Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

Impact Fuzzy Ideal Extension in Terms of Gamma Semigroup

[1]Vasantha.Gadipally, [2]Dr. Sri Lakshmi T
[1]GITAM, Vishakapatnam, AP, INDIA, vgadipal@gitam.in
[2]GITAM, Vishakapatnam, AP, INDIA, stalasil@gitam.in

Article History:
Received: 12-11-2024
Revised: 24-12-2024
Accepted: 16-01-2025

Abstract:
Our exploration into the properties of some fuzzy Semigroups and ideals is a deeply collaborative effort involving the contributions of many researchers in the field. This collective endeavour, which builds on the work of our peers, aims to deepen our understanding of fuzzy ideal semigroup and fuzzy extension in terms of gamma semigroups. Using a star zeta, we define, characterise and describe the different classes of fuzzy ideal extension in the gamma Semigroup, resulting from our shared research efforts. Investigating star zeta and fuzzy ideal properties and their results is a testament to the power of shared knowledge in our academic community. The description of many properties of fuzzy prime ideal and fuzzy semiprime in the gamma function further underscores the collaborative nature of scholarly research, making each member of our community feel included and valued.

Keywords: Fuzzy subsemigroups, Fuzzy ideal, fuzzy prime ideal, fuzzy ideal extension Semigroup, star zeta of fuzzy Semigroup.

1. Introduction

The work aims to explain the terms used by the authors and a quick review of semigroup theory. This study examined the fundamental properties and variations of semigroups. Early authors described an extensive range of semigroup kinds and verified various procedures. We studied the roots of semigroups and worked on perplexing assertions in regular semigroups. We found properties related to the semigroup theory written by four critical writers. Anton Schushewitch is the first semigroup theorist in history. In 1941, Clifford proved that 'if S is a collection of groups, then it is a semilattice of completely simple semigroups [1-3].' Furthermore, he proved that 'a band is a semilattice of rectangular bands.' Vagner did, however, provide the opposite semigroup [4-7]. This paper introduces and defines a new concept of Ṱ-norms and Š-norms (notations used to represent certain operations in fuzzy set theory) and their properties, which we denote throughout the paper, bringing a novel perspective to the field.

Zadeh introduced the essential concept of a fuzzy set in 1965 [8,9], leading to insightful discoveries and practical uses in various scientific fields. This seminal work paved the way for numerous writers who wrote after that time, confirming the necessity and value of the concept. Rosenfeld extended several group results to include ambiguous groupings. Additionally, he suggested noting fuzzy groups [10]. Wu studied and introduced traditional fuzzy subgroups. Rosenfeld also demonstrated that the fuzzy subgroup's homomorphic image equals one. Therefore, Anthony and Sherwood should have utilised these properties to study fuzzy homomorphisms. Initially, Authors Imtiaz, A., Alolaiyan, H., and Shuaib investigated the idea of Applications of conjunctive complex fuzzy or vague cosets and their interrelation within fuzzy normal subgroups, proving how impact in Sylow theory[11-19]. These historical developments in the field of fuzzy set theory and algebraic structures have laid the foundation for our current research. For example, the introduction of fuzzy (subgroupoids) subgroups and fuzzy (left, fight) ideals in the seminal publication of [20-24] Yiarayong marked the beginning of the study of fuzzy algebraic structures. Fuzzy set theory has now been extended to semigroups by several writers. For instance, Mursaleen, Srivastava, and Sunil [32] investigated particular novel spaces of statistically convergent and strongly summable sequences of fuzzy numbers.[25-31] Jun, Song, and Muhiuddin established the concept of hybrid structure in a set of parameters over an initial universe set as a parallel circuit of fuzzy sets and soft sets (or hesitant fuzzy sets) [34-38]. They used it on linear spaces and BCK/BCI algebras. A Russian scientist (1999) presented the soft set theory as a novel mathematical technique for handling uncertainties.[33] Torra, 2010; Torra & Narukawa, a generalisation of Zadeh's fuzzy or vague set. The hesitant fuzzy set is handy for expressing people's hesitancy in daily life, and it is a convenient tool to deal with uncertainty, which can be accurately and perfectly described in terms of decision-makers' opinions.

