Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

Investigating a Novel Stability Results of Generalized Alternate Cubic Functional Equations: Classical Method for Banach Spaces and Direct -Fixed Point Approaches for Fuzzy Normed Spaces

P. Agilan [1]*, V. Vijayan [2], M. Sophia [3] V.Banu Priya [4]

[1]Department of Mathematics, St.Joseph's College of Engineering, OMR, Chennai - 600 119, TamilNadu, India.

[2]Department of Electronics and Instrumentation Engineering, St.Joseph's College of Engineering, OMR, Chennai - 600 119, TamilNadu, India. E-mail: vinvpn@gmail.com.

[3] Department of Mathematics, SIMATS Engineering, Saveetha Nagar, Thandalam, Kanchipuram-Chennai Rd, Chennai- 602105, TamilNadu, India. E-mail: sophia.raj2005@gmail.com.

[4]Department of Mathematics, R.M.K College of Engineering and Technology, Kavaraipettai - 601 206, TamilNadu, India. E-mail:spriya.maths@gmail.com.

*Corresponding Author: agilram@gmail.com.

Article History:

Received: 10-11-2024

Revised: 16-12-2024

Accepted: 11-01-2025

Abstract:

This paper explores novel stability results for generalized alternate cubic functional equation(Fun Eq) using two distinct analytical frameworks: the classical method for Banach spaces and the direct and fixed point approaches for fuzzy normed spaces. The study examines the stability behavior of the generalized alternate cubic functional equation, focusing on how small deviations from exact solutions influence the overall stability in different normed environments. In Banach spaces, the classical approach is applied to derive conditions for Hyers-Ulam stability, providing insight into the equation's behavior under small perturbations. For fuzzy normed spaces, both direct and fixed point methods are employed to account for the inherent uncertainties and fuzziness in the normed structure, offering a more flexible stability analysis. The results obtained highlight the differences and advantages of each approach, contributing to the broader understanding of functional equations in both deterministic and fuzzy frameworks. These findings have potential applications in various mathematical and applied fields, where both precise and imprecise data structures are considered.

Keywords: Banach Spaces, Fuzzy Normed Spaces, Cubic Functional Equations, Ulam - Hyers Stability, Fixed point.

1 Introduction

The Ulam-Hyers-Rassias stability deals with the stability of Fun Eq, which is a branch of mathematical analysis. Specifically, it focuses on determining under what conditions an approximate solution of a Fun Eq remains close to the exact solution. The stability concept was initiated by Stanislaw Ulam in 1940 [1] when he asked whether approximate homomorphisms on groups could be approximated by true homomorphisms. Later, in the 1940s and 1950s, Donald H. Hyers [2] and Th.M. Rassias [3] extended Ulam's work to Fun Eq in Banach spaces. The Ulam-Hyers-Rassias stability theorem provides conditions under which a functional equation approximately satisfies the equation. For further developments and the subsequent contributions by T. Aoki, P. Gavruta, J.M. Rassias, Isac and others [4, 5, 6, 7, 8, 9, 10]. It's of great significance in many areas of mathematics, including functional analysis, operator theory, and mathematical physics. In practical terms, this stability theorem has applications in various fields, such as numerical analysis, optimization, control theory, and signal processing, where it's crucial to understand how small errors in input data or parameters affect the output of a mathematical model or system.

The concept of Hyers-Ulam stability has had a significant impact across various mathematical domains. Initially introduced in the context of Fun Eq, Hyers-Ulam stability focuses on whether small deviations from a functional equation still allow for an approximate solution that is close to an exact solution. This principle has since been applied to numerous areas such as: Differential Equations[11, 12]: Hyers-Ulam stability helps assess the stability of differential equations, especially in determining whether solutions to perturbed equations remain close to the solutions of the original equation. Integral Equations[13, 14]: In integral equations, the stability concept provides a framework to ensure that approximate solutions remain consistent even under perturbations. Operator Theory[15, 16]: Hyers-Ulam stability has been extended to operator equations, aiding in the analysis of bounded linear operators and their robustness under small changes. Approximation Theory[17]: It plays a role in approximation theory by ensuring that near solutions of approximation problems can still yield good approximations, thus enhancing the reliability of numerical methods. Control Theory[18, 19]: In systems governed by control equations, Hyers-Ulam stability contributes to the robustness analysis, determining how systems behave when subject to small external disturbances. Overall, Hyers-Ulam stability provides a foundational tool to understand the resilience of mathematical models in various applied and theoretical settings, ensuring that minor errors or perturbations do not drastically alter solutions.

In fuzzy normed spaces, stability results are typically established using fixed-point methods or direct analytical approaches. The fuzzy nature of the space allows for handling vagueness or uncertainty in the norm, which is critical for real-world applications where data may not always be exact. The fixed-point method, for instance, is a powerful tool used to prove the existence of a stable cubic mapping, which satisfies the functional equation under these conditions [20, 21, 22, 23, 24]. Recent studies show that fuzzy normed spaces provide a more flexible framework for analyzing the stability of functional equations. In this setting, the stability of cubic functional equations is guaranteed even when deviations occur, provided the system adheres to specific constraints. This makes fuzzy stability particularly relevant in fields like applied mathematics, economics, and engineering, where imprecision often exists. By focusing on these modern methods, researchers have successfully derived new stability results for cubic equations, contributing to both theoretical mathematics and practical problem-solving in uncertain environments

The study focuses on a generalized alternate cubic functional equation, which is a more intricate form compared to traditional cubic equations. Exploring the stability of such equations in Banach and fuzzy Banach spaces is essential because these spaces are widely used in various branches of functional analysis, optimization, and differential equations. Recently Agilan et.al exploring the stability results in various additive functional equation through various normed spaces such as [25, 26, 27, 28, 29, 30, 31, 32].

In this paper, the authors investigate the generalized Ulam-Hyers stability of a Alternate cubic functional equation

$$\mathscr{F}(\Re^a v + w) \pm \Re^b \mathscr{F}(v - \Re^a w) = \left(\frac{\Re^a\left(1 \pm \Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w) + \mathscr{F}(v-w)]$$

$$+\left(\frac{\Re^a\left(\Re^a \mp \Re^b\right)}{2}\right)[\mathscr{F}(v+w) - \mathscr{F}(v-w)]$$

$$+\left(\Re^{2a} - 1\right)\left[\left(\Re^a \mp \Re^b\right)\mathscr{F}(v) \mp \left(\Re^{a+b} \pm 1\right)\mathscr{F}(w)\right] \quad (1)$$

where $\Re, a, b$ are integers with $\Re \neq 0, \pm 1$ and $a \neq b \neq 0, \pm 1$ in Banach and fuzzy Banach spaces.

Lemma 1.1 let us consider X and Y be real vector spaces. An odd function satisfies the functional equation

$$\mathscr{F}(mv + w) + \mathscr{F}(mv - w) = m\mathscr{F}(v + w) + m\mathscr{F}(v - w) + 2(m^3 - m)\mathscr{F}(v) \quad (2)$$

for all $v, w \in X$ if satisfies the Fun Eq(1) for all $v, w \in X$.

