. Numerous researchers have examined and debated a great deal of material for the operators. This work introduces a new subclass $\mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)$ of the function class for univalent functions defined by the Raducanu-Orhan differential operator connected with pascal distribution series. Our goal in this work is to further our understanding and make inferences regarding the functions that are a part of these new subclass. Furthermore, the convexity of the subclass, growth and distortion, radius of starlike, extreme points, and integral means of inequalities are obtained. All this research was performed inside an open unit disc.

Keywords: Analytic function, univalent function, differential operator, subordination, coefficient inequality, starlike and convexity.

1. Introduction

Consider that the class $\mathcal{A}$ of univalent function has the following form

$$f(\xi) = \xi + \sum_{v=2}^{\infty} a_v \xi^v, \quad \xi \in \mathbb{U} := \{\xi \in \mathbb{C} : |\xi| < 1\}, \tag{1}$$

which is analytic in the unit disc $\mathbb{U}$, and

$$g(\xi) = \xi + \sum_{v=2}^{\infty} b_v \xi^v, \xi \epsilon \mathbb{U} \tag{2}$$

Then the convolution of (1) and (2) is represented by

$$(f * g)(\xi) = \xi + \sum_{v=2}^{\infty} a_v b_v \xi^v, \quad \xi \in \mathbb{U} \tag{3}$$

Let $f(\xi) \in K(\alpha)$ then $f(\xi)$ is convex of order $\alpha, (0 \leq \alpha < 1)$ in $\mathbb{U}$, iff

$$Re\left(\frac{\xi f''(\xi)}{f'(\xi} + 1\right) > \alpha, \xi \in \mathbb{U}.$$

Let $f(\xi) \in S^*(\alpha),$ then $f(\xi)$ is starlike of order $\alpha, (0 \leq \alpha < 1)$ in $\mathbb{U}$, iff

$$Re\left(\frac{\xi f'(\xi)}{f(\xi)}\right) > \alpha, \xi \in \mathbb{U}.$$

The class $K(\alpha)$ and $S^*(\alpha)$ introduced by Roberston [10]. After that many authors introduced the various subclass of starlike and convex functions connected with some differential operators.

The Schwarz function in $\mathbb{U}$, $\omega(\xi)$ exists if and only if $f(\xi)$ and $g(\xi)$ are analytic. It is our claim that $f(\xi)$ is subordinate to $g(\xi)$; that is $f(\xi) \prec g(\xi)$. In this case,

$\omega(0) = 0$ and $|\omega| < 1$ such that $f(\xi) = g(\omega(\xi))$ as proven, $f(\xi) \prec g(\xi)$ and

$f(\mathbb{U}) \subset g(\mathbb{U})$ Implied by $f(0) = g(0)$.

Ma and Minda [13] used the idea of subordination to create various sub classes of radii of convexity and starlikeness. To achieve this goal, a univalent function $\phi(\xi)$ is taken into consideration. This function is analytic and defined on $\mathbb{U}$ with a positive real portion, such that $\phi'(0) > 0$ and $\phi(0) = 1$.

For $f(\xi) \in \mathcal{A}$, Raducanu-Orhan [4] introduced the differential operator

$$\mathcal{Q}^{n}_{\delta,\mu} f(\xi) = \mathcal{Q}_{\delta,\mu}(\mathcal{Q}^{n-1}_{\delta,\mu}) = \xi + \sum_{\upsilon=2}^{\infty}[1 + (\upsilon - 1)(\delta - \mu + \upsilon\delta\mu)]^{n} a_{v}\xi^{\upsilon}, \quad (4)$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup 0$, $\mathbb{N} = \{1, 2, ...,\}$, $\mu, \delta \geq 0$, $\xi \in \mathbb{U}$.

Remark: $\mathcal{Q}^{n}_{\delta,0} = \mathcal{D}^{n}$ yields the operator of Al-Oboudi derivative [5], $\mathcal{Q}^{n}_{1,0} = \mathcal{D}^{n}$ is the Salagean derivative operator [7].

Recent studies have focused on a subclass of univalent functions associated with distribution series. These include the Borel, Pascal, Binomial, Poisson, Geometric, exponential, and generalized distributions as well as a generalized discrete probability distribution. In recent years, various sub class of univalent functions related to pascal distribution series have been studied by the following authors, B.A.Frasin et al.[2], S.Porwal [9], Anitha LakshmiNarayanan et al [1], G.Murugusundramoorthy [6], R.M.El-Ashwah, W.Y.Kota [10], T.Bulboaca and G.Murugusundramoorthy [12], B.A.Frasin et al. [3]. By examining the subclasses, researchers hope to gain a deeper understanding of the structure and behaviour of analytic functions, hence advancing their knowledge of complex analysis and its applications, which provides an extensive investigation of this area of study.