We introduce and define a new concept of Ț-norms and Š-norms (notations used to represent certain operations in fuzzy set theory) and their properties, which we denote throughout the paper, bringing a novel perspective to the field. This paper introduces the star of zeta, a key concept in our research, and its application to fuzzy ideal extension in gamma Semigroups. This extension is significant as it provides a new perspective on the properties and extensions of fuzzy semigroups. It has practical applications in various scientific fields, such as data analysis, pattern recognition, and decision-making under uncertainty. We aim to inspire and motivate further research and innovation by highlighting these practical applications. We present the notions of fuzzy subsemigroups and discuss their potential real-world applications. Using these notions, we consider characterisations of sub-semigroups and extensions of fuzzy semigroups. We also introduce the concept of the dot(product) with properties and star zeta with properties developed in this paper and discuss characterisations of the fuzzy ideal of gamma semigroups.

2. Objectives

This study comprehensively explores semigroups' fundamental properties and variations to review semigroup theory and its historical evolution. By examining the contributions of significant theorists, this study aims to comprehend the evolution of semigroups and their structural classifications. Fuzzy algebraic structures are also analysed in the research, including fuzzy subgroups, fuzzy ideals and their extensions, and the use of fuzzy set theory with semigroups. This study provides fresh insights into semigroup features by presenting and describing new concepts, including fuzzy ideal extension, star zeta, Ț-norms, Š-norms, and the dot product in the context of gamma semigroups. With two theorems and eleven lemmas that expand our theoretical and applied understanding of semigroup structures, the study provides strong mathematical evidence in favour of these concepts. The paper also highlights the valuable applications of fuzzy semigroups, emphasising how they can be used in various scientific domains. This work intends to stimulate additional research and creativity by characterising sub-semigroups and expanding fuzzy semigroup theory, promoting a deeper investigation of algebraic structures and their computing applications.

3. Preliminaries

Definition 3.1: [10] A non-empty set □ is called a Ternary Semigroup if there exists a mapping □x□x□ → □ is defined by abc ∈ □ and $(abc)de = a(bcd)e = ab(cde)$ For all a,b,c,d,e ∈ □.

Example: Let □ = { x√5 / x∈ Z- } where Z- is the set of negative odd integers. Then □ is a ternary Semigroup under usual multiplication.

Definition 3.2: [12] A non-empty set L of a ternary Semigroup □ is called

(i) A left ideal of □ if □□L ⊆ L

(ii) An interior ideal of □ if □L□ ⊆ L

(iii) A proper ideal of □ L□□ ⊆ L

(iv) An ideal of □ if L is a left ideal, a right ideal and an interior ideal of □

An ideal L of a ternary Semigroup □ is a proper ideal if L. ≠ □.

Definition 3.3: Let ζ be a fuzzy subset of Ternary Semigroup □, then (□,ζ) is called fuzzy Ternary Semigroup if ζ(pqr) ≥ min{ζ(p),ζ(q),ζ(r) } for all p,q,r,∈ □.

Example: Consider set □ = {p,q,r} with the ternary operation Θ and assigned membership values of fuzzy set ζ as follows.

Table of ternary operation:

Θ	p	q	r
p	p	q	r
q	q	r	p
r	r	p	q

Table-1(Caylley's table of p, q r)

And ζ(p) = 0.8 ; ζ(q) = 0.6; ζ(r) = 0.4

The graph shows this ternary operation and membership function as visualisation. Let us assume that nodes represent elements p,q,r, and edges represent the ternary operation; the membership values can be indicated next to the nodes. A textual representation of the fuzzy ternary semigroup is below.

Nodes: p (0.8); q (0.6); r (0.4)

Edges (ternary operation) :

(p, p, p) it tends to p; (p, q, r) it tends to q; (p, r, q) is tends to r; (q, q, q) is tends to r; (r, r, r) is tends to p

(q, r, p) is tends to q and so on.