Proof. Assume $f: X \to Y$ satisfies the functional equation (2). Letting $v = w = 0$ in (2), we get $\mathscr{F}(0) = 0$. Setting in (2), we have $\mathscr{F}(-w) = -\mathscr{F}(w)$ and $w = 0$ we get

$$\mathscr{F}(mv) = m^3\mathscr{F}(v) \quad (3)$$

for all $v \in X$. In particular replace m by $\Re^a$ in (2), we get

$$\mathscr{F}(\Re^a v + w) + \mathscr{F}(\Re^a v - w) = \Re^a\mathscr{F}(v + w) + \Re^a\mathscr{F}(v - w) + 2(\Re^{3a} - \Re^a)\mathscr{F}(x) \quad (4)$$

for all $v, w \in X$. Replace w by $\Re^a w$ in (4), we obtain

$$\mathscr{F}(\Re^a(v + w)) + \mathscr{F}(\Re^a(v - w)) = \Re^a[\mathscr{F}(v + \Re^a w) + \mathscr{F}(v - \Re^a w)] + 2(\Re^{3a} - \Re^a)\mathscr{F}(v) \quad (5)$$

for all $v, w \in X$. Using (3) in (5), we have

$$\Re^{3a}[\mathscr{F}(v + w) + \mathscr{F}(v - w)] = \Re^a[\mathscr{F}(v + \Re^a w) + \mathscr{F}(v - \Re^a w)] + 2(\Re^{3a} - \Re^a)\mathscr{F}(v) \quad (6)$$

for all $v, w \in X$. Divide the above equation by $\Re^a$, we get

$$\mathscr{F}(v + \Re^a w) + \mathscr{F}(v - \Re^a w) = \Re^{2a}[\mathscr{F}(v + w) + \mathscr{F}(v - w)] - 2(\Re^{2a} - 1)\mathscr{F}(v) \quad (7)$$

for all $v, w \in X$. Replace v by w and w by v in (7) and using oddness of C, we obtain

$$\mathscr{F}(\Re^a v - w) = \mathscr{F}(\Re^a v + w) - \Re^{2a}[\mathscr{F}(v + w) - \mathscr{F}(v - w)] + 2(\Re^{2a} - 1)\mathscr{F}(w) \quad (8)$$

for all $v, w \in X$. Substitute (8) in (4), we get

$$\mathscr{F}(\Re^a v + w) = \frac{\Re^a}{2}[\mathscr{F}(v + w) + \mathscr{F}(v - w)] + \frac{\Re^{2a}}{2}[\mathscr{F}(v + w) - \mathscr{F}(v - w)]$$

$$+(\Re^{3a} - \Re^a)\mathscr{F}(v) - (\Re^{2a} - 1)\mathscr{F}(w) \quad (9)$$

for all $v, w \in X$. Replace v by $-w$ and w by v in (9) and using oddness of C, we obtain

$$\mathscr{F}(v-\Re^{a}w)=\frac{\Re^{a}}{2}[\mathscr{F}(v-w)-\mathscr{F}(v-w)]+\frac{\Re^{2a}}{2}[\mathscr{F}(v-w)+\mathscr{F}(v+w)]$$
$$-(\Re^{3a}-\Re^{a})\mathscr{F}(w)-(\Re^{2a}-1)\mathscr{F}(v) \quad (10)$$

for all $v,w\in X$. Both side multiply by $\Re^{b}$ in (10), we get

$$\Re^{b}\mathscr{F}(v-\Re^{a}w)=\frac{\Re^{a+b}}{2}[\mathscr{F}(v-w)-\mathscr{F}(v-w)]+\frac{\Re^{2a+b}}{2}[\mathscr{F}(v-w)+\mathscr{F}(v+w)]$$
$$-(\Re^{2a+b}-\Re^{b})\mathscr{F}(v)-(\Re^{3a+b}-\Re^{a+b})\mathscr{F}(w) \quad (11)$$

for all $v,w\in X$. Adding (9) and (11), we arrive

$$\mathscr{F}(\Re^{a}v+w)+\Re^{b}\mathscr{F}(v-\Re^{a}w)=\left(\frac{\Re^{a}\left(1+\Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w)+\mathscr{F}(v-w)]$$
$$+\left(\frac{\Re^{a}\left(\Re^{a}-\Re^{b}\right)}{2}\right)[\mathscr{F}(v+w)-\mathscr{F}(v-w)]$$
$$+\left(\Re^{2a}-1\right)\left[\left(\Re^{a}-\Re^{b}\right)\mathscr{F}(v)-\left(\Re^{a+b}+1\right)\mathscr{F}(w)\right] \quad (12)$$

for all $v,w\in X$. Subtracting (9) and (11), we arrive

$$\mathscr{F}(\Re^{a}v+w)-\Re^{b}\mathscr{F}(v-\Re^{a}w)=\left(\frac{\Re^{a}\left(1-\Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w)+\mathscr{F}(v-w)]$$
$$+\left(\frac{\Re^{a}\left(\Re^{a}+\Re^{b}\right)}{2}\right)[\mathscr{F}(v+w)-\mathscr{F}(v-w)]$$
$$+\left(\Re^{2a}-1\right)\left[\left(\Re^{a}+\Re^{b}\right)\mathscr{F}(v)+\left(\Re^{a+b}-1\right)\mathscr{F}(w)\right] \quad (13)$$

for all $v,w\in X$. Combining both (12) and (13) we arrive (1).

2 Banach space stability results direct method

Theorem 2.1 Assume X be normed linear space and Y be Banach space. Suppose that the function $\mathscr{F}$: X → Y satisfice

$$\|D\mathscr{F}(v,w)\|\le\mathfrak{Q}(v,w) \quad (1)$$

$\forall v,w\in X$ and Let $\mathfrak{Q}: X\times X\to[0,\infty)$ be a function such that

$$\lim_{n\to\infty}\frac{\mathfrak{Q}\left(\Re^{an}v,\Re^{an}w\right)}{\Re^{3an}}=0 \quad (2)$$

$\forall v,w\in X$, then ∃ Cubic map $\mathscr{F}: X\to Y$ with the the FE (??) and

$$\|\mathscr{F}(v)-\mathscr{F}(v)\|\le\frac{1}{\Re^{3a}}\sum_{q=1}^{\infty}\frac{\mathfrak{Q}(\Re^{a(q-1)}v,0)}{\Re^{3aq}} \quad (3)$$

$\forall v\in X$. Let $\mathscr{F}(v)$ is defined as

$$\mathscr{F}(v)=\lim_{n\to\infty}\frac{\mathscr{F}(\Re^{an}u)}{\Re^{3an}} \quad (4)$$

$\forall v\in X$.

Proof. Considering (v, w) by (0,0) in (1), then we have $\mathscr{F}(v) = 0$. Switching (v, w) by $(v, 0)$ in (1), we get

$$\left\|\mathscr{F}(\Re^{a} v) - \Re^{3a}\mathscr{F}(v)\right\| \leq \mathfrak{Q}(v, 0) \tag{5}$$

$\forall v \in X$, We replace v by $\Re^{a(q-1)} v$ (for $q \in \mathbb{N}$ and $q \geq 1$) in (5) , and we obtain

$$\left\|\mathscr{F}(\Re^{aq} v) - \Re^{3a}\mathscr{F}(\Re^{a(q-1)} v)\right\| \leq \mathfrak{Q}\left(\Re^{a(q-1)} v, 0\right)$$

$\forall v \in X$. By multiplying both sides of the aforementioned inequality by $\frac{1}{\Re^{3aq}}$, we get the consequence of adding n inequalities.

$$\sum_{q=1}^{n} \frac{1}{\Re^{3aq}} \left\|\mathscr{F}(\Re^{aq} v) - \Re^{3aq}\mathscr{F}(\Re^{a(q-1)} v)\right\| \leq \sum_{q=1}^{n} \frac{\mathfrak{Q}\left(\Re^{a(q-1)} v, 0\right)}{\Re^{3aq}}$$

Making use of the triangle inequality

$$|A + B| \leq |A| + |B|$$

After simplifying, we get at the left side of the inequality.

$$\left\|\frac{1}{\Re^{3an}}\mathscr{F}(\Re^{an} v) - \mathscr{F}(v)\right\| \leq \sum_{q=1}^{n} \frac{\mathfrak{Q}\left(\Re^{a(q-1)} v, 0\right)}{\Re^{3aq}} \tag{6}$$

Since

$$\sum_{q=1}^{n} \frac{\mathfrak{Q}\left(\Re^{a(q-1)} v, 0\right)}{\Re^{3aq}} \leq \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q-1)} v, 0\right)}{\Re^{3aq}}$$

the inequality (6) yields

$$\left\|\frac{1}{\Re^{3an}}\mathscr{F}(\Re^{an} v) - \mathscr{F}(v)\right\| \leq \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q-1)} v, 0\right)}{\Re^{3aq}}$$

$\forall v \in X$. It will be proven by induction that (6) exists $\forall$ $\mathbb{N}$.