2. The Subclass $\mathcal{PQ}^{n,r}_{q,\delta,\mu}(\theta)$

The probabilities $(1 - q)^r$, $\frac{q^2 r(r+1)(1-q)^r}{2!}$, $\frac{qr(1-q)^r}{1!}$, $\frac{q^3 r(r+1)(r+2)(1-q)^r}{3!}, \ldots$ correspond to a variable x with values of 0,1,2, and 3, respectively, where q, and r are called the parameters, and thus

$$\mathcal{P}(\mathrm{x} = x) = \binom{n+r-1}{r-1} q^{x}(1-q)^{r}, \ x \in \{0,1,2,3,\ldots\} \quad (5)$$

According to S.M.El-Deepa et al.[11],the power series of equation (6) is examined, with its coefficients representing probabilities of the pascal distribution, that is

$$\mathcal{P}^{r}_{q}(\xi) = \xi + \sum_{v=2}^{\infty}\binom{v+r-2}{r-1} q^{v-1}(1-q)^{r}\xi^{v}, \xi \in \mathbb{U}, r \geq 1, 0 \leq q \leq 1 \quad (6)$$

And the linear opeerator $\mathcal{D}^{r}_{q}: \mathcal{A} \to \mathcal{A}$ is defined by

$$\mathcal{D}_q^r(f(\xi)) = \mathcal{P}_q^r * f(\xi) = \xi + \sum_{v=2}^{\infty} \binom{v+2-r}{r-1} q^{v-1}(1-q)^r a_v \xi^v \ , \xi \in \ \mathbb{U} \tag{7}$$

By using the convolution (Hadamard product) of two equations (4) and (7), the linear operator $\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi) : \ \mathcal{A} \to \mathcal{A}$ is defined by

$$\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi) = \xi + \sum_{v=2}^{\infty} a_v c_v \xi^v \ , \tag{8}$$

Where $c_v = [1 + (v-1)(\delta - \mu + v\delta\mu)]^n \binom{v+r-2}{r-1} q^{v-1}(1-q)^r$. the new subclass is defined in the following definitions:

Definition 2.1. Let $\mathcal{PD}_{q,\delta,\mu}^{n,r}(\theta)$ represents a class of f in $\mathcal{A}$. Then

$$\mathrm{Re}\left(1 + \frac{1}{b}\left(\frac{\xi(\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi))'}{\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi)} - 1\right)\right) > \theta \tag{9}$$

Where r ≥1, $0 \le q \le 1, \mu,\ \delta \ge 0, n \in \mathbb{N}_0$ $0 \le \theta < 1$, $b \in \mathbb{C} - \{0\}$, and $\xi \in \mathbb{U}$.

Theorem 2.2 (Coefficient Inequalities)

Let (1) define $f(\xi) \in \mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)$.

Then $\sum_{v=2}^{\infty} \phi_v \, c_v \, |a_v| \le (1-\theta)|b|,$ (10)

Where $\phi_v = |1 - b - v + \theta b|$, $c_v = [1 + (v-1)(\delta - \mu + v\delta\mu)]^n \binom{v+r-2}{r-1} q^{v-1}(1-q)^r$,

$r \ge 1, 0 \le q \le 1, \mu, \delta \ge 0, n \in \mathbb{N}_0, 0 \le \theta < 1, b \in \mathbb{C} - \{0\}$and $\xi \in \mathbb{U}$.

Proof:

Let

$$F(z) = 1 + \frac{1}{b}\left(\frac{\xi(\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi))'}{\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi)} - 1\right) - \theta,$$

$$= 1 + \left(\frac{\xi(\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi))'}{\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi)} - \frac{1}{b}\right) - \theta$$

$$= 1 + \left(\frac{\xi\left(\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi)\right)' - b\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi) - \theta b\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi)}{b\mathcal{PD}_{q,\delta,\mu}^{n,r} f(\xi)}\right)$$

By the condition of the class, $F(z) \prec \frac{1+z}{1-z}$.