Using the above steps, we can draw nodes and edges with labelled membership values and operation results. This visualisation technique is a helpful tool for understanding the relationships and operations within a fuzzy Ternary Semigroup. This Section provides a step-by-step guide on creating such visualisations and a detailed example of the algorithm.

```
import networkx as nx
import matplotlib.pyplot as plt

# Create graph
G = nx.Graph()

# Add nodes
G.add_node("a")
G.add_node("b")
G.add_node("c")

# Add edges with labels
G.add_edge("a", "b", weight=0.5)
G.add_edge("b", "c", weight=0.8)
G.add_edge("c", "a", weight=0.2)

# Draw graph
pos = nx.spring_layout(G)
nx.draw(G, pos,
with_labels=True)
edge_labels =
nx.get_edge_attributes(G,
'weight')
nx.draw_networkx_edge_labels(G,
pos, edge_labels=edge_labels)
plt.show()
```

Figure (Algorithm)

Definition 3.4: A non-empty set □ is called ternary gamma semigroup if there exists a map □Γ□Γ□. → □ is ternary gamma semigroup; then it is satisfies

(i) (p αq β r)γ s δ t = p α(q β r γ s)δ t = p αq β(r γ s δ t)
(ii) p α q β r ∈ □
for all p ,q , r, s, t ∈ □ and α,β, γ,δ ∈ Γ.

Definition 3.5: Let ζ be a fuzzy subset of the ternary gamma semigroup □, then it is called fuzzy ternary gamma semigroup if $\zeta_{(p\alpha q\beta r)} \geq \vee_{(p\alpha q\beta r)} min\,\{\zeta(p), \zeta(q), \zeta(r)\}$.If ζ is the fuzzy interior ideal of ternary gamma Semigroup □, then ζ(pαqβr) ≥ ζ(q); if ζ is the fuzzy left ideal of ternary gamma Semigroup □, then ζ(pαqβr) ≥ ζ(r); if ζ is the fuzzy right ideal of ternary gamma Semigroup □, then ζ(pαqβr) ≥ ζ(p).

Definition 3.6: A non-empty set Ƀ of □ is called generalised bi-ideal if Ƀ□Ƀ□Ƀ ⊆ Ƀ.The generalised bi-ideal Ƀ of □ is called bi-ideal if ɃɃɃ ⊆ Ƀ. A mapping ζ:Ӿ → [0,1] is called a fuzzy set of Ӿ. The fuzzy set ζ of □ is called generalised fuzzy bi-ideal if ζ(pαqβr) ≥ min{ζ(p),ζ(q),ζ(r)}. The generalised fuzzy bi-ideal ζ of □ is called a fuzzy bi-ideal if ζ(pαqβr) ≥ min{ζ(p),ζ(q),ζ(r)}. The fuzzy set ζ of □ is called generalised anti-fuzzy bi-ideal if ζ(pαqβr) ≤ max{ζ(p),ζ(q),ζ(r)}. The generalised fuzzy bi-ideal ζ of □ is called an anti-fuzzy bi-ideal if ζ(pαqβr) ≤ max{ζ(p),ζ(q),ζ(r)}.

4. Methods

4.1.Star fuzzing Semigroups:

let ξ ∈ F_T and x ∈ S then we defined ξ_* = {x∈s such that ξ(x) = ξ(0)}

1. Lemma:

Let ξ ∈ F_T (S) then (i) $(\xi^i)_* \subseteq \xi_*$ for all i ∈∞ (ii) $(\xi^{(i)})_* \subseteq \xi_*$

Let x ∈ $(\xi^i)_*$

Then $\xi^i(x) = \xi^i(0) = \xi(0)$

∵$\xi^i(x) \leq \xi(1)$

ξ(x) = ξ(0)

Then x∈ξ_*

Here $(\xi^{i})_* \subseteq \xi_*$

Similarly we can prove (ii) $(\xi^{(i)})_* \subseteq \xi_*$

2. Lemma:

Let $\xi \in F(S)$ and $K \in N$ then $\xi^{K+1}(x_1, x_2, \ldots, x_{k+1}) = \Lambda_{i=1}^{K+1} \xi(x_i)$