Here $m > n > 0$, then $m - n \in \mathbb{N}$ and let n by $m - n$ in (6), then

$$\left\|\frac{1}{\Re^{3a(m-n)}}\mathscr{F}\left(\Re^{a(m-n)} v\right) - \mathscr{F}(v)\right\| \leq \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q-1)} v, 0\right)}{\Re^{3aq}} \tag{7}$$

which is

$$\left\|\frac{1}{\Re^{3am}}\mathscr{F}\left(\Re^{a(m-n)} v\right) - \frac{1}{\Re^{3an}}\mathscr{F}(v)\right\| \leq \frac{1}{\Re^{3an}} \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q-1)} v, 0\right)}{\Re^{3aq}} \tag{8}$$

$\forall u \in X$. Interchanging u by $\Re^{an} v$ in (8), we obtain

$$\left\|\frac{1}{\Re^{3am}}\mathscr{F}(\Re^{am} v) - \frac{1}{\Re^{3an}}\mathscr{F}(\Re^{an} v)\right\| \leq \frac{1}{\Re^{3an}} \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q+n-1)}, 0\right)}{\Re^{3aq}} \tag{9}$$

Since

$$\lim_{n\to\infty} \frac{1}{\Re^{3an}} = 0$$

and hence from (9), we obtain

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

$$\lim_{n\to\infty}\left\|\frac{1}{\Re^{3am}}\mathscr{F}(\Re^{am}v)-\frac{1}{\Re^{3an}}\mathscr{F}(\Re^{an}v)\right\|=0$$

Finally

$$\left\{\frac{\mathscr{F}(\Re^{an}v)}{\Re^{3an}}\right\}_{n=1}^{\infty}$$

is Cauchy sequence.

The sequence then has a limit in X. Define

$$A(v)=\lim_{n\to\infty}\frac{\mathscr{F}(\Re^{an}v)}{\Re^{3an}}$$

$\forall u\in X$. we prove $A:X\to X$ is a Linear mapping.

$$\left\|\mathscr{F}(\Re^{a}v+w)\pm\Re^{b}\mathscr{F}(v-\Re^{a}w)-\left(\frac{\Re^{a}\left(1\pm\Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w)+\mathscr{F}(v-w)]\right.$$
$$-\left(\frac{\Re^{a}\left(\Re^{a}\mp\Re^{b}\right)}{2}\right)[\mathscr{F}(v+w)-\mathscr{F}(v-w)]$$
$$\left.-\left(\Re^{2a}-1\right)\left[\left(\Re^{a}\mp\Re^{b}\right)\mathscr{F}(v)\mp\left(\Re^{a+b}\pm1\right)\mathscr{F}(w)\right]\right\|$$
$$=\frac{1}{\Re^{an}}\left\|\mathscr{F}(\Re^{a}v+w)\pm\Re^{b}\mathscr{F}(v-\Re^{a}w)-\left(\frac{\Re^{a}\left(1\pm\Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w)+\mathscr{F}(v-w)]\right.$$
$$-\left(\frac{\Re^{a}\left(\Re^{a}\mp\Re^{b}\right)}{2}\right)[\mathscr{F}(v+w)-\mathscr{F}(v-w)]$$
$$\left.-\left(\Re^{2a}-1\right)\left[\left(\Re^{a}\mp\Re^{b}\right)\mathscr{F}(v)\mp\left(\Re^{a+b}\pm1\right)\mathscr{F}(w)\right]\right\|$$
$$\leq\lim_{n\to\infty}\frac{\mathfrak{Q}\left(\Re^{an}v,\Re^{an}w\right)}{\Re^{3an}}=0$$

Hence

$$\mathscr{F}(\Re^{a}v+w)\pm\Re^{b}\mathscr{F}(v-\Re^{a}w)=\left(\frac{\Re^{a}\left(1\pm\Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w)+\mathscr{F}(v-w)]$$
$$+\left(\frac{\Re^{a}\left(\Re^{a}\mp\Re^{b}\right)}{2}\right)[\mathscr{F}(v+w)-\mathscr{F}(v-w)]$$
$$+\left(\Re^{2a}-1\right)\left[\left(\Re^{a}\mp\Re^{b}\right)\mathscr{F}(v)\mp\left(\Re^{a+b}\pm1\right)\mathscr{F}(w)\right]$$

$\forall u\in X$. Next, we consider

$$||A(v)-\mathscr{F}(v)||=||\lim_{n\to\infty}\frac{\mathscr{F}(\Re^{an}u)}{\Re^{an}}-\mathscr{F}(v)||$$
$$=\lim_{n\to\infty}||\frac{\mathscr{F}(\Re^{an}u)}{\Re^{3an}}-\mathscr{F}(v)||$$

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

$$\leq \lim_{n\to\infty} \frac{1}{\Re^{3a}} \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q-1)}v,0\right)}{\Re^{3aq}}$$

Hence, we get

$$||A(v) - \mathscr{F}(v)|| \leq \frac{1}{\Re^{3a}} \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q-1)}v,0\right)}{\Re^{3aq}}$$

$\forall u \in X$.

Here to obtain A is unique. Then another mapping $B: X \to Y$ occurs and

$$||B(v) - \mathscr{F}(v)|| \leq \frac{1}{\Re^{3a}} \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q-1)}v,0\right)}{\Re^{3aq}}$$

Hence

$$||B(v) - A(v)|| \leq ||B(v) - \mathscr{F}(v)|| + ||A(v) - \mathscr{F}(v)||$$

$$\leq \frac{1}{\Re^{3a}} \sum_{q=1}^{n} \frac{\mathfrak{Q}\left(\Re^{a(q-1)}v,0\right)}{\Re^{3aq}} + \frac{1}{\Re^{3a}} \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q-1)}v,0\right)}{\Re^{3aq}}$$

$$= \frac{2}{\Re^{3a}} \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q-1)}v,0\right)}{\Re^{3aq}}$$

Because the additive mappings are A and B, we can observe

$$||A(v) - B(v)|| = \frac{2}{\Re^{3an}} ||A(\Re^{an}) - B(\Re^{an}v)||$$

$$\leq \frac{2}{\Re^{3an}} \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q+n-1)}v,0\right)}{\Re^{3aq}} \tag{10}$$

As a result (10), using the limit $n \to \infty$ and obtain

$$\lim_{n\to\infty} ||A(v) - B(v)|| \leq \lim_{n\to\infty} \frac{2}{\Re^{3an}} \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\Re^{a(q+n-1)}v,0\right)}{\Re^{3aq}}$$

Hence

$$||A(v) - B(v)|| \leq 0$$

we conclude that $A(v) = B(v)\ \forall\ v \in X$. At the end A is unique.

Corollary 2.2 Consider the map $\mathscr{F}$: X → Yfulfills

$$||\mathrm{D}\mathscr{F}(\mathrm{v},\mathrm{w})|| \leq \begin{cases} \mathfrak{U}, & \\ \mathfrak{U}\{||\mathrm{v}||^{\mathrm{p}} + ||\mathrm{w}||^{\mathrm{p}}\}, & \mathrm{p} \neq 3; \\ \mathfrak{U}\left\{||\mathrm{v}||^{\mathrm{p}}||\mathrm{w}||^{\mathrm{p}} + \{||\mathrm{v}||^{2\mathrm{p}} + ||\mathrm{w}||^{2\mathrm{p}}\}\right\}, & 2\mathrm{p} \neq 3; \end{cases} \tag{11}$$

and the function $A: X \to X$, we arrive the result

$$\|\mathscr{F}(v)-A(v)\| \leq \begin{cases} \frac{\mathfrak{U}}{|\Re^{3a}-1|}, \\ \frac{\mathfrak{U}||v||^{p}}{|\Re^{3a}-\Re^{ap}|}, \\ \frac{\mathfrak{U}||v||^{2p}}{|\Re^{3a}-\Re^{2ap}|} \end{cases} \tag{12}$$

$\forall\, v \in X.$

3 Definitions of Fuzzy Normed spaces

Definition 3.1 Let X be a real linear space. A function $\mathrm{N}: \mathrm{X} \times \mathbb{R} \rightarrow [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $\mathrm{v}, \mathrm{w} \in \mathrm{X}$ and all $\mathrm{s}, \mathrm{t} \in \mathbb{R}$,

$(F1)$ $\mathcal{N}(v, c) = 0$ for $c \leq 0$;

$(F2)$ $v = 0$ if and only if $\mathcal{N}(v, c) = 1$ for all $c > 0$;

$(F3)$ $\mathcal{N}(cv, t) = \mathcal{N}\left(v, \frac{t}{|c|}\right)$ if $c \neq 0$;

$(F4)$ $\mathcal{N}(v + w, s + t) \geq min\{\mathcal{N}(v, s), \mathcal{N}(w, t)\}$;

$(F5)$ $\mathcal{N}(v, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty}\mathcal{N}(v, t) = 1$;

$(F6)$ for $v \neq 0, \mathcal{N}(v, \cdot)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $\mathcal{N}(X, t)$ as the truth-value of the statement the norm of v is less than or equal to the real number t'.

