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Controllability and Observability of Matrix Sylvester Adjoint Dynamic Impulsive Systems on Time Scales

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Abstract:

This paper investigates the controllability and observability of matrix Sylvester adjoint dynamic impulsive systems within the framework of time scales. By applying the vectorization operator, the system is reformulated into an equivalent Kronecker product-based dynamic impulsive system, enabling more efficient analysis. The study derives necessary and sufficient conditions for controllability and observability through the adjoint matrix approach, highlighting its analytical significance. Additionally, the research incorporates the Gramian matrix to establish results for impulsive dynamic systems on time scales, providing a comprehensive criterion for evaluating system properties. This unified framework bridges discrete and continuous-time dynamics, enabling the modeling and analysis of hybrid systems with abrupt state transitions. The findings advance the mathematical understanding of Sylvester matrix systems and offer practical insights into the design and control of complex impulsive systems across varied applications. This work contributes to the growing field of time-scale calculus and its applications in dynamic system analysis.

Keywords: Controllability, observability, Kronecker product, Time scales.

Mathematics Subject Classification: 93B05, 93B07,39A12, 18A40, 34N05.

1. Introduction

The study of adjoint matrix Sylvester dynamic impulsive systems on time scales combines the strengths of continuous and discrete systems, offering a unified framework to analyze complex real-world phenomena. This approach effectively models systems that exhibit both abrupt changes and continuous evolution, making it particularly relevant in fields such as engineering, biology, and economics. The use of adjoint matrix techniques brings significant advantages, such as simplifying the process of solving linear dynamic equations, reducing computational complexity, and enhancing analytical clarity. Matrices also provide a compact representation of system dynamics, facilitating efficient manipulation, scalability, and the application of powerful algebraic methods. By incorporating the time scales framework, this method seamlessly integrates hybrid systems, enabling comprehensive analyses of controllability, stability, and optimization under impulsive effects. Consequently, this framework bridges the gap between discrete and continuous dynamics, advancing both theoretical mathematics and practical applications.

Impulsive differential equations play a crucial role in describing systems with sudden changes in state, governed by continuous dynamics interspersed with jump criteria. These equations provide a logical and robust framework for modeling abrupt transitions commonly observed in real-world processes. Their versatility has led to extensive research and development in this area. For instance, in [4], solutions to fractional equations such as the Fitzhugh-Nagumo equation, the Newell-Whitehead-Segel equation, and the Zeldovich equation were explored, demonstrating their practical significance. Similarly, [5] addressed an inverse coefficient problem for the conformable time-diffusion equation, retrieving time-dependent diffusion coefficients with precision. Furthermore, [6] proposed a novel mathematical formation model using the fractional Atangana-Baleanu-Caputo derivative, showcasing the reliability and computational efficiency of this method. Applications of impulsive systems extend to biological models, as demonstrated in [8], where insect population dynamics were analyzed using exponential, hyperbolic, and trigonometric functions to solve second-order linear dynamic equations. Additionally, [9] discussed stability conditions that ensure input-to-state stability for hybrid systems, even under instability. The concept of controllability further enhances system stabilization by constraining behavior through the analysis of linear and nonlinear operators [10].

Matrix-based approaches further bolster the study of impulsive systems. They allow for efficient representation of multi-dimensional systems and enable the use of spectral analysis, eigenvalue computation, and matrix decompositions to study system properties like stability and controllability. The study of impulsive systems on time scales has also proven the existence and uniqueness of solutions for nonlinear impulsive dynamic equations [11]. In [13], the properties of impulsive Dirac systems on Sturmian time scales were examined, including the construction of self-adjoint operators. Additionally, [14] introduced a new transition matrix to analyze the controllability and observability of impulsive systems on time scales. This research highlights the flexibility and adaptability of the impulsive framework, enabling its application to nonuniform time domains [15].

In this paper, we address the sufficient and necessary controllability and observability conditions for matrix Sylvester adjoint dynamic impulsive systems over various time scales.

$$\begin{cases} X^{\Delta}(t) = P(t)X(t) + X(t)Q(t) + \mu(t)P(t)X(t)Q(t) + T_1(t)U(t)T_2^*(t) \\ X(t_k^+) = (I + L_k)X(t_k), \quad t = t_k \text{ k = 1,2,3....} \\ Y(t) = C(t)X(t) + D(t)U(t) \\ X(t_0) = X_0. \end{cases} \tag{1.1}$$

Where $X(t)$ is an $n \times n$ matrix, $U(t)$ is m$\times n$ input pricewise rd-continuous matrix called control input and $Y(t)$ is $p \times n$ output rd-continuous. Here $P(t), Q(t), T_1(t), T_2(t)$ and L_k are $n \times n, n \times n, n \times n, n \times m$ and $n \times n$ rd-continuous matrices respectively. C(t), D(t) are rd-continuous matrices of order $p \times n$ and $p \times m$ respectively. $X^{\Delta}(t)$ is the generalized Delta derivative of X, and t is from a time scales $\mathbb{T}$, which is a non-empty closed subset of $\mathbb{R}$ and μ is a graininess function. when $Q = P^*$(* denotes the transpose of matrix) equation (1.1) is called matrix Lyapunov dynamical system on time scale.

2. Preliminaries

We provide some preliminary information to help you understand the notation used in this paper. A summary of the time scales can be found in [6, 7]. A nonempty closed subset of the R real line is called a time scale $\mathbb{T}$. We usually write $\mathbb{T}^k = \mathbb{T}\{max\mathbb{T}\}$ if $max\mathbb{T} < \infty$, otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 2.1[6] Let $f: \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$ the delta derivative of $f^{\Delta}(t)$ is the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighbourhood U of τ such that

$$|[f(\sigma(\tau)) - f(s)] - f^{\Delta}(\tau)[\sigma(\tau) - s]| \leq \varepsilon|\sigma(\tau) - s|, \text{ for all } s \in U$$

Definition 2.2.[7]: The regressive function y(t) mapping from $\mathbb{T}$ to $\mathbb{R}$ is defined as$1 + \mu(t)y(t) \neq 0\ \forall\ t \in \mathbb{T}$. The combination of all regressive and right dense continuous function is represented as $\mathscr{R} = \mathscr{R}(t) = \mathscr{R}(\mathbb{T}, \mathbb{R})$. Similarly all positively regressive function is denoted by $\mathscr{R}^+ = \mathscr{R}^+(\mathbb{T}, \mathbb{R}) = \{y \in \mathscr{R}: 1 + \mu(t)y(t) > 0, \forall\ t \in \mathbb{T}\}$