A schwarz function $\omega(z)$ exist, and $\omega(0) = 0$ as $F(z) = \frac{1+\omega(z)}{1-\omega(z)}$, where $|\omega| < 1$.

$$\therefore \omega(z) = \frac{F(z)-1}{F(z)+1}.$$

We know that $|\omega(z)| = \left|\frac{F(z)-1}{F(z)+1}\right| < 1$.

Then

$$\left|\frac{F(z)-1}{F(z)+1}\right| = \left|\frac{\xi\left(\mathcal{PD}_{q,\delta,\mu}^{n,r}f(\xi)\right)' - (1+\theta b)\mathcal{PD}_{q,\delta,\mu}^{n,r}f(\xi)}{\xi\left(\mathcal{PD}_{q,\delta,\mu}^{n,r}f(\xi)\right)' - (1+\theta b-2b)\mathcal{PD}_{q,\delta,\mu}^{n,r}f(\xi)}\right|$$

$$= \left|\frac{z+\sum_{v=2}^{\infty} vc_v a_v z^v - (1+\theta b)z - \sum_{v=2}^{\infty}(1+\theta b)c_v a_v z^v}{z+\sum_{k=2}^{\infty} vc_v a_v z^v - (1+\theta b-2b)z - \sum_{v=2}^{\infty}(1+\theta b-2b)c_v a_v z^v}\right|$$

$$= \left|\frac{\theta b + \sum_{v=2}^{\infty}(1+\theta b - v)c_v a_v z^{v-1}}{(2-\theta)b + \sum_{v=2}^{\infty}(1+\theta b-2b-v)c_v a_v z^{v-1}}\right|$$

$$\leq \left|\frac{\theta|b| + \sum_{v=2}^{\infty}|(1+\theta b - v||c_v a_v||z^{v-1}|}{(2-\theta)|b| - \sum_{v=2}^{\infty}|(1+\theta b-2b-v)||c_v a_v||z^{v-1}|}\right|.$$

Which is bounded by 1, if

$$\theta|b| + \sum_{v=2}^{\infty}|(1+\theta b - v)|c_v|a_v| \leq (2-\theta)|b| - \sum_{v=2}^{\infty}|(1+\theta b - 2b - v)|\, c_v|a_v|.$$

$$\sum_{v=2}^{\infty}|(1+\theta b - b - v)|c_v|a_v| \leq (1-\theta)|b|.$$

Hence equation (10) holds.

Corollary 2.3

Let $f \in \mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)$ then we have $a_v \leq \frac{(1-\theta)|b|}{\phi_v c_v}$ and $f(\xi) = \xi + \frac{(1-\theta)|b|}{\phi_v c_v}\xi^v$, $v = 2,3,4,\ldots$ (11)

equals itself.

The function $f \epsilon \mathcal{A}$ is the subclass $\overline{\mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)} \subset \mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)$, which we define. The extreme points of the subclass $\overline{\mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)}$ are now determined.

Theorem 2.4(Extreme points)

Let $f1(\xi) = \xi, f_v(\xi) = \xi + \sum_{v=2}^{\infty}\eta_v \frac{(1-\theta)|b|}{\phi_v c_v}\xi^v, v \geq 2.$ Then $f \epsilon \mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)$ strictly if

$f(\xi) = \sum_{v=1}^{\infty}\eta_v f_v(\xi)$, Where $\eta_v > 0$ and $\sum_{v=1}^{\infty}\eta_v = 1$.

Proof:

Let

$$f(\xi) = \sum_{v=1}^{\infty}\eta_v f_v(\xi)$$

$$= \xi + \sum_{v=2}^{\infty}\eta_v \frac{(1-\theta)|b|}{\phi_v c_v}\xi^v$$

$$= \sum_{v=2}^{\infty}\eta_v \frac{(1-\theta)|b|}{\phi_v c_v}\phi_v c_v$$

$$= (1-\theta|b|\sum_{v=1}^{\infty}\eta_v$$

$= (1-\theta\,)|b|(1-\eta_1) < (1-\theta)|b|,$

Which shows that $f \epsilon\, \mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)$.

Conversely, suppose that $f \epsilon\, \overline{\mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)}$. since $|a_v| \le \frac{(1-\theta)|b|}{\phi_v c_v}$, $v = 2,3, ..$

Let

$$\eta_v \le \frac{\phi_v c_v}{(1-\theta)|b|}\ , \eta_1 = 1-\sum_{v=2}^{\infty}\eta_v.$$

Then we obtain $f(\xi) = \sum_{v=1}^{\infty}\eta_v f_v(\xi)$.