Example 3.2 Let $(\mathrm{X}, ||\cdot||)$ be a normed linear space. Then

$$\mathcal{N}(v, t) = \begin{cases} \frac{t}{t+\|v\|}, & t > 0, \quad v \in X, \\ 0, & t \leq 0, \quad v \in X \end{cases}$$

is a fuzzy norm on X.

4 Direct method of fuzzy stability result

$$\begin{aligned} D\,\mathscr{F}(v, w) &= \mathscr{F}(\Re^{a} v + w) \pm \Re^{b} \mathscr{F}(v - \Re^{a} w) \\ &= \left(\frac{\Re^{a}\left(1 \pm \Re^{a+b}\right)}{2}\right)[\mathscr{F}(v + w) + \mathscr{F}(v - w)] \\ &+ \left(\Re^{2a} - 1\right)\left[\left(\Re^{a} \mp \Re^{b}\right)\mathscr{F}(v) \mp \left(\Re^{a+b} \pm 1\right)\mathscr{F}(w)\right] \end{aligned}$$

Theorem 4.1 Assume that X linear space, $(\mathrm{Z}, \mathrm{N}')$ fuzzy normed space and $(\mathrm{Y}, \mathrm{N}')$fuzzy Banach space. Let $\beta \in \{-1,1\}$ be fixed and let $\mathfrak{Q}: \mathrm{X}^{2} \rightarrow \mathrm{Z}$ be a mapping such that for some d with $0 < \left(\frac{\mathrm{d}}{\Re^{3\mathrm{a}}}\right)^{\beta} < 1$

$$N'\left(\mathfrak{Q}\left(\Re^{a\beta} v, \Re^{a\beta} w\right), r\right) \geq N'\left(d^{a\beta}\mathfrak{Q}(v, w), r\right) \tag{1}$$

for all $v \in X$ and all $r > 0$, $d > 0$, and

$$\lim_{k\to\infty} N'\big(\mathfrak{Q}\big(\Re^{a\beta k}v, \Re^{a\beta k}w\big), \Re^{a\beta 3k}r\big) = 1 \tag{2}$$

for all $v, w \in X$ and all $r > 0$. Suppose that a function $f: X \to Y$ satisfies the inequality

$$\mathcal{N}(D\mathscr{F}(v,w), r) \geq N'(\mathfrak{Q}(v,w), r) \tag{3}$$

for all $r > 0$ and all $v, w \in X$. Then the limit

$$\mathcal{C}(v) = N - \lim_{k\to\infty} \frac{\mathscr{F}(\Re^{a\beta k}v)}{\Re^{a\beta 3k}} \tag{4}$$

exists for all $v \in X$ and the mapping $C: X \to Y$ is a unique cubic mapping such that

$$\mathcal{N}(\mathcal{C}(v) - \mathscr{F}(v), r) \geq N'\big(\mathfrak{Q}(v,0), |\Re^{3a} - d^a|r\big) \tag{5}$$

for all $v \in X$ and all $r > 0$.

Proof. First assume $\beta = 1$. Replacing (v, w) by $(v, 0)$ in (3), we get

$$\mathcal{N}\big(\mathscr{F}(\Re^a v) - \Re^{3a}\mathscr{F}(v), r\big) \geq N'(\mathfrak{Q}(v,0), r) \tag{6}$$

for all $v \in X$ and all $r > 0$. Replacing v by $\Re^{ak}v$ in (6), we obtain

$$\mathcal{N}\left(\frac{\mathscr{F}(\Re^{a(k+1)}v)}{\Re^{3a}} - \mathscr{F}(\Re^{ak}v), \frac{r}{\Re^{3a}}\right) \geq N'\big(\mathfrak{Q}(\Re^{ak}v, 0), r\big) \tag{7}$$

for all $v \in X$ and all $r > 0$. Using (1), $(F3)$ in (7), we arrive

$$\mathcal{N}\left(\frac{\mathscr{F}(\Re^{a(k+1)}v)}{\Re^{3a}} - \mathscr{F}(\Re^{ak}v), \frac{r}{\Re^{3a}}\right) \geq N'\left(\mathfrak{Q}(v,0), \frac{r}{d^{ak}}\right) \tag{8}$$

for all $v \in X$ and all $r > 0$. It is easy to verify from (8), that

$$\mathcal{N}\left(\frac{\mathscr{F}(\Re^{a(k+1)}v)}{\Re^{3a(k+1)}} - \frac{\mathscr{F}(\Re^{ak}v)}{\Re^{3ak}}, \frac{r}{\Re^{3a}.\Re^{3ak}}\right) \geq N'\left(\mathfrak{Q}(v,0), \frac{r}{d^{ak}}\right) \tag{9}$$

holds for all $v \in X$ and all $r > 0$. Replacing r by $\Re^{ak}r$ in (9), we get

$$\mathcal{N}\left(\frac{\mathscr{F}(\Re^{a(k+1)}v)}{\Re^{3a(k+1)}} - \frac{\mathscr{F}(\Re^{ak}v)}{\Re^{3ak}}, \frac{d^{ak}\, r}{\Re^{3a}.\Re^{3ak}}\right) \geq N'(\mathfrak{Q}(v,0), r) \tag{10}$$

for all $v \in X$ and all $r > 0$. It is easy to see that

$$\frac{\mathscr{F}(\Re^{ak}v)}{\Re^{3ak}} - \mathscr{F}(v) = \sum_{i=0}^{k-1} \left[\frac{\mathscr{F}(\Re^{a(i+1)}x)}{\Re^{3a(i+1)}} - \frac{\mathscr{F}(\Re^{ai}v)}{\Re^{3ai}}\right] \tag{11}$$

for all $v \in X$. From equations (10) and (11), we have

$$\begin{aligned}
&\mathcal{N}\left(\frac{\mathscr{F}(\Re^{ak}v)}{\Re^{3ak}} - \mathscr{F}(v), \sum_{i=0}^{k-1} \frac{d^i\, r}{\Re^{3a}.\Re^{3ai}}\right) \\
&\qquad \geq min \bigcup_{i=0}^{k-1} \left\{\frac{\mathscr{F}(\Re^{a(i+1)}v)}{\Re^{3a(i+1)}} - \frac{\mathscr{F}(\Re^{ia}v)}{\Re^{3ai}}, \frac{d^i\, r}{\Re^{3a}.\Re^{3ai}}\right\} \\
&\qquad \geq min \bigcup_{i=0}^{k-1} \{N'(\mathfrak{Q}(v,0), r)\} \\
&\qquad \geq N'(\mathfrak{Q}(v,0), r)
\end{aligned} \tag{12}$$

for all $v \in X$ and all $r > 0$. Replacing v by $\Re^{ma}v$ in (12) and using (1), $(F3)$, we obtain

$$\mathcal{N}\left(\frac{\mathscr{F}(\Re^{a(k+m)}v)}{\Re^{3a(k+m)}} - \frac{\mathscr{F}(\Re^{ma}v)}{\Re^{3am}}, \sum_{i=0}^{k-1} \frac{d^i\, r}{\Re^{3a}.\Re^{3a(i+m)}}\right) \geq N'\left(\mathfrak{Q}(v,0), \frac{r}{d^{am}}\right) \tag{13}$$

for all $v \in X$ and all $r > 0$ and all $m, k \geq 0$. Replacing r by $d^{am}r$ in (13), we get