Definition 2.3[6] If $F: \mathbb{T}^k \to \mathbb{R}$ is said to be anti-derivative of $f: \mathbb{T}^k \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ fulfilled, for all $t \in \mathbb{T}^k$, then

$$\int_a^t f(s)\Delta s = F(t) - F(a)$$

Definition 2.4 Let the matrices are $A \in C^{m\times n}(\mathbb{R}^{m\times n})$ and $B \in C^{p\times q}(\mathbb{R}^{p\times q})$ the the Kronecker product of A and B. we have defined to be the partitioned matrix written $(A \otimes B)$ is

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n2}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{11}B & a_{12}B & \cdots & a_{1n}B \end{bmatrix}$$

is an $mp \times nq$ matrix is in $C^{m\times n}(\mathbb{R}^{m\times n})$.

Definition 2.5. Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, we denote

$$\hat{A} = VecA = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}, \text{ where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \ (1 \leq j \leq n)$$

Here we converted the linear matrix Sylvester dynamic impulsive system on time scales to an equivalent KP dynamic impulsive system on time scales using vectorization operator. The dynamical system is

$$\begin{cases} z^{\Delta}(t) = G(t)z(t) + A(t)\widehat{U}(t), \ \mathrm{t} \in [t_{k-1}, t_k)_{\mathbb{T}} \\ z(t_k^+) = [I_n \otimes R_k]z(t_k), t = t_k, \ \mathrm{k} = 1,2,3\ldots \\ \hat{y}(t) = (\mathrm{I} \otimes C)(t)z(t) + (I \otimes D)\widehat{U}(t) \\ z(t_0) = z_0. \end{cases} \tag{2.1}$$

Where z(t) = Vec X(t), $\widehat{U}(t) =$ Vec U(t), $\hat{y}(t) =$ Vec Y(t), $R_k = (I_n + L_k)$ and G(t)= $[Q^* \otimes I + I \otimes P + \mu(t)(Q^* \otimes P)]$ A(t)= $[T_2^* \otimes T_1]$.

Now rearranged the linear adjoint dynamic system (2.1) as follows

$$\begin{cases} z^{\Delta}(t) = -G_k^T(t)z^{\sigma}(t) + A_k(t)\widehat{U}(t), \ \mathrm{t} \in [t_{k-1}, t_k)_{\mathbb{T}} \\ z(t_k^+) = [I_n \otimes R_k]z(t_k), t = t_k, \ \mathrm{k} = 1,2,3\ldots \\ \hat{y}(t) = (\mathrm{I} \otimes C_k)(t)z^{\sigma}(t) + (I \otimes D_k)\widehat{U}(t) \\ z(t_0) = z_0. \end{cases} \tag{2.2}$$

Remark 2.1.[3]. Clearly observe that, the matrix valued function X(t) is a solution (2.1) on $\mathbb{T}$ if and only if the vector valued function $z(t) = VecX(t)$ is a solution of the system (2.1) on $\mathbb{T}$.

Theorem 2.1.[7]. If $G \in C_{rd}\mathscr{R}\left(\mathbb{T}_+, M_{n^2\times n^2}(\mathbb{R})\right)$ and $l \in C_{rd}\left(\mathbb{T}_+, M_{n^2\times 1}(\mathbb{R})\right)$, then for each $(\tau, \eta) \in \mathbb{T}_+ \times \mathbb{R}^{n^2}$ the initial value problem

$$z^{\Delta}(t) = G(t)z(t) + l(t),\ \mathrm{z}(\tau) = \eta,$$

has a unique solution $z\colon \mathbb{T}_{(\tau)} \to \mathbb{R}^{n^2}$.

Lemma 2.1.[7]. If $G \in C_{rd}\mathscr{R}\left(\mathbb{T}_{+}, M_{n^2}(\mathbb{R})\right)$ and $l \in C_{rd}\left(\mathbb{T}_{+}, M_{n^2\times 1}(\mathbb{R})\right)$, then for each $(\tau, \eta) \in \mathbb{T}_{+} \times \mathbb{R}^{n^2}$ the initial value problem

$$z^{\Delta}(t) = G(t)z(t) + l(t),\ \mathrm{z}(\tau) = \eta,$$

has one and only one solution $z\colon \mathbb{T}_{(\tau)} \to \mathbb{R}^{n^2}$ is given by

$$z(t) = \psi_{G}(t,\tau)\eta + \int_{\tau}^{t} \psi_{G}\big(t, \sigma(s)\big) l(s) \Delta s\,, t \geq \tau.$$

Lemma 2.2.[7]. If $G \in C_{rd}\mathscr{R}\left(\mathbb{T}_{+}, M_{n^2}(\mathbb{R})\right)$ and $l \in C_{rd}\left(\mathbb{T}_{+}, M_{n^2\times 1}(\mathbb{R})\right)$, then for each $(\tau, \eta) \in \mathbb{T}_{+} \times \mathbb{R}^{n^2}$ the initial value problem

$$z^{\Delta}(t) = -G^{T}(t)z^{\sigma}(t) + l(t),\ \mathrm{z}(\tau) = \eta,$$

has one and only one solution $z\colon \mathbb{T}_{(\tau)} \to \mathbb{R}^{n^2}$ is given by

$$z(t) = \psi_{\ominus G^{T}}(t,\tau)\eta + \int_{t_0}^{t} \psi_{\ominus G^{T}}\big(t, \sigma(s)\big) l(s) \Delta s\,, t \in \mathbb{T}_{(\tau)}.$$

Proposition 2.1. [8] The system (2.2) with $G_k \in M_{n^2}(R)$ constant, there exist scalar functions $\chi_0(t,\tau), \chi_1(t,\tau), \ldots\ldots \chi_{n^2-1}(t,\tau) \in C_{rd}^{\infty}(\mathbb{T}_{+}, \mathbb{R})$ such that the one and only one solution is given by

$$e_{G_k^T}(t,\tau) = \sum_{i=0}^{n^2-1} \chi_i\,(t,\tau)G^{i}.$$

3. Complete Controllability

In this section, we present the controllability in time variant and time invariant adjoint dynamic system (3) on time scales.