Definition 2.5.(Little wood subordination theorem [8])

Considering that f and g in $\mathbb{U}$ are analytic and that $f(\xi) \prec g(\xi)$, then

$\int_0^{2\pi}|f(\xi)|^{\mu}\, d\theta \le \int_0^{2\pi}|g(\xi)|^{\mu}\, d\theta, \mu > 0,$ and $\xi = re^{i\theta}, 0 < r < 1$.

Theorem 2.6(Integral means of inequalities)

If $f \epsilon\, \mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)$ and suppose that $g(\xi) = \xi + \frac{(1-\theta)|b|\varepsilon_v}{\phi_v c_v}\xi^v, v = 2,3, \dots, |\varepsilon_v| = 1.$

if $\omega(\xi)$ is real it is given by $(\omega(\xi))^{v-1} = \frac{\phi_v c_v}{(1-\theta)|b|\varepsilon_v}\sum_{v=2}^{\infty} a_v \xi^{v-1}$.

Then$\int_0^{2\pi}|f(\xi)|^{\mu}\, d\theta \le \int_0^{2\pi}|g(\xi)|^{\mu}\, d\theta$, for $\xi = re^{i\theta}, 0 < r < 1, \mu > 0$.

We need to demonstrate that to finish the theorem

$$\int_0^{2\pi}\left|1+\sum_{v=2}^{\infty}a_v\,\xi^{v-1}\right|^{\mu} d\theta \le \int_0^{2\pi}\left|1+\frac{(1-\theta)|b|\,\epsilon_v}{\phi_v c_v}\,\xi^{v-1}\right|^{\mu} d\theta.$$

The Littlewood subordination theorem can be used to demonstrate that

$$1+\sum_{v=2}^{\infty}a_v\xi^{v-1} \prec 1+\frac{(1+\theta)b\epsilon_v}{\phi_v c_v}\xi^{v-1}.$$

Let

$$1+\sum_{v=2}^{\infty}a_v\xi^{v-1} \prec 1+\frac{(1+\theta)|b|\epsilon_v}{\phi_v c_v}(\omega(\xi))^{v-1}$$

Therefore

$$(\omega(\xi))^{v-1} = \frac{\phi_v c_v}{(1-\theta)|b|\epsilon_v}\sum_{v=2}^{\infty}a_v\xi^{v-1}$$

Hence ω(0)=0.

Furthermore, if $f \epsilon \mathcal{A}$ satisfy $\phi_v c_v \leq (1-\theta)|b|$.

$$|\omega(\xi)|^{v-1} = \left|\frac{\phi_v c_v}{(1-\theta)|b|\epsilon_v}\right| \sum_{v=2}^{\infty} |a_v| \left|\xi^{v-1}\right| \leq |\xi| < 1.$$

Theorem 2.7(Convex of order θ)

Let $f \epsilon\ \mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)$. Then f is convex of order θ in $|\xi| < R_3$, w*h*ere

$$R_3 := \inf \left(\frac{(1-\theta)\phi_v c_v}{v(v-\theta)(1-\theta)|b|}\right)^{\frac{1}{v-1}}, (v \geq 2) \tag{12}$$

Proof:

If $|\xi| < R_3$ and the inequality (12) are valid, it is demonstrated that $\left|\frac{\xi f''(\xi)}{f'(\xi)}\right| \leq 1-\theta.$ (13)

It is adequate to show that

$$|\xi| \leq \left(\frac{(1-\theta)\phi_v c_v}{v(v-\theta)(1-\theta)|b|}\right)^{\frac{1}{v-1}}, (v \geq 2).$$

From (13), we obtain

$$\left|\frac{\sum_{v=2}^{\infty} v(v-1)a_v\xi^{v-1}}{1+\sum_{v=2}^{\infty} va_v\xi^{v-1}}\right| \leq 1-\theta.$$

$$\sum_{v=2}^{\infty}(v(v-1)a_v\xi^{v-1} \leq 1 + \sum_{v=2}^{\infty} va_v\xi^{v-1} - \theta - \theta\sum_{v=2}^{\infty} va_v|\xi|^{v-1}$$

$$\sum_{v=2}^{\infty}(v^2-\theta v)a_v|\xi|^{v-1} \leq (1-\theta)\ .$$

$$|\xi| \leq \left(\frac{1-\theta}{(v^2-\theta v)a_v}\right)^{\frac{1}{v-1}}\ , (v \geq 2).$$

$$|\xi| \leq \left(\frac{(1-\theta)\phi_v c_v}{v(v-\theta)(1-\theta)|b|}\right)^{\frac{1}{v-1}}\ .$$