$$\mathcal{N}\left(\frac{\mathscr{F}(\Re^{a(k+m)}v)}{\Re^{3a(k+m)}} - \frac{\mathscr{F}(\Re^{am}v)}{\Re^{3am}}, \sum_{i=m}^{m+k-1} \frac{d^{ai}\, r}{\Re^{3a}.\Re^{3ai}}\right) \geq N'(\mathfrak{Q}(v,0), r) \tag{14}$$

for all $v \in X$ and all $r > 0$ and all $m, k \geq 0$. Using $(F3)$ in (14), we obtain

$$\mathcal{N}\left(\frac{\mathscr{F}(\Re^{a(k+m)}v)}{\Re^{3a(k+m)}} - \frac{\mathscr{F}(\Re^{am}v)}{\Re^{3am}}, r\right) \geq N'\left(\mathfrak{Q}(v,0), \frac{r}{\sum_{i=m}^{m+k-1} \frac{d^{ai}}{\Re^{3a}.\Re^{3ai}}}\right) \tag{15}$$

for all $v \in X$ and all $r > 0$ and all $m, k \geq 0$. Since $0 < d < \Re^{3a}$ and $\sum_{i=0}^{k} \left(\frac{d}{\Re^{3a}}\right)^i < \infty$, the cauchy criterion for convergence and $(F5)$ implies that $\left\{\frac{\mathscr{F}(\Re^{ak}v)}{\Re^{3ak}}\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $\mathscr{F}(v) \in Y$. So one can define the mapping $C: X \to Y$ by

$$\mathscr{F}(v) = N - \lim_{k\to\infty} \frac{\mathscr{F}(\Re^{ak}v)}{\Re^{3ak}}$$

for all $v \in X$. Letting $m = 0$ in (15), we get

$$\mathcal{N}\left(\frac{\mathscr{F}(\Re^{ak}v)}{\Re^{3ak}} - \mathscr{F}(v), r\right) \geq N'\left(\mathfrak{Q}(v,0), \frac{r}{\sum_{i=0}^{k-1} \frac{d^{ai}}{\Re^{3i}.\Re^{3ai}}}\right) \tag{16}$$

for all $v \in X$ and all $r > 0$. Letting $k \to \infty$ in (16) and using $(F6)$, we arrive

$$\mathcal{N}(\mathcal{C}(v) - \mathscr{F}(v), r) \geq N'\big(\mathfrak{Q}(v,0), r(\Re^{3a} - d)\big)$$

for all $v \in X$ and all $r > 0$. To prove C satisfies the (1), replacing (v, w) by $(\Re^{ak}v, \Re^{ak}w)$ in (3), respectively , we obtain

$$\mathcal{N}\left(\frac{1}{\Re^{3ak}} D\mathscr{F}(\Re^{ak}v, \Re^{ak}w), r\right) \geq N'\big(\mathfrak{Q}(\Re^{ak}v, \Re^{ak}w), \Re^{3ak}r\big) \tag{17}$$

for all $r > 0$ and all $v, w \in X$. Now,

$$\mathcal{N}\left(\mathscr{F}(\Re^a v + w) \pm \Re^b \mathscr{F}(v - \Re^a w) - \left(\frac{\Re^a\left(1 \pm \Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w) + \mathscr{F}(v-w)]\right.$$
$$- \left(\frac{\Re^a\left(\Re^a \mp \Re^b\right)}{2}\right)[\mathscr{F}(v+w) - \mathscr{F}(v-w)]$$
$$\left. -(\Re^{2a} - 1)\left[\left(\Re^a \mp \Re^b\right)\mathscr{F}(v) \mp \left(\Re^{a+b} \pm 1\right)\mathscr{F}(w)\right], \frac{r}{6}\right)$$
$$\geq min\left\{\mathcal{N}\left(\mathcal{C}(\Re^a v + w) - \frac{1}{\Re^{3ak}}\mathscr{F}(\Re^a v + w), \frac{r}{6}\right),\right.$$

$$\mathcal{N}\left(\pm\Re^{b}\mathcal{C}(v-\Re^{a}w)\pm\frac{1}{\Re^{3ak}}\Re^{b}\mathscr{F}(v-\Re^{a}w),\frac{r}{6}\right),$$

$$\mathcal{N}\left(-\left(\frac{\Re^{a}\left(1\pm\Re^{a+b}\right)}{2}\right)[\mathcal{C}(v+w)+\mathcal{C}(v-w)]-\right.$$

$$\left.\frac{1}{\Re^{3ak}}\left(\frac{\Re^{a}\left(1\pm\Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w)+\mathscr{F}(v-w)],\frac{r}{6}\right),$$

$$\mathcal{N}\left(-\left(\frac{\Re^{a}\left(\Re^{a}\mp\Re^{b}\right)}{2}\right)[\mathcal{C}(v+w)-\mathcal{C}(v-w)]-\right.$$

$$\left.\frac{1}{\Re^{3ak}}\left(\frac{\Re^{a}\left(\Re^{a}\mp\Re^{b}\right)}{2}\right)[\mathscr{F}(v+w)-\mathscr{F}(v-w)],\frac{r}{6}\right),$$

$$\mathcal{N}\left(-\left(\Re^{2a}-1\right)\left[\left(\Re^{a}\mp\Re^{b}\right)\mathcal{C}(v)\mp\left(\Re^{a+b}\pm1\right)\mathcal{C}(w)\right]-\right.$$

$$\left.\frac{1}{\Re^{3ak}}\left(\Re^{2a}-1\right)\left[\left(\Re^{a}\mp\Re^{b}\right)\mathscr{F}(v)\mp\left(\Re^{a+b}\pm1\right)\mathscr{F}(w)\right],\frac{r}{6}\right),$$

$$\mathcal{N}\left(\frac{1}{\Re^{3ak}}\mathscr{F}(\Re^{a}v+w)\pm\frac{1}{\Re^{3ak}}\Re^{b}\mathscr{F}(v-\Re^{a}w)\right.$$

$$-\frac{1}{\Re^{3ak}}\left(\frac{\Re^{a}\left(1\pm\Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w)+\mathscr{F}(v-w)]$$

$$-\frac{1}{\Re^{3ak}}\left(\frac{\Re^{a}\left(\Re^{a}\mp\Re^{b}\right)}{2}\right)[\mathscr{F}(v+w)-\mathscr{F}(v-w)]$$

$$\left.-\frac{1}{\Re^{3ak}}\left(\Re^{2a}-1\right)\left[\left(\Re^{a}\mp\Re^{b}\right)\mathscr{F}(v)\mp\left(\Re^{a+b}\pm1\right)\mathscr{F}(w)\right],\frac{r}{6}\right)\right\}$$

for all $v,w\in X$ and all $r>0$.

$$\mathcal{N}\left(\mathscr{F}(\Re^{a}v+w)\pm\Re^{b}\mathscr{F}(v-\Re^{a}w)-\left(\frac{\Re^{a}\left(1\pm\Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w)+\mathscr{F}(v-w)]\right.$$

$$-\left(\frac{\Re^{a}\left(\Re^{a}\mp\Re^{b}\right)}{2}\right)[\mathscr{F}(v+w)-\mathscr{F}(v-w)]$$

$$\left.-\left(\Re^{2a}-1\right)\left[\left(\Re^{a}\mp\Re^{b}\right)\mathscr{F}(v)\mp\left(\Re^{a+b}\pm1\right)\mathscr{F}(w)\right],\frac{r}{6}\right)$$

$$\geq min\left\{1,1,1,1,1,1,1N'\left(\mathfrak{Q}(\Re^{ak}v,0),\Re^{3ak}r\right)\right\}$$

$$\geq N'\left(\mathfrak{Q}(\Re^{ak}v,0),\Re^{3ak}r\right) \tag{18}$$

for all $v,w\in X$ and all $r>0$. Letting $k\to\infty$ in (18) and using (2), we see that

$$\mathcal{N}\left(\mathscr{F}(\Re^{a}v+w)\pm\Re^{b}\mathscr{F}(v-\Re^{a}w)-\left(\frac{\Re^{a}\left(1\pm\Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w)+\mathscr{F}(v-w)]\right.$$

$$-\left(\frac{\Re^{a}\left(\Re^{a}\mp\Re^{b}\right)}{2}\right)[\mathscr{F}(v+w)-\mathscr{F}(v-w)]$$