Lemma 3.1. for any $\in [t_{l-1}, t_l)_{\mathbb{T}}, l = 1,2,\ldots,k$ the solution of initial value problem (2.2) is given by

$$z(t) = \begin{cases} \psi_{G_1}^{T}(t_0,t)z_0 + \displaystyle\int_{t_0}^{t} \psi_{G_1}^{T}(\tau,t)A_1(\tau)\widehat{U}(\tau)\Delta\tau, l = 1 \\ \psi_{G_l}^{T}(t_{l-1},t)\{ \displaystyle\prod_{j=l-1}^{1}\left[I_n \otimes R_j\right] \prod_{j=l-1}^{1} \psi_{G_j}^{T}\big(t_{j-1},t_j\big)z_0 + \\ \displaystyle\sum_{j=1}^{l-2}\left(\prod_{i=l-1}^{j}\left[I_n \otimes R_j\right] \prod_{i=l-1}^{j+1} \psi_{G_i}^{T}(t_{i-1},t_i) \int_{t_{j-1}}^{t_j} \psi_{G_j}^{T}\big(\tau,t_j\big)A_j(\tau)\widehat{U}(\tau)\Delta\tau \right) \\ +[I_n \otimes R_{l-1}] \displaystyle\int_{t_{l-2}}^{t_{l-1}} \psi_{G_{l-1}}^{T}(\tau,t_{l-1})A_{l-1}(\tau)\widehat{U}(\tau)\Delta\tau \,\} \\ + \displaystyle\int_{t_{l-1}}^{t} \psi_{G_l}^{T}(\tau,t)A_l(\tau)\widehat{U}(\tau)\Delta\tau\,, l = 2,3,\cdots,k. \end{cases} \tag{3.1}$$

Proof: Form lemma 2.2 for $t \in [t_0, t_1]_{\mathbb{T}}$, we have

$$z(t) = \psi_{G_1}^{T}(t_0,t)z_0 + \int_{t_0}^{t} \psi_{G_1}^{T}(\tau,t)A_1(\tau)\widehat{U}(\tau)\Delta\tau,$$

similarly, for $t \in [t_{l-1}, t_l)_{\mathbb{T}}$, we have

$$z(t) = \psi^T_{G_l}(t_{l-1}, t)\{z(t_l)\} + \int_{t_{l-1}}^{t} \psi^T_{G_l}(\tau, t) A_l(\tau)\widehat{U}(\tau)\Delta\tau\,,$$

Also, form the system (2.2), we have

$$\begin{aligned} z(t_l) = {} & \prod_{j=l-1}^{1} [I_n \otimes R_j] \prod_{j=l-1}^{1} \psi^T_{G_j}(t_{j-1}, t_j) z_0 \\ & + \sum_{\square=1}^{\square-2} \left(\prod_{\square=\square-1}^{\square} [\square_\square \otimes \square_\square] \prod_{\square=\square-1}^{\square+1} \square^{\square}_{\square_\square}(\square_{\square-1}, \square_\square) \int_{\square_{\square-1}}^{\square_\square} \square^{\square}_{\square_\square}(\square, \square_\square)\square_\square(\square)\widehat{\square}(\square)\Delta\square \right) \\ & + [I_n \otimes R_{l-1}] \int_{t_{l-2}}^{t_{l-1}} \psi^T_{G_{l-1}}(\tau, t_{l-1}) A_{l-1}(\tau)\widehat{U}(\tau)\Delta\tau\,, l = 2,3,\cdots,k, \end{aligned}$$

therefore for $t \in (t_{l-1}, t_l)_{\mathbb{T}}$, we have

$$\begin{aligned} z(t) = {} & \psi^T_{G_l}(t_{l-1}, t) \left\{ \prod_{j=l-1}^{1} [I_n \otimes R_j] \prod_{j=l-1}^{1} \psi^T_{G_j}(t_{j-1}, t_j) z_0 \right. \\ & + \sum_{\square=1}^{\square-2} \left(\prod_{\square=\square-1}^{\square} [\square_\square \otimes \square_\square] \prod_{\square=\square-1}^{\square+1} \square^{\square}_{\square_\square}(\square_{\square-1}, \square_\square) \int_{\square_{\square-1}}^{\square_\square} \square^{\square}_{\square_\square}(\square, \square_\square)\square_\square(\square)\widehat{\square}(\square)\Delta\square \right) \\ & \left. + [I_n \otimes R_{l-1}] \int_{t_{l-2}}^{t_{l-1}} \psi^T_{G_{l-1}}(\tau, t_{l-1}) A_{l-1}(\tau)\widehat{U}(\tau)\Delta\tau \right\} + \int_{t_{l-1}}^{t} \psi^T_{G_l}(\tau, t) A_l(\tau)\widehat{U}(\tau)\Delta\tau. \end{aligned}$$

After repeating the above same process, we get the desire results.

Theorem 3.1.

i. If there exist at least k$\in \{1,2,\dots,l\}$ such that

$$rank\,\{H_k(t_{k-1}, t_k, t_f)\} = n^2$$

then the impulsive system (2.2) is controllable on $[t_0, t_1]_{\mathbb{T}}(t_f \in [t_{k-1}, t_k)_{\mathbb{T}})$.

ii. Suppose that $(I_n \otimes R_j) \neq -1,\ j = 1,2,\dots,k$. If impulsive system (2.2) is controllable on $[t_0, t_1]_{\mathbb{T}}(t_f \in [t_{k-1}, t_k)_{\mathbb{T}})$, then

$rank\,\{H_1, \dots, H_l\} = n^2$.