If $x_1, x_2, \ldots, x_k \in S$

Then $\xi^K (x_1, x_2, \ldots, x_k) \geq \Lambda_{(i=1)}^{K} \xi(x_i)$

Clearly, the result is valid for K = 1

Assume that it is valid for $K \geq 1$

Now $\xi^{(K+1)} (x_1, x_2, \ldots, x_{k+1})$

$= \xi^K . \xi(x_1, x_2, \ldots, x_{k+1}) \wedge \xi(x_{(k+1)})$

$= \Lambda_{(i=1)}^{K} \xi(x_i) \wedge \xi(x_{(k+1)})$

$= \Lambda_{(i=1)}^{K} \xi(x_i)$

3. Lemma:

Let $\xi, \mu \in F(S)$ Prove that $\xi_* \cap \mu_* \subseteq (\xi \cap \mu)_*$

Let $x \in \xi_* \cap \mu_*$

Then $\xi(x) = \xi(0)$ and $\mu(x) = \mu(0)$

Now $(\xi \cap \mu)(x) = \xi(x) \cap \mu(x)$

$= \xi(0) \cap \mu(0)$

$= (\xi \cap \mu)(0)$

$x \in (\xi \cap \mu)_*$

Hence $\xi_* \cap \mu_* \subseteq (\xi \cap \mu)_*$

Definition 4.1: Let $\xi, \zeta \in F^S$ define $\zeta \oplus \xi$ as follows

$(\zeta \oplus \xi)(x) = \vee \{ \zeta(y) \wedge \xi(z) / y \in S, y + z = x \}$

$(\zeta \odot \xi)(x) = \vee \{ \Lambda_{i=1}^{n} \zeta(r_i) \wedge \xi(x_i) / r_i \in R_1\ x_i \in S, h \leq i \leq n, n \in N, \sum_{i=1}^{h} r_i x_i = x \}$

4. Lemma:

1. $(r\zeta)(rx) \geq \zeta(x); \ \forall x \in S$
2. $\xi(rx) \geq \zeta(x) \ \forall x \in S \Rightarrow r\zeta \subseteq \xi$
3. $(r\zeta + s\xi)(rx + sy) \geq \zeta(x) \wedge \xi(y) \ \forall x, y \in S$
4. $\xi(rx + sy) \geq \zeta(x) \wedge \sigma(y) \ \forall x, y \in S \Rightarrow r\zeta + S\sigma \subseteq \xi$

Proof:

(1). $(x\zeta)(rx) \geq \zeta(x) \ \forall x \in S$

$$(x\zeta)(rx) = V\{\zeta(y) / y \in S, ry = rx\}$$

$\geq \zeta(x) \ \forall x \in S$

(2). $If \ \xi(rx) \geq \zeta(x) \ \forall x \in S$

Then $(r\zeta)(x) = V\{\zeta(y) / y \in S, ry = x\}$

$\leq V\{\xi(ry) / y \in S, ry = x\}$

$\leq \xi(x) \ \forall x \in y$

$r\zeta \subseteq \xi$

$If \ \ r\zeta \subseteq \xi$

Then $\xi(rx) \geq r\zeta(rx)$

$\geq \zeta(x)$

$\forall x \in S$

(3). By definition of rζ consider ⊕

$(r\zeta + s\xi)(rx + sy) \geq (r\zeta)(rx) \wedge (s\xi)(sy)$

$\geq \zeta(x) \wedge \xi(y) \ \forall x, y \in S$

(4). Suppose that $\xi(rx + sy) \geq \zeta(x) \wedge \sigma(y)$

$\forall x, y \in S$

Then $(r\zeta + s\sigma)(z) = V\{(r\zeta)(u) \wedge (s\sigma)((v)) / (u, v) \in S; u + v = z\}$

$= V\{(V\{\zeta(x) \big| x \in S, rx = u\}) \wedge (V\{\sigma(y) / y \in S, sy = v\}) / u, v \in S, u + v = z\}$

$= V\{\zeta(x) \wedge \sigma(y) | x, y \in S, rx + sy = z\}$

$= \xi(x) \ \forall z \in S$

Hence $r\zeta + s\sigma \subseteq \xi$

Conversely, suppose that.