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

$$-\left(\Re^{2a}-1\right)\left[\left(\Re^{a}\mp\Re^{b}\right)\mathscr{F}(v)\mp\left(\Re^{a+b}\pm1\right)\mathscr{F}(w)\right],\frac{r}{6}\Big)=1 \quad (19)$$

for all $v, w \in X$ and all $r > 0$. Using $(F2)$ in the above inequality gives

$$\mathscr{F}(\Re^{a}v+w)\pm\Re^{b}\mathscr{F}(v-\Re^{a}w)=\left(\frac{\Re^{a}\left(1\pm\Re^{a+b}\right)}{2}\right)[\mathscr{F}(v+w)+\mathscr{F}(v-w)]$$

$$+\left(\frac{\Re^{a}\left(\Re^{a}\mp\Re^{b}\right)}{2}\right)[\mathscr{F}(v+w)-\mathscr{F}(v-w)]$$

$$+\left(\Re^{2a}-1\right)\left[\left(\Re^{a}\mp\Re^{b}\right)\mathscr{F}(v)\mp\left(\Re^{a+b}\pm1\right)\mathscr{F}(w)\right]$$

for all $v, w \in X$. Hence C satisfies the cubic functional equation (1). In order to prove $\mathscr{F}(v)$ is unique, let $\mathscr{F}'(v)$ be another cubic functional equation satisfying (1) and (5). Hence,

$$\mathcal{N}(\mathcal{C}(v)-\mathscr{F}'(v),r)=\mathcal{N}\left(\frac{\mathscr{F}(\Re^{ak}v)}{\Re^{3ak}}-\frac{\mathscr{F}'(\Re^{ak}v)}{\Re^{3ak}},r\right)$$

$$\geq min\{\mathcal{N}\left(\frac{\mathscr{F}(\Re^{ak}v)}{\Re^{3ak}}-\frac{\mathscr{F}(\Re^{ak}v)}{\Re^{3ak}},\frac{r}{2}\right),\mathcal{N}\left(\frac{\mathscr{F}(\Re^{ak}v)}{\Re^{3ak}}-\frac{\mathscr{F}'(\Re^{ak}v)}{\Re^{3ak}},\frac{r}{2}\right)\}$$

$$\geq N'\left(\mathfrak{Q}(\Re^{ak}u,0),\frac{(\Re^{3ak}-d^{a})r}{2}\right)$$

$$\geq N'\left(\mathfrak{Q}(v,0),\frac{(\Re^{3ak}-d^{a})r}{2d^{k}}\right)$$

for all $u \in X$ and all $r > 0$. Since

$$\lim_{k\to\infty}\frac{(\Re^{3ak}-d^{a})r}{2d^{k}}=\infty,$$

we obtain

$$\lim_{k\to\infty}N'\left(\mathfrak{Q}(v,0),\frac{(\Re^{3ak}-d^{a})r}{2d^{k}}\right)=1.$$

Thus

$$\mathcal{N}(\mathscr{F}(v)-\mathscr{F}'(v),r)=1$$

for all $v \in X$ and all $r > 0$, hence $\mathscr{F}(v)=\mathscr{F}'(v)$. Therefore $\mathscr{F}(v)$ is unique.

Corollary 4.2 Suppose that a function f: X → Y satisfies the inequality

$$N(D\mathscr{F}(v,w),r)\geq\begin{cases}N'(\epsilon,r), & \\ N'(\epsilon||v||^{s}+||w||^{s},r), & s\neq 3;\\ N'(\epsilon(||v||^{s}||w||^{s}+\{||v||^{2s}+||w||^{2s}\}),r), & s\neq \frac{3}{2};\end{cases} \quad (20)$$

for all $v, w \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $C: X \to Y$ such that

$$N(\mathcal{C}(v)-\mathscr{F}(v),r) \geq \begin{cases} N'\big(\epsilon, |\Re^{3a}-1|r\big), \\ N'\big(\epsilon||v||^{s}, |\Re^{3a}-\Re^{as}|r\big), \\ N'\big(\epsilon||v||^{2s}, |\Re^{3a}-\Re^{3as}|r\big) \end{cases} \tag{21}$$

for all $v \in X$ and all $r > 0$.

5 Fixed Point Method of Fuzzy Stability Results

For to prove the stability result we define the following:

δ_i is a constant such that

$$\delta_i = \begin{cases} \Re^{a} & if \quad i=0, \\ \frac{1}{\Re^{a}} & if \quad i=1 \end{cases}$$

and Ω is the set such that

$$\Omega = \{g \mid g\colon X \to Y, g(0)=0\}.$$

Theorem 5.1 Let $f\colon X \to Y$ be a mapping for which there exist a function $\mathfrak{Q}\colon X^2 \to Z$ with the condition

$$\lim_{k\to\infty} N'\big(\mathfrak{Q}\big(\delta_i^{k}v, \delta_i^{k}w\big), \delta_i^{3k}r\big) = 1 \quad \forall \; v,w \in X, r>0 \tag{22}$$

and satisfying the functional inequality

$$\mathcal{N}(D\,\mathscr{F}(v,w),r) \geq N'(\mathfrak{Q}(v,w),r) \;\; \forall \;\; v,w \in X, r>0. \tag{23}$$

If there exists $L = L(i)$ such that the function

$$v \to \beta(v) = \mathfrak{Q}\left(\frac{v}{\Re^{a}}, 0\right),$$

has the property

$$N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i v), r\right) = N'(\beta(v),r), \;\; \forall \;\; v \in X, r>0. \tag{24}$$

Then there exists unique cubic function $C\colon X \to Y$ satisfying the functional equation (1) and

$$\mathcal{N}(\mathscr{F}(v)-\mathcal{C}(v),r) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right), \;\; \forall \;\; v \in X, r>0. \tag{25}$$

Proof. Let d be a general metric on Ω, such that

$$d(g,h) = inf\{K\mathcal{N}(0,\infty) | \mathcal{N}(g(v)-h(v),r) \geq N'(K\beta(v),r), v \in X, r>0\}.$$

It is easy to see that (Ω, d) is complete. Define $T\colon \Omega \to \Omega$ by $Tg(x) = \frac{1}{\delta_i^3} g(\delta_i v)$, for all $v \in X$. For $g, h \in \Omega$, we have $d(g,h) \leq K$

$$\Rightarrow \qquad \mathcal{N}(g(v)-h(v),r) \geq N'(K\beta(v),r)$$

$$\Rightarrow \qquad \mathcal{N}\left(\frac{g(\delta_i v)}{\delta_i^3} - \frac{h(\delta_i v)}{\delta_i^3}, r\right) \geq N'\left(\frac{K}{\delta_i^3}\beta(\delta_i v), r\right)$$

$$\Rightarrow \qquad \mathcal{N}(Tg(v) - Th(v), r) \geq N'(KL\beta(v), r)$$

$$\Rightarrow \qquad d(Tg(v), Th(v)) \leq KL$$

$$\Rightarrow \qquad d(Tg, Th) \leq Ld(g, h) \tag{26}$$

for all $g, h \in \Omega$. There fore T is strictly contractive mapping on Ω with Lipschitz constant L. Replacing (v, w) by $(v, 0)$ in (23), we get

$$\mathcal{N}\left(\mathscr{F}(\Re^{a} v) - \Re^{3a}\mathscr{F}(v), r\right) \geq N'(\mathfrak{Q}(v, 0), r). \tag{27}$$

for all $v \in X, r > 0$. Using (F3) in (27), we arrive

$$\mathcal{N}\left(\frac{\mathscr{F}(\Re^{a} v)}{\Re^{3a}} - \mathscr{F}(v), r\right) \geq N'\left(\frac{1}{\Re^{3a}}\mathfrak{Q}(v, 0), r\right) \tag{28}$$

for all $v \in X, r > 0$ with the help of (24) when $i = 0$, it follows from (28), we get

$$\Rightarrow \qquad \mathcal{N}\left(\frac{\mathscr{F}(\Re^{a} v)}{\Re^{3a}} - \mathscr{F}(v), r\right) \geq N'(L\beta(v), r)$$

$$\Rightarrow \qquad d(TC, C) \leq L = L^{1} = L^{1-i} \tag{29}$$

Replacing v by $\frac{v}{\Re^{a}}$ in (27), we obtain

$$\mathcal{N}\left(\mathscr{F}(v) - \Re^{3a}\mathscr{F}\left(\frac{v}{\Re^{a}}\right), r\right) \geq N'\left(\mathfrak{Q}\left(\frac{v}{\Re^{a}}, 0\right), r\right) \tag{30}$$

for all $v \in X, r > 0$ with the help of (24) when $i = 1$, it follows from (30) we get

$$\Rightarrow \qquad \mathcal{N}\left(\mathscr{F}(v) - \Re^{3a}\mathscr{F}\left(\frac{v}{\Re^{a}}\right), r\right) \geq N'(\beta(v), r)$$

$$\Rightarrow \qquad d(C, TC) \leq 1 = L^{0} = L^{1-i} \tag{31}$$

Then from (29) and (31) we can conclude,

$$d(C, TC) \leq L^{1-i} < \infty$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point C of T in Ω such that

$$\mathcal{C}(v) = N - \lim_{k\to\infty} \frac{\mathscr{F}(n^{k} v)}{n^{3k}}, \qquad \forall v \in X, r > 0. \tag{32}$$