Proof: (i). Let k$\in \{1,2,\dots,l\}$ such that the $rank\,\{H_k(t_{k-1}, t_k, t_f)\} = n^2$ i.e., the matrix $H_k(t_{k-1}, t_k, t_f)$ is invertible then for a given $z_0 \in \mathbb{R}^{n^2}$, we choose a control function given as

$$\widehat{U}(t) = \begin{cases} -A_1{}^T(t)\psi_{G_1}\left(t,t_f\right)H_1^{-1}\psi_{G_1}^T\left(t_0,t_f\right)z_0, \quad for\ t \in [t_0,t_1]_{\mathbb{T}}, 2 \le k \le l-1, \\ -A_k{}^T(t)\psi_{G_k}\left(t,t_f\right)H_k^{-1}\psi_{G_k}^T\left(t_{k-1},t_f\right)\prod_{j=k-1}^{1}\left[I_n \otimes R_j\right]\prod_{j=k-1}^{1}\psi_{G_k}^T\left(t_{j-1},t_j\right)z_0, for\ t \in [t_{k-1}, \\ 0 \qquad if\ t \in \left[t_0,t_f\right]_{\mathbb{T}} \backslash [t_{k-1},t_k)_{\mathbb{T}}, \\ -A_l{}^T(t)\psi_{G_l}\left(t,t_f\right)H_l^{-1}\psi_{G_l}^T\left(t_{l-1},t_f\right)\prod_{j=l-1}^{1}\left[I_n \otimes R_j\right]\prod_{j=l-1}^{1}\psi_{G_j}^T\left(t_{j-1},t_j\right)z_0, for\ t \in [t_{l-1}, \\ 0 \qquad if\ t \in \left[t_0,t_f\right]_{\mathbb{T}} \backslash [t_{l-1},t_l)_{\mathbb{T}}, \end{cases} \tag{3.5}$$

obviously, the control function $\widehat{U}(t)$ is a piecewise rd-continuous on $[t_0,t_1]_{\mathbb{T}}$. By Lemma 3.1. we obtain

$$z\left(t_f\right) = \psi_{G_1}^T\left(t_0,t_f\right)z_0 - \int_{t_0}^{t_f} \psi_{G_1}^T\left(\tau,t_f\right)A_1(\tau)A_1{}^T(\tau)\psi_{G_1}\left(\tau,t_f\right)H_1^{-1}\psi_{G_1}^T\left(t_0,t_f\right)z_0\Delta\tau,$$

by using equation (3.2), we have

$$z\left(t_f\right) = \psi_{G_1}^T\left(t_0,t_f\right)z_0 - H_1H_1^{-1}\psi_{G_1}^T\left(t_0,t_f\right)z_0 = 0\ for\ t \in [t_0,t_1]_{\mathbb{T}},$$

then the system (2.2) is a controllable on $[t_0,t_1]_{\mathbb{T}}$.

Next, for $2 \le k \le l-1$, and $t \in [t_{k-1},t_k)_{\mathbb{T}}$,

$$\begin{aligned} z\left(t_f\right) = {} & \psi_{G_k}^T\left(t_{k-1},t_f\right)\prod_{j=k-1}^{1}\left[I_n \otimes R_j\right]\prod_{j=k-1}^{1}\psi_{G_j}^T\left(t_{j-1},t_j\right)z_0 \\ & - \int_{t_{k-1}}^{t} \psi_{G_k}^T\left(\tau,t_f\right)A_k(\tau)A_k{}^T(\tau)\psi_{G_k}\left(\tau,t_f\right)H_k^{-1}\psi_{G_k}^T\left(t_{k-1},t_f\right)\prod_{j=k-1}^{1}\left[I_n \right. \\ & \left. \otimes R_j\right]\prod_{j=k-1}^{1}\psi_{G_j}^T\left(t_{j-1},t_j\right)z_0\Delta\tau, \end{aligned}$$

it follows that from equation (3.3)

$$z\left(t_f\right) = 0\ for\ t \in [t_{k-1},t_k)_{\mathbb{T}},$$

and similarly, we have

$$z\left(t_f\right) = 0\ for\ t \in [t_{l-1},t_l)_{\mathbb{T}},$$

thus, the system (2.2) is a controllable on $\left[t_0,t_f\right]_{\mathbb{T}}$. So (i) holds.

(ii). Suppose that (2.2) is controllable on $\left[t_0,t_f\right]_{\mathbb{T}}$. We have to show that

$$rank\ \{H_1,\dots,H_l\} = n^2$$

Assume that

$$rank\ \{H_1,\dots,H_l\} < n^2$$

Then, there exists a non-zero $z_\alpha \neq 0 \in \mathbb{R}^{n^2}$ such that

$$z_\alpha^T H_j\left(t_{j-1}, t_j, t_f\right) z_\alpha = 0, j = 1,2, \dots, l.$$

For j=1

$$z_\alpha^T H_1 z_\alpha = \int_{t_0}^{t_f} z_\alpha^T \psi_{G_1}^T\left(\tau, t_f\right) A_1(\tau) A_1{}^T(\tau) \psi_{G_1}\left(\tau, t_f\right) z_\alpha \Delta\tau,$$

as $z_\alpha^T \psi_{G_1}^T\left(t, t_f\right) A_1(t)$ is rd-continuous functions so

$$\left\| z_\alpha^T \psi_{G_1}^T\left(\tau, t_f\right) A_1(t) \right\|^2 = 0.$$

Which according to

$$A_1{}^T(\tau) \psi_{G_1}^T\left(t, t_f\right) z_\alpha = 0, t \in [t_0, t_1]_{\mathbb{T}}. \tag{3.6}$$

For $k = 2,3, \cdots, l-1$.

$$z_\alpha^T H_k z_\alpha = \int_{t_{k-1}}^{t_f} z_\alpha^T \psi_{G_k}^T\left(\tau, t_f\right) A_k(\tau) A_k{}^T(\tau) \psi_{G_k}\left(\tau, t_f\right) z_\alpha \Delta\tau = 0,$$

$$z_\alpha^T \psi_{G_k}^T\left(t, t_f\right) A_k(t) A_k{}^T(t) \psi_{G_k}\left(t, t_f\right) z_\alpha = \left\| z_\alpha^T \psi_{G_k}^T\left(t, t_f\right) A_k(t) \right\|^2$$

$$A_k{}^T(t) \psi_{G_k}^T\left(t, t_f\right) z_\alpha = 0, t \in [t_{k-1}, t_k)_{\mathbb{T}}, \tag{3.7}$$

Similarly,

$$A_l{}^T(t) \psi_{G_l}^T\left(t, t_f\right) z_\alpha = 0, t \in [t_{l-1}, t_l)_{\mathbb{T}}, \tag{3.8}$$