Theorem 2.8 (Starlike of order θ)

Let $f \epsilon\ \mathcal{P}\ \mathcal{Q}_{q,\delta,\mu}^{n,r}(\theta)$. Then f is starlike of order θ in $|\xi| < R_2$, where

$$R_2 := \inf \left(\frac{(1-\theta)\phi_v c_v}{(v-\theta)(1-\theta)|b|}\right)^{\frac{1}{v-1}}, (v \geq 2) \tag{14}$$

Proof:

If $|\xi|<R_2$ and the inequality (14) are valid it is demonstrated that

$$\left|\frac{\xi f'(\xi)}{f(\xi)} - 1\right| \leq 1-\theta \tag{15}$$

It is adequate to show that

$$|\xi| \leq \left(\frac{(1-\theta)\phi_v c_v}{(v-\theta)(1-\theta)|b|}\right)^{\frac{1}{v-1}}, (v \geq 2)$$

From (15), we obtain

$$\left|\frac{\xi + \sum_{v=2}^{\infty} v a_v \xi^v}{\xi + \sum_{v=2}^{\infty} a_v \xi^v} - 1\right| \leq 1 - \theta$$

$$\sum_{v=2}^{\infty}(1 - v)a_v|\xi|^{v-1} \leq (\theta - 1)(1 + \sum_{v=2}^{\infty} a_v|\xi|^{v-1})$$

$$\sum_{v=2}^{\infty}(\theta - v)a_v|\xi|^{v-1} \leq (\theta - 1)\xi.$$

$$|\xi| \leq \left(\frac{(1-\theta)\phi_v c_v}{(v-\theta)(1-\theta)|b|}\right)^{\frac{1}{v-1}}, (v \geq 2).$$

Theorem 2.9(Close-to convex of order θ)

Let $f \epsilon \mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)$, then f is close-to-convex of order θ $(0 \leq \theta < 1)$ in the disc $|\xi| < R_3$, were

$$R_3 := \inf\left(\frac{(1-\theta)\phi_v c_v}{v(1-\theta)|b|}\right)^{\frac{1}{v-1}}, (v \geq 2) \tag{16}$$

Proof:

If $|\xi| < R_3$ and the inequality (16) are valid, it is demonstrated that

$$|f'(\xi) - 1| < 1 - \theta. \tag{17}$$

It is adequate to show that

$$|\xi| \leq \left(\frac{(1-\theta)\phi_v c_v}{(v)(1-\theta)|b|}\right)^{\frac{1}{v-1}}, (v \geq 2)$$

From (17), we obtain

$$\left|1 + \sum_{v=2}^{\infty} v a_v \xi^{v-1} - 1\right| < 1 - \theta$$

$$\sum_{v=2}^{\infty} v a_v |\xi|^{v-1} < 1 - \theta$$

$$|\xi|^{v-1} < \frac{1-\theta}{v a_v}$$

$$|\xi| \leq \left(\frac{(1-\theta)\phi_v c_v}{(v)(1-\theta)|b|}\right)^{\frac{1}{v-1}}, (v \geq 2).$$

Theorem 2.10 (Growth theorem)

Let $f(\xi = \xi + \sum_{v=2}^{\infty}|a_v|\xi^v$ belongs to class $f \epsilon \mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)$. Then for $|\xi| = r^*$, we *h*ave

$$r^* - \frac{(1-\theta)|b|}{|(\theta b-b-1)|c_2} r^{*2} \leq |f(\xi)| \leq r^* + \frac{(1+\theta)|b|}{|(\theta b-b-1)|c_2} r^{*2}, \tag{18}$$

Where $c_2 = (1 + (\delta - \mu + 2\delta\mu))^n rq(1-q)^r$.

Proof:

Since

$$a_v \leq \frac{(1-\theta)|b|}{\phi_v c_v}$$

$$f(\xi) = \xi + \sum_{v=2}^{\infty} a_v \xi^v$$

$$|f(\xi)| \leq r^* + \sum_{v=2}^{\infty} a_v (r^*)^v$$

$$|f(\xi)| \leq r^* + \left(\sum_{v=2}^{\infty} \frac{(1-\theta)|b|}{\phi_v c_v}\right)(r^*)^v .$$

$$|f(\xi)| \leq r^* + \left(\frac{(1-\theta)|b|}{\phi_v c_v}\right)(r^*)^v .$$

$$|f(\xi)| \leq r^* + \left(\frac{(1-\theta)|b|}{|(\theta b-b-1)|c_2}\right)(r^*)^2.$$