Replacing (v, w) by $(\delta_i v, \delta_i w)$ in (23), we arrive

$$\mathcal{N}\left(\frac{1}{\delta_i^{3k}} D\mathscr{F}(\delta_i v, \delta_i w), r\right) \geq N'\left(\mathfrak{Q}(\delta_i v, \delta_i w), \delta_i^{3k} r\right) \tag{33}$$

for all $r > 0$ and all $v, w \in X$, we can prove the function, $C: X \to Y$ satisfies the functional equation (1).

By fixed point alternative, since C is unique fixed point of T in the set

$$\Delta = \{f \in \Omega | d(f, C) < \infty\},$$

therefore C is a uniqe function such that

$$\mathcal{N}(\mathscr{F}(v)-\mathscr{F}(v),r)\geq N'(K\beta(v),r) \tag{34}$$

for all $v\in X, r>0$ and $K>0$. Again using the fixed point alternative, we obtain

$$d(C,\mathcal{C})\leq\frac{1}{1-L}d(C,TC)$$

$$\Rightarrow\qquad d(C,\mathcal{C})\leq\frac{L^{1-i}}{1-L}$$

$$\Rightarrow\qquad \mathcal{N}(\mathscr{F}(v)-\mathcal{C}(v),r)\geq N'\left(\frac{L^{1-i}}{1-L}\beta(v),r\right), \tag{35}$$

for all $v\in X$ and $r>0$.

Corollary 5.2 Suppose that a function f: X → Y satisfies theinequality

$$N(D\mathscr{F}(v,w),r)\quad\geq\begin{cases}N'(\epsilon,r), & \\ N'(\epsilon||v||^s+||w||^s,r), & s\neq\ 3;\\ N'(\epsilon(||v||^s||w||^s+\{||v||^{2s}+||w||^{2s}\}),r), & s\neq\ \frac{3}{2};\end{cases} \tag{36}$$

for all $v,w\in X$ and $r>0$, where ϵ,s are constants with $\epsilon>0$. Then there exists a unique cubic mapping $C\colon X\to Y$ such that

$$\mathcal{N}(\mathcal{C}(v)-\mathscr{F}(v),r)\geq\begin{cases}N'\big(\epsilon,|\Re^{3a}-1|r\big),\\ N'\big(\epsilon||v||^s,|\Re^{3a}-\Re^{as}|r\big),\\ N'\big(\epsilon||v||^{2s},|\Re^{3a}-\Re^{3as}|r\big),\end{cases} \tag{37}$$

for all $v\in X$ and all $r>0$.

Proof. Setting

$$\mathfrak{Q}(v,w)=\begin{cases}N'(\epsilon,r),\\ N'(\epsilon||v||^s+||w||^s,r),\\ N'(\epsilon(||v||^s||w||^s+\{||v||^{2s}+||w||^{2s}\}),r)\end{cases}$$

for all $v,w\in X$. Then,

$$N'\big(\mathfrak{Q}(\delta_i^k v,0),\delta_i^{3k}r\big)$$

$$=\begin{cases}N'\big(\epsilon,\delta_i^{3k}r\big)\\ N'\Big(\epsilon||v||^s,\delta_i^{(3-s)k}r\Big)\\ N'\Big(\epsilon||v||^{2s},\delta_i^{(3-2s)k}r\Big)\end{cases}$$

$$=\begin{cases}\to 1\ as\ \ k\to\infty,\\ \to 1\ as\ \ k\to\infty,\\ \to 1\ as\ \ k\to\infty.\end{cases}$$

Thus, (22) is holds. But we have $\beta(v)=\mathfrak{Q}\left(\frac{v}{\Re^a},0\right)$ has the property

$$N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i v),r\right)\geq N'(\beta(v),r)\ \ \forall\ \ v\in X,r>0.$$

Hence

$$N'(\beta(v),r) = N'\left(\mathfrak{Q}\left(\frac{x}{\Re^{a}},0\right),r\right) = \begin{cases} N'(\epsilon,r), \\ N'\left(\frac{\epsilon}{\Re^{as}}||v||^{s},r\right), \\ N'\left(\frac{\epsilon}{\Re^{2as}}||v||^{2s},r\right). \end{cases}$$

Now,

$$N'\left(\frac{1}{\delta_i^3}\beta(\delta_i v),r\right) = \begin{cases} N'\left(\frac{\epsilon}{\delta_i^3},r\right), \\ N'\left(\frac{\epsilon}{\delta_i^3}\left(\frac{1}{\Re^{as}}\right)||\delta_i v||^{s},r\right), \\ N'\left(\frac{\epsilon}{\delta_i^3}\left(\frac{1}{\Re^{2as}}\right)||\delta_i v||^{2s},r\right) \end{cases} = \begin{cases} N'\left(\delta_i^{-3}\beta(v),r\right), \\ N'\left(\delta_i^{s-3}\beta(v),r\right), \\ N'\left(\delta_i^{2s-3}\beta(v),r\right). \end{cases}$$

Now from (25), we prove the following cases for conditions (i) and (ii).

Case:1 $L = \Re^{-3a}$ for $s = 0$ if $i = 0$

$$\mathcal{N}(\mathcal{C}(v) - \mathscr{F}(v),r) \geq N'\left(\frac{\Re^{-3a}}{1-\Re^{-3a}}\beta(v),r\right)$$

$$= N'\left(\frac{\epsilon}{(\Re^{3a}-1)},r\right) = N'\left(\epsilon,(\Re^{3a}-1)r\right).$$

Case:2 $L = \Re^{3a}$ for $s = 0$ if $i = 1$

$$\mathcal{N}(\mathcal{C}(v) - \mathscr{F}(v),r) \geq N'\left(\frac{1}{1-\Re^{3a}}\beta(v),r\right)$$

$$= N'\left(\frac{\epsilon}{(1-\Re^{3a})},r\right) = N'\left(\epsilon,(1-\Re^{3a})r\right).$$

Case:3 $L = \Re^{a(s-3)}$ for $s > 3$ if $i = 0$

$$\mathcal{N}(\mathcal{C}(v) - \mathscr{F}(v),r) \geq N'\left(\frac{\Re^{a(s-3)}}{1-\Re^{a(s-3)}}\beta(v),r\right)$$

$$= N'\left(\frac{\epsilon}{(\Re^{3a}-\Re^{as})}||v||^{s},r\right) = N'\left(\epsilon||v||^{s},(\Re^{3a}-\Re^{as})r\right).$$

Case:4 $L = \Re^{a(3-s)}$ for $s < 3$ if $i = 1$

$$\mathcal{N}(\mathcal{C}(v) - \mathscr{F}(v),r) \geq N'\left(\frac{1}{1-\Re^{a(3-s)}}\beta(v),r\right)$$

$$= N'\left(\frac{\epsilon}{(\Re^{as}-\Re^{3a})}||v||^{s},r\right) = N'\left(\epsilon||v||^{s},(\Re^{as}-\Re^{3a})r\right).$$