However, the impulsive system (2.2) is controllability on $[t_0, t_1]_{\mathbb{T}}$, and so choosing $z_0 = z_\alpha$, there exists a piecewise rd-continuous control function $\widehat{U}(t)$ such that

$$0 = z\left(t_f\right) = \psi_{G_1}^T\left(t_0, t_f\right) z_\alpha + \int_{t_0}^{t_f} \psi_{G_1}^T\left(\tau, t_f\right) A_1(\tau) \widehat{U}(\tau) \Delta\tau; l = 1 \tag{3.9}$$

Multiply through by z_α^T in (3.9) and by using the transpose of the equations (3.6), we have

$$\psi_{G_1}^T\left(t_0, t_f\right) z_\alpha^T z_\alpha = 0. \tag{3.10}$$

Similarly,

$$
\begin{aligned}
z\left(t_f\right) = \psi_{G_l}^T\left(t_{l-1}, t_f\right) \Bigg\{ & \prod_{j=l-1}^{1} \left[I_n \otimes R_j\right] \prod_{j=l-1}^{1} \psi_{G_j}^T\left(t_{j-1}, t_j\right) z_\alpha \\
& + \sum_{\square=1}^{\square-2} \Bigg(\prod_{\square=\square-1}^{1} [\square_{\square} \\
& \otimes R_j] \prod_{i=l-1}^{j+1} \psi_{G_i}^T(t_{i-1}, t_i) \int_{t_{j-1}}^{t_j} \psi_{G_j}^T\left(\tau, t_j\right) A_j(\tau) \widehat{U}(\tau) \Delta\tau \Bigg) \\
& + [I_n \otimes R_{l-1}] \int_{t_{l-2}}^{t_{l-1}} \psi_{G_{l-1}}^T(\tau, t_{l-1}) A_{l-1}(\tau) \widehat{U}(\tau) \Delta\tau \Bigg\} \\
+ \int_{t_{l-1}}^{t_f} \psi_{G_l}^T\left(\tau, t_f\right) A_l(\tau) \widehat{U}(\tau) \Delta\tau, l = 2,3, \ldots
\end{aligned}
\tag{3.11}
$$

Multiply by $\psi_{G_1}^T(t_1, t_2)\psi_{G_2}^T(t_2, t_3)\ldots\psi_{G_k}^T\left(t_{l-1}, t_f\right)$ and z_α^T in the equation (3.11), using equations (3.7), and (3.8), we have

$$
\prod_{i=2}^{l} \left[I_n \otimes R_j\right] z_\alpha^T z_\alpha = 0, \tag{3.12}
$$

From equations (3.10) and (3.12), according to that $z_\alpha^T z_\alpha = 0$. This contradicts $z_\alpha \neq 0$ and so, we conclude that

$$
rank\, \{H_1, \ldots, H_l\} = n^2
$$

Theorem 3.2. Suppose that $(I_n \otimes R_j) \neq -1, j = 1,2, \ldots, k$ and $G_k(t) = G_k, A_k(t) = A_k$ are constant matrices. Then, the system (2.2) is a controllable on $\left[t_0, t_f\right]_{\mathbb{T}} (t \in [t_{l-1}, t_l)_{\mathbb{T}})$, if and only if

$$
rank\, \{M_1, M_2, \ldots, M_l\} = n^2 \tag{3.13}
$$

Since, $M_j = \left[{A_j}^T \; {A_j}^T G_j \; \ldots \; {A_j}^T {G_j}^{n^2-1}\right], j = 1,2, \ldots, l.$

Proof: Suppose that the system (2.2) is a controllability on $\left[t_0, t_f\right]_{\mathbb{T}}$. If the rank condition (3.13) does not hold, if there exist $z_\alpha \in \mathbb{R}^{n^2}$ with $z_\alpha \neq 0$, such that

$$
A_j {G_j}^i z_\alpha = 0. \tag{3.14}
$$

For, $\mathrm{j} = 1, \ldots, \mathrm{k}, \mathrm{i} = 0,1, \ldots, n^2 - 1$

We Consider

$$
H_1\left(t_0, t_f, t_f\right) z_\alpha = \int_{t_0}^{t_f} e_{G_1}^T\left(\tau, t_f\right) A_1 {A_1}^T e_{G_1}\left(\tau, t_f\right) z_\alpha \Delta\tau.
$$

From equation (3.14) and using Proposition 2.1, we have

$$H_1\big(t_0, t_f, t_f\big)z_\alpha = \int_{t_0}^{t_f} e^T_{G_1}\big(\tau, t_f\big)\, A_1 {A_1}^T \sum_{i=0}^{n^2-1} \chi_{1i}\big(\tau, t_f\big)\, {G_1}^i z_\alpha \Delta\tau,$$

$$= \int_{t_0}^{t_f} e^T_{G_1}\big(\tau, t_f\big)\, A_1 \sum_{i=0}^{n^2-1} \chi_{1i}\big(\tau, t_f\big)\, {A_1}^T {G_1}^i z_\alpha \Delta\tau = 0.$$

By again equation (3.14) and using Proposition 2.1, according to

$$H_k\big(t_{k-1}, t_k, t_f\big)z_\alpha = \int_{t_{k-1}}^{t_f} e^T_{G_k}\big(\tau, t_f\big)\, A_k {A_k}^T e_{G_k}\big(\tau, t_f\big) z_\alpha \Delta\tau$$

$$= \int_{t_{k-1}}^{t_f} e^T_{G_k}\big(\tau, t_f\big)\, A_k \sum_{i=0}^{n^2-1} \chi_{1i}\big(\tau, t_f\big)\, {A_k}^T {G_k}^i z_\alpha \Delta\tau = 0.$$

For $2 \leq k \leq l-1$, similarly, $H_l\big(t_{l-1}, t_l, t_f\big)z_\alpha = 0$, according to

$$rank\ \{M_1, \dots, M_l\} < n^2$$

Hence, it is contradicting the conclusion (ii) of Theorem (3.1) and thus, we can conclude that the condition (3.13) is true.