Similarly,

$$|f(\xi)| \geq r^* - \left(\frac{(1-\theta)|b|}{|(\theta b-b-1)|c_2}\right)(r^*)^2.$$

Theorem 2.11(Distortion theorem)

Let $f(\xi) = \xi + \sum_{v=2}^{\infty}|a_v|\xi^v$ belong to class $f \epsilon \mathcal{PQ}_{q,\delta,\mu}^{n,r}(\theta)$, then for $|\xi| = r^*$, we have

$$1 - \frac{2(1-\theta)|b|}{|(\theta b-b-1)|c_2} r^* \leq |f'(\xi)| \leq 1 + \frac{2(1-\theta)|b|}{|(\theta b-b-1)|c_2} r^* \tag{19}$$

Proof. Since

$$a_v \leq \frac{(1-\theta)|b|}{\phi_v c_v}$$

$$f(\xi) = \xi \sum_{v=2}^{\infty} a_v \xi^v$$

$$|f\xi)| \leq 1 + \sum_{v=2}^{\infty} v a_v |\xi|^{v-1}$$

$$|f'\xi)| \leq 1 + \frac{2(1-\theta)|b|}{|(\theta b-b-1)|c_2} r^*$$

Similarly

$$|f'\xi)| \geq 1 + \frac{2(1-\theta)|b|}{|(\theta b - b - 1)|c_2} r^*.$$

3. Conclusion

In conclusion, this study has provided a thorough examination of the coefficient challenges inherent in the newly defined subclass of univalent functions in U, as outlined in Definition (2.1). Key properties such as the radius of starlikeness, extreme points, development and distortion, convexity, and integral means of inequalities were explored, enhancing our understanding of the subclass's behaviour. The findings contribute valuable insights into the composition and properties of analytic functions. Moreover, the study suggests promising directions for future research, including the analysis of Hankel determinants for orders between two and three, as well as further investigations and estimates related to the Fekete-Szegö functional problem. These avenues present exciting opportunities for advancing the field.

References

[1] Anitha Lakshminarayanan, Ramachandran Chellakutti, Bul boaca Teodor, Certain subclasses of spirallike univalent functions related to Poisson distribution series, Turk- ish J. Math., 45(3), 1449-1458, (2021).

[2] B. A. Frasin, On certain subclasses of analytic functions associated with Poisson distribution series, Acta Univ. Sapientiae Math., 11(1), 78-86, (2019).

[3] B.A. Frasin, G. Murugusundaramoorthy, Sibel Yalçin, Subclass of analytic functions associated with Pascal distribution series, SERIES III - MATEMATICS, INFOR- MATICS, PHYSICS, 13(62), 521-528, (2021).

[4] D. Raducanu and H. Orhan, Subclass of analytic functions defined by a generalized differential operator, International Journal of Mathematics and Mathematical Anal- ysis, 4(1-4), 1-15, (2010).

[5] F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, International Journal Of Mathematics and Mathematical Sciences, 2004(27), 1429-1436, (2004).

[6] G. Murugusundaramoorthy, Subclasses of starlike and convex functions involving Poisson distribution series, Afr. Mat., 28(7-8), 1357-1366, (2017).

[7] G. Salagean, Subclasses of Univalent Functions, Lecture Notes in Maths, Springer- Verlag, Berlin, 1013, 362-372, (1983).

[8] J. E. Littlewood, On inequalities in the theory of functions, Proceedings of London Mathematical Society, 23(1), 481-519, (1925).

[9] Porwal Saurabh, Kumar Manish, A unifed study on star like and convex functions associated with Poisson distribution series, Afr. Mat., 27(5-6), 1021 -1027, (2016).

[10] R. M. El-Ashwah, W. Y. Kota, Some condition on a Poisson distribution series to be in subclasses of univalent functions, Acta Universitatis Apulensis, 51, 89-103, (2017).

[11] S. M. El-Deeb, T. Bulboaca and J. Dziok, Pascal Distribution Series Connected with Certain Subclasses of Univalent Functions, KYUNGPOOK Math. J., 59, 301-314, (2019).

[12] T. Bulboaca and G. Murugusundaramoorthy, Univalent functions with positive co-efficient involving Pascal distribution series, Commun. Korean Math. Soc., 35(3), 867-877, (2020).

[13] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, proc. of the conf. on complex analysis, 157-169, (1994).