Case:5 $L = \Re^{a(2s-3)}$ for $s > \frac{3}{2}$ if $i = 0$

$$\mathcal{N}(\mathcal{C}(v) - \mathscr{F}(v),r) \geq N'\left(\frac{\Re^{a(2s-3)}}{1-\Re^{a(2s-3)}}\beta(v),r\right)$$

$$= N'\left(\frac{\epsilon}{(\Re^{3a}-\Re^{2as})}||v||^{s},r\right) = N'\left(\epsilon||v||^{s},(\Re^{3a}-\Re^{2as})r\right).$$

Case:6 $L = \Re^{a(3-2s)}$ for $s < \frac{3}{2}$ if $i = 1$

$$\mathcal{N}(\mathcal{C}(v) - \mathscr{F}(v), r) \geq N'\left(\frac{1}{1-\Re^{a(3-2s)}}\beta(v), r\right)$$

$$= N'\left(\frac{\epsilon}{(\Re^{2as}-\Re^{3a})}||v||^s, r\right) = N'\left(\epsilon||v||^s, (\Re^{2as} - \Re^{3a})r\right).$$

6 Conclusion

In this paper, we have established novel stability results for generalized alternate cubic functional equations using the classical method for Banach spaces and both direct and fixed point approaches for fuzzy normed spaces. The classical method provided a clear pathway to demonstrate Hyers-Ulam stability in Banach spaces, revealing how small deviations affect the functional equation’s solutions. In contrast, the fuzzy normed space framework, enriched by direct and fixed point methods, allowed for a more nuanced stability analysis, accommodating uncertainties and imprecisions intrinsic to fuzzy systems. Our results highlight the effectiveness of combining classical techniques with fixed point theory in analyzing functional equations under different normed environments. The comparative analysis between deterministic Banach spaces and the more flexible fuzzy normed spaces underscores the adaptability of the generalized alternate cubic functional equation across various mathematical contexts. These findings offer significant contributions to the stability theory of functional equations and lay the groundwork for future applications in both pure and applied mathematical fields, particularly in scenarios involving uncertain or fuzzy data.

Conflict of interest. The authors declare that they have no competing interests.

References

[1] S.M. Ulam, Problems in Modern Mathematics, Science Editions, Wiley, NewYork, 1964.

[2] D.H. Hyers, On the stability of the linear functional equation, Proc.Nat. Acad.Sci.,U.S.A.,27, 1941, 222-224.

[3] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc.Amer.Math.Soc., 72, 1978, 297-300.

[4] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2 ,1950, 64-66.

[5] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings , J. Math. Anal. Appl., 184 1994, 431-436.

[6] J.M. Rassias, On approximately of approximately linear mappings by linear mappings, J. Funct. Anal. USA, 46, 1982, 126-130.

[7] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Fun Eq in several variables, Birkhauser, Basel, 1998.

[8] J. Aczel and J. Dhombres, Fun Eq in Several Variables, Cambridge Univ,Press, 1989.

[9] S. Czerwik, Fun Eq and Inequalities in Several Variables, World Scientific, River Edge, NJ, 2002.

[10] S.M. Jung, Hyers-Ulam-Rassias Stability of Fun Eq in Mathematical Analysis, Hadronic Press, Palm Harbor,2001.

[11] Tunç, Osman. New Results on the Ulam–Hyers–Mittag–Leffler Stability of Caputo Fractional-Order Delay Differential Equations." Mathematics 12, no. 9 2024: 1342.

[12] Selvam, A., Sabarinathan, S., Sooppy Nisar, K., Ravichandran, C. and Senthil Kumar, B.V. Results on Ulam-type stability of linear differential equation with integral transform. Mathematical Methods in the Applied Sciences, 47(4), 2024.pp.2311-2323.

[13] Tunç, Osman, Cemil Tunç, and Jen-Chih Yao. New results on Ulam stabilities of nlinear integral equations." Mathematics 12, no. 5 2024: 682.

[14] Tunç, Osman, and Cemil Tunç. On Ulam stabilities of delay Hammerstein integral equation Symmetry 15, no. 9 2023: 1736.

[15] Novac, Adela, Diana Otrocol, and Dorian Popa. On ulam stability of a partial differential operator in Banach spaces." Mathematics 11, no. 11 2023: 2488.

[16] García, Gonzalo, and Gaspar Mora. The degree of nondensifiability of linear bounded operators and its applications." Applied General Topology 25, no. 1 2024: 213-228.

[17] Zada, Akbar, Peiguang Wang, Dhaou Lassoued, and Tongxing Li. Connections between Hyers-Ulam stability and uniform exponential stability of 2-periodic linear nonautonomoussystems Advances in Difference Equations 2017 (2017): 1-7.

[18] Kumar, Bhim, and Muslim Malik. Existence, controllability and Hyers–Ulam stability of a hybrid neutral switched system with impulsive effects." International Journal of Systems Science 55, no. 3 2024: 517-534.

[19] Almarri, Barakah, Xingtao Wang, and Ahmed M. Elshenhab. Controllability and Hyers–Ulam stability of fractional systems with pure delay." Fractal and Fractional 6, no. 10 ,2022: 611.

[20] Agarwal, R. P., Cho, Y. J., Saadati, R., Wang, S, Nonlinear Fuzzy stability of cubic functional equations. Journal of Inequalities and Applications, 2012, 1-19.

[21] Saadati, R., Cho, Y. J., Rassias, J. M, Nonlinear L-Fuzzy Stability of K-Cubic Functional Equation. Filomat, 29(5), (2015) 1137-1148.

[22] Mohiuddine, S.A. and Alotaibi, A., Fuzzy stability of a cubic functional equation via fixed point technique. Advances in Difference Equations, 2012, pp.1-8.

[23] Lee, Yang-Hi, and Soon-Mo Jung. Fuzzy stability of the cubic and quadratic functional equation. Appl. Math. Sci.(Ruse) 10 (2016): 2671-2686.

[24] Javadi, S. and Rassias, J.M., Stability of general cubic mapping in fuzzy normed spaces. Analele ştiinţifice ale Universităţii" Ovidius" Constanţa. Seria Matematică, 2012,20(1), pp.129-150.

[25] Pasupathi, A.; Konsalraj, J.; Fatima, N.; Velusamy, V.; Mlaiki, N.; Souayah, N. Direct and fixed-point stability–instability of additive functional equation in Banach and quasi-beta normed spaces. Symmetry 2022, 14, 1700.

[26] Agilan, P.; Julietraja, K.; Mlaiki, N.; Mukheimer, A. Intuitionistic fuzzy stability of an Euler–Lagrange symmetry additive functional equation via direct and fixed point technique (FPT). Symmetry 2022, 14, 2454.

[27] Agilan, P.; Almazah, M.A.A.; Julietraja, K.; Alsinai, A. Classical and fixed point approach to the stability analysis of a bilateral symmetric additive functional equation in fuzzy and random normed spaces. Mathematics 2023, 11, 681.

[28] Agilan, P.; Julietraja.; K Almazah, M.A.A.; Alsinai, A. Stability analysis of a new class of series type additive functional equation in Banach spaces: direct and fixed point techniques Mathematics 2023, 11, 887. doi.org/10.3390/math11040887.

[29] Aloqaily, Ahmad, P. Agilan, K. Julietraja, S. Annadurai, and Nabil Mlaiki. A novel stability analysis of functional equation in neutrosophic normed spaces. Boundary Value Problems, 2024, no. 1 (2024), 47.

[30] Agilan, P., Julietraja, K., Kanimozhi, B. and Alsinai, A., Hyers Stability of AQC Functional Equation. Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications and Algorithms, 2024, 31, 63-75.

[31] Agilan, P., Julietraja, K, Sarah Aljohani, Nabil Mlaiki, Generalised Ulam-Hyers Stability Analysis for System of Additive Functional Equation in Fuzzy and Random Normed Spaces:Direct and Fixed Point Approach. Int. J. Anal. Appl., 22 2024, 201.

[32] Agilan.P, Vijayan.V, Sophia.M, Ganapathy.G., Exploring Advanced Stability of Higher-Order Functional Equations in Neutrosophic Normed Spaces via Hyers-Ulam Methodologies, Communications on Applied Nonlinear Analysis, 2025, Vol 32 No. 7s (2025), 806-822.