Conversely, assume that the condition (3.13) is satisfied. If the impulsive system (2.2) is not controllable on $t \in \big[t_0, t_f\big]_{\mathbb{T}} (t \in [t_{l-1}, t_l)_{\mathbb{T}})$, then it follows that from the conclusion (i) of Theorem 3.1., that the matrices $H_1\big(t_0, t_f, t_f\big)$, $H_k\big(t_{k-1}, t_k, t_f\big)$ and $H_l\big(t_{l-1}, t_l, t_f\big)$ are not invertible. If there exist $z_\alpha \in \mathbb{R}^{n^2}$ with $z_\alpha \neq 0$, such that

$$z^T_\alpha H_1\big(t_0, t_f, t_f\big)z_\alpha = \int_{t_0}^{t_f} z^T_\alpha e^T_{G_1}\big(\tau, t_f\big)\, A_1 {A_1}^T e_{G_1}\big(\tau, t_f\big) z_\alpha \Delta\tau = 0,$$

$$z^T_\alpha H_k\big(t_{k-1}, t_k, t_f\big)z_\alpha = \int_{t_{k-1}}^{t_f} z^T_\alpha e^T_{G_k}\big(\tau, t_f\big)\, A_k {A_k}^T e_{G_k}\big(\tau, t_f\big) z_\alpha \Delta\tau = 0, 2 \leq k \leq l-1$$

$$z^T_\alpha H_l\big(t_{l-1}, t_l, t_f\big)z_\alpha = \int_{t_{l-1}}^{t_f} z^T_\alpha e^T_{G_l}\big(\tau, t_f\big)\, A_l {A_l}^T e_{G_l}\big(\tau, t_f\big) z_\alpha \Delta\tau = 0,$$

Exactly same as in proof of Theorem 3.1., according to

$${A_1}^T e_{G_1}\big(t, t_f\big) z_\alpha = 0. for\ t \in \big[t_0, t_f\big]_{\mathbb{T}} \tag{3.15}$$

$${A_k}^T e_{G_k}\big(t, t_f\big) z_\alpha = 0. for\ t \in [t_{k-1}, t_k)_{\mathbb{T}}. \tag{3.16}$$

Where $2 \leq k \leq l-1$, and

$${A_l}^T e_{G_l}\big(t, t_f\big) z_\alpha = 0. for\ t \in [t_{l-1}, t_l)_{\mathbb{T}} \tag{3.17}$$

Differentiating equations (3.13), (3.14) and (3.15) i^{th} times, where $(0 \leq i \leq n^2 - 1)$, we obtain

$$A_1{}^T G_1{}^i e_{G_1}(\tau, t_f) z_\alpha = 0. for\ t \in [t_0, t_1]_{\mathbb{T}}. \tag{3.18}$$

$$A_k{}^T G_k{}^i e_{G_k}(\tau, t_f) z_\alpha = 0. for\ t \in [t_{k-1}, t_k)_{\mathbb{T}}. \tag{3.19}$$

Where $2 \leq k \leq l-1$, and

$$A_l{}^T G_l{}^i e_{G_l}(\tau, t_f) z_\alpha = 0. for\ t \in [t_{l-1}, t_l)_{\mathbb{T}}. \tag{3.20}$$

If we take $t = t_f$ in equations (3.18), (3.19) and (3.20), then it follows that $A_j{}^T G_j{}^i z_\alpha = 0, for\ j = 1, \dots, k$ and $i = 0,1, \dots, n^2 - 1$. Which implies that the rank condition (3.11) fails, which gives contradiction. So, the impulsive system (2.1) is controllable on $t \in [t_0, t_f]_{\mathbb{T}} (t_f \in [t_{l-1}, t_l)_{\mathbb{T}})$. So that the system (2.2) is controllable by theorem 3.2.

4. Complete Observability

In this section, we present the observability in time variant and time invariant adjoint dynamic system (2.3) on time scales.

Definition 4.1. The system (2.2) is said to be completely observability on $[t_0, t_f]_{\mathbb{T}} (t_f > t_0)$ if any initial state $z(t_0) = z_0 \in \mathbb{R}^{n^2}$ is uniquely determined by the corresponding system input $\widehat{U}(t)$ and the system output y(t) for $[t_0, t_f]_{\mathbb{T}}$.

Theorem 4.1. Suppose that $[I_n \otimes R_j] \geq 0, j = 1,2, \dots, l$. Then, the impulsive system (2.2) is observable on $t \in [t_0, t_f]_{\mathbb{T}} (t_f \in [t_{l-1}, t_l)_{\mathbb{T}})$ if and only if the matrix

$$W(t_0, t_f) \coloneqq W(t_0, t_0, t_1) + \sum_{j=2}^{l-1} \prod_{i=1}^{j} [I_n \otimes R_i] W(t_0, t_{j-1}, t_j) + \prod_{i=1}^{l} [I_n \otimes R_i] W(t_0, t_{l-1}, t_f)$$

is invertible, where

$$W(t_0, t_0, t_1) \coloneqq \int_{t_0}^{t_1} \psi_{G_1}(t_0, \tau) (\mathrm{I} \otimes C_1)^T(\tau) (\mathrm{I} \otimes C_1)(\tau) \psi_{G_1}^T(t_0, \tau) \Delta\tau,$$

$$W(t_0, t_{j-1}, t_j) \coloneqq \int_{t_{j-1}}^{t_j} \Omega_j(t_0, \tau) (\mathrm{I} \otimes C_j)^T(\tau) (\mathrm{I} \otimes C_j)(\tau) \Omega_j^T(t_0, \tau) \Delta\tau, j = 2, \dots, l-1,$$

and

$$W(t_0, t_{l-1}, t_f) \coloneqq \int_{t_{l-1}}^{t_f} \Omega_l(t_0, \tau,) (\mathrm{I} \otimes C_l)^T(\tau) (\mathrm{I} \otimes C_l)(\tau) \Omega_l^T(t_0, \tau) \Delta\tau,$$

with

$$\Omega_j^T(t_0, \tau) = \psi_{G_j}^T(t_{j-1}, \tau) \psi_{G_{j-1}}^T(t_{j-2}, t_{j-1}) \dots \psi_{G_j}^T(t_0, t_1), j = 1, \dots, k$$

Proof: Assume that the matrix $W(t_0, t_f)_{\mathbb{T}}$ is invertible. From the system (2.2) and the equation (3.1), we have

For $t \in [t_0, t_f]_{\mathbb{T}}$

$$\begin{aligned} y(t) = (\mathrm{I} \otimes C_1)(t)\psi_{G_1}^{T}(t_0,\tau)z_0 + (\mathrm{I} \otimes C_1)(t) \int_{t_0}^{t_1} \psi_{G_1}^{T}(\tau,t)A_1(\tau)\widehat{U}(t)\Delta\tau \\ + (\mathrm{I} \otimes D_1)\widehat{U}(t), \end{aligned} \tag{4.1}$$

and

for $t \in (t_{k-1}, t_k]_{\mathbb{T}}, k = 2,3, \dots l.$

$$\begin{aligned} y(t) = (\mathrm{I} \otimes C_k)(t)\psi_{G_k}^{T}(t_{k-1},t)\{ \prod_{j=k-1}^{1} \left[I_n \otimes R_j\right] \prod_{j=k-1}^{1} \psi_{G_j}^{T}\left(t_{j-1},t_j\right) z_0 (\mathrm{I} \otimes C_k)(t) \\ + \sum_{j=1}^{k-2} \left(\prod_{i=k-1}^{k} [I_n \otimes R_i] \prod_{i=k-1}^{k+1} \psi_{G_i}^{T}(t_{i-1},t_i) \int_{t_{j-1}}^{t_j} \psi_{G_j}^{T}\left(\tau,t_j\right) A_j(\tau)\widehat{U}(\tau)\Delta\tau \right) (\mathrm{I} \otimes C_k)(t) \\ + [I_n \otimes R_{l-1}] \int_{t_{k-2}}^{t_{k-1}} \psi_{G_{k-1}}^{T}(\tau,t_{k-1})\, A_{k-1}(\tau)\widehat{U}(\tau)\Delta\tau \} (\mathrm{I} \otimes C_k)(t) \\ + \int_{t_{k-1}}^{t} \psi_{G_k}^{T}(\tau,t)\, A_k(\tau)\widehat{U}(\tau)\Delta\tau + (\mathrm{I} \otimes D_k)\widehat{U}(t) \end{aligned} \tag{4.2}$$

From the Definition 4.1., that the observability of the system (2.2) is

$$y(t) = \begin{cases} (\mathrm{I} \otimes C_k)(t)\psi_{G_k}^{T}(t_0,t)z_0, t \in [t_0,t_1]_{\mathbb{T}} \\ \prod_{j=k-1}^{1} \left[I_n \otimes R_j\right](\mathrm{I} \otimes C_k)\psi_{G_k}^{T}(t_0,t)\, z_0, t \in (t_{k-1},t_k]_{\mathbb{T}}, k = 2,3, \dots, l \end{cases} \tag{4.3}$$

as $\widehat{U}(t) = 0$. Now multiply by $\Omega_k(t_0,t)(\mathrm{I} \otimes C_k)^{T}(t)$ to both sides of the equation (4.3) and integrating with respect to $t_0\ to\ t_f$, we get

$$\begin{aligned} \int_{\square_0}^{\square_\square} \Omega_\square(\square_0,\square,)(\mathrm{I} \otimes \square_\square)^{\square}(\square)\square(\square)\Delta\square \\ = \Bigg[\int_{\square_0}^{\square_1} \square_{\square_1}(\square_0,\square,)(\mathrm{I} \otimes \square_1)^{\square}(\square)(\mathrm{I} \otimes \square_1)(\square)\square_{\square_1}^{\square}(\square_0,\square)\Delta\square \\ + \sum_{\square=2}^{\square-1} \prod_{\square=\square}^{\square} [\square_\square \otimes \square_\square] \int_{\square_{\square-1}}^{\square_\square} \Omega_\square(\square_0,\square,)(\mathrm{I} \otimes \square_\square)^{\square}(\square)(\mathrm{I} \otimes \square_\square)(\square)\Omega_\square^{\square}(\square_0,\square)\Delta\square \\ + \prod_{i=1}^{k} [I_n \otimes R_i] \int_{t_{l-1}}^{t_f} \Omega_l(t_0,\tau,)(\mathrm{I} \otimes C_l)^{T}(\tau)(\mathrm{I} \otimes C_l)(\tau)\Omega_l^{T}(t_0,\tau)\Delta\tau \Bigg] z_0 \end{aligned}$$

and so,

$$\int_{t_0}^{t_f} \Omega_k(t_0,\tau,)(\mathrm{I}\otimes C_k)^T(\tau)y(\tau)\Delta\tau = W(t_0,t_f)z_0\,. \quad (4.4)$$

Obviously, the left-hand side of equation (4.4) depends on $y(t), t \in [t_0, t_f]_{\mathbb{T}}$. Since the matrix $W(t_0,t_f)$ is invertible, then from linear algebraic equations (4.4) we deduce that $z(t_0) = z_0$ is a uniquely determined by the corresponding system output $y(t), t \in [t_0, t_f]_{\mathbb{T}}$.

Conversely, assume that the matrix $W(t_0,t_f)$ is not invertible, then there exists a nonzero $z_\alpha \in \mathbb{R}^{n^2}$, such that

$$z_\alpha^T W(t_0,t_f)z_\alpha = 0.$$

Since, $[I_n \otimes R_j] \geq 0,\ j = 1,2,\dots,l,\quad W(t_0,t_0,t_1),\quad W(t_0,t_{j-1},t_j)\ for\ j = 2,3,\dots l-1$ and $W(t_0,t_{l-1},t_f)$ are positive semidefinite matrices, we get

$$\begin{gathered} z_\alpha^T W(t_0,t_0,t_1)z_\alpha = 0. \\ z_\alpha^T W(t_0,t_{j-1},t_j)z_\alpha = 0.\ \ for\ j = 2,\dots l-1 \\ z_\alpha^T W(t_0,t_{l-1},t_f)z_\alpha = 0. \end{gathered} \quad (4.5)$$

We choose $z_0 = z_\alpha$. Thus, from equations (4.3) and (4.5), according to

$$\begin{aligned} \int_{t_0}^{t_f} y^T(\tau)y(\tau)\Delta\tau &= \int_{t_0}^{t_1} z_\alpha^T \psi_{G_1}(t_0,\tau,)(\mathrm{I}\otimes C_1)^T(\tau)(\mathrm{I}\otimes C_1)(\tau)\psi_{G_1}^T(t_0,\tau)z_\alpha\Delta\tau \\ &+\sum_{j=2}^{l-1}\left[\prod_{i=1}^{j}[I_n\otimes R_i]\right]^2 \int_{t_{j-1}}^{t_j} z_\alpha^T\Omega_j(t_0,\tau,)(\mathrm{I}\otimes C_j)^T(\tau)(\mathrm{I}\otimes C_j)(\tau)\Omega_j^T(t_0,\tau)z_\alpha\Delta\tau \\ &+\left[\prod_{i=1}^{l}[I_n\otimes R_i]\right]^2 \int_{t_{l-1}}^{t_f} z_\alpha^T\Omega_l(t_0,\tau,)(\mathrm{I}\otimes C_l)^T(\tau)(\mathrm{I}\otimes C_l)(\tau)\Omega_l^T(t_0,\tau)z_\alpha\Delta\tau. \end{aligned}$$

implies

$$\int_{t_0}^{t_f} \|y(\tau)\|^2\Delta\tau = 0.$$

According to

$$0 = y(t) = \begin{cases} (\mathrm{I}\otimes C_1)(t)\psi_{G_1}^T(t_0,t)z_0, t \in [t_0,t_1]_{\mathbb{T}} \\ \prod_{i=1}^{k}[I_n\otimes R_i](\mathrm{I}\otimes C_k)\Omega_k^T(t_0,t)\,z_0, t \in (t_{k-1},t_k]_{\mathbb{T}}, k = 2,\dots,l-1, \\ \prod_{i=1}^{l}[I_n\otimes R_i](\mathrm{I}\otimes C_l)\Omega_l^T(t_0,t)\,z_0, t \in (t_{l-1},t_l]_{\mathbb{T}}. \end{cases}$$

The last equality implies, by Definition 4.1., that the system (2.2) is not observable on $t \in [t_0, t_f]_{\mathbb{T}} (t_f \in [t_{l-1}, t_l)_{\mathbb{T}})$.

Theorem 4.2. Assume that $[I_n \otimes R_j] \geq 0, j = 1,2,\dots,l$ and $G_k(t) = G_k, (\mathrm{I} \otimes C_k)(t) = (\mathrm{I} \otimes C_k)$ are constant matrices. Then, the system (2.2) is observable on $t \in [t_0, t_f]_{\mathbb{T}} (t \in [t_{l-1}, t_l)_{\mathbb{T}})$, if and only if $rank\ (S) = n^2$.

Let us define the following matrix

$$S = \begin{bmatrix} (\mathrm{I} \otimes G_{1j}) \\ (\mathrm{I} \otimes G_{1k}) P_j^T \\ \vdots \\ (\mathrm{I} \otimes G_k) (P_j^T)^{n^2-1} \end{bmatrix} \tag{4.6}$$

Proof: Assume that $rank\ (S) = n^2$. And we aim to show that the system (2.2) is observability on $t \in [t_0, t_f]_{\mathbb{T}} (t \in [t_{l-1}, t_l)_{\mathbb{T}})$. If otherwise, namely the system (2.2) is not observability then by Theorem 4.1., according to the matrix $W(t_0, t_f)$ is not invertible, which leads to that there exists a nonzero vector $z_\alpha \neq 0$. Then by using Theorem 4.1., we have

$$z_\alpha^T W(t_0, t_0, t_1) z_\alpha = \int_{t_0}^{t_1} z_\alpha^T e_{G_1}(t_0, \tau)(\mathrm{I} \otimes C_1)(\mathrm{I} \otimes C_1)^T e_{G_1}^T(t_0, \tau) z_\alpha \Delta\tau$$

$$= \int_{t_0}^{t_1} \left[(\mathrm{I} \otimes C_1) e_{G_1}^T(t_0, \tau) z_\alpha\right]^T \left[(\mathrm{I} \otimes C_1) e_{G_1}^T(t_0, \tau) z_\alpha\right]^T \Delta\tau,$$

Similarly,

$$(\mathrm{I} \otimes C_j) \Omega_j^T(t_0, t) z_\alpha = 0, j = 1, \dots, l-1, \tag{4.8}$$

and

$$(\mathrm{I} \otimes C_l) \Omega_l^T(t_0, t) z_\alpha = 0, \tag{4.9}$$

Where $\Omega_j^T(t_0, t) = e_{G_j}^T(t_{j-1}, t) e_{G_{j-1}}^T(t_{j-2}, t_{j-1}) \dots e_{G_j}^T(t_0, t_1)$.

Obviously, at $t = t_0$, we obtain $(\mathrm{I} \otimes C_j) z_\alpha = 0, for\ \ j = 1, \dots, l-1,$ and differentiating the equations (4.7), (4.8) and (4.9) $n^2 - 1$ times and evaluating the results at $t = t_0$ gives

$$(\mathrm{I} \otimes C_j) G_j^i z_\alpha = 0, i = 0,1,\dots,n^2-1, j = 1,2,\dots,l \tag{4.10}$$

Therefore, by the equations (4.6) and (4.9) we have $S z_\alpha = 0$, And furthermore, $z_\alpha \neq 0$ implies that $rank\ (S) < n^2$ which leads to a contradiction with the assumptions that $rank\ (S) = n^2$.

Conversely, we assume that $rank\ (S) < n^2$. Thus, there exists $z_\alpha \neq 0$ such that $S z_\alpha = 0$, which leads to the equation (4.10).

From equation (4.10) and using Proposition 2.1. we obtain

$$W(t_0, t_0, t_1) z_\alpha = \int_{t_0}^{t_1} \sum_{i=0}^{n^2-1} \chi_{1i}(t_0, \tau) e_{G_1}(t_0, \tau)(\mathrm{I} \otimes C_1)^T (\mathrm{I} \otimes C_1) e_{G_1}^T(t_0, \tau) z_\alpha \Delta\tau$$

$$= \int_{t_1}^{t_1} \sum_{i=0}^{n^2-1} \chi_{1i}(t_0,\tau) e_{G_1}(t_0,\tau)(\mathrm{I}\otimes C_1)^T(\mathrm{I}\otimes C_1) e_{G_1}^T(t_0,\tau) z_\alpha \Delta\tau = 0,$$

Similarly, for $j = 1,2,\ldots,l-1$

$$W\big(t_0, t_{j-1}, t_j\big) z_\alpha = 0,$$

and

$$W(t_0, t_{l-1}, t_l) z_\alpha = 0.$$

The equation (4.10) yields $W\big(t_0, t_f\big) z_\alpha = 0$. Since $z_\alpha \neq 0$, the matrix $W\big(t_0, t_f\big)$ is not invertible. Hence the system (2.2) is not observable, and it is contradicting with the assumption of observability.

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