$r\zeta + s\sigma \subseteq \xi$

$\xi(rx + sy) \geq (r\zeta + s\sigma)(rx + sy)$

$\geq (r\zeta)(rx) \wedge (s\sigma)(sy)$

$\geq \zeta(x) \wedge \sigma(y)[\because (1)]$

$\forall\, x, y \in S$

5. Lemma: Prove that $(\zeta \odot \sigma)(x+y) \geq (\zeta \odot \sigma)(x) \wedge (\zeta \odot \sigma)(y)(\zeta \odot \sigma)(rx) = V\{\Lambda^n_{(i=1)}\, \zeta(s_i) \wedge \zeta(z_i) s_i, z \in S; 1 \leq i \leq n, n \in N, \sum_{i=1}^n s_i\, z_i = rx\} \geq V\{\Lambda^n_{(i=1)} \left(\zeta(r\, r_i) \wedge \sigma(x_i\,)\right) r_i, x_i \in S; 1 \leq i \leq n, n \in N, \sum_{i=1}^n (rr_i) x_i = rx\}$

$$\geq V\{\Lambda^n_{(i=1)} \left(\zeta(\, r_i) \wedge \sigma(x_i\,)\right) r_i, x_i \in S; 1 \leq i \leq n, n \in N, \sum_{i=1}^n (rr_i) x_i = rx\}$$

$$= (\zeta \odot \sigma)(x) \ldots\ldots\ldots (1)$$

Similarly $(\zeta \odot \sigma)(ry) \geq (\zeta \odot \sigma)(y) \ldots\ldots\ldots\ldots (2)$

By definition additive of (1) & (2)

$$(\zeta \odot \sigma)(rx + ry) \geq (\zeta \odot \sigma)(x) \wedge (\zeta \odot \sigma)(y)$$

Put $r = 1$

$$(\zeta \odot \sigma)(x+y) \geq (\zeta \odot \sigma)(x) \wedge (\zeta \odot \sigma)(y)$$

Definition 4.2: [13] A fuzzy ideal ζ of a Γ- Semigroup is called a fuzzy prime ideal if

$\underset{(\gamma \in \Gamma)}{inf} \zeta(x\gamma y) = max\, \{\, \zeta(x), \zeta(y)\, \}$ for all x,y ∈ and γ∈ Γ.

Example: Let be the set of all 1x2 matrices, Γ be the set of all 2x1 matrices, then is Γ-Semigroup where p,q∈; α,β∈Γ Which denotes the usual matrix product. Let ζ: → [0,1] be defined by

$$\zeta(p) = \{ \begin{matrix} 0.1 & if\ [0 \quad 0] \\ 0.3 & 0.w \end{matrix}$$

let $p = [1 \quad 0]$; $q = [0 \quad 0]$ and $\gamma = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then $\zeta(p\gamma q) = \zeta\left([1 \quad 0] \begin{bmatrix} 1 \\ 1 \end{bmatrix} [0 \quad 0]\right)$

$$= \zeta([1][0 \quad 0])$$

$$= \zeta([0 \quad 0]) = 0.1$$

$Max\, \{\, \zeta(p), \zeta(q)\, \} = max\, \{0.3, 0.1\} = 0.3$

$\zeta(x\gamma y) \leq max\, \{\, \zeta(x), \zeta(y)\, \}$

Definition 4.3: A fuzzy ideal ζ of a ternary Γ- Semigroup is called a fuzzy prime ideal if inf┆(γ∈Γ) ζ(pαqβr) = max { ζ(p), ζ(q),ζ(r) } for all p,q,r ∈ and γ∈ Γ.

Example: Let be the set of all 1x2 matrices, Γ be the set of all 2x1 matrices, then is ternary Γ-Semigroup where p,q∈; α,β∈Γ which denotes the usual matrix product. Let ζ: → [0,1] be defined by

$$\zeta(p) = \begin{cases} 0.5 & if\ [0 \quad 0] \\ 0.9 & if\ [1 \quad 2] \\ 0.7 & 0.w \end{cases}$$

let $p = [0 \quad 0]$; $q = [1 \quad 2]$; $r = [2 \quad 1]$ $and\ \alpha = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

then $\zeta(p\alpha q\beta r) = \zeta\left([0 \quad 0] \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \quad 2] \begin{bmatrix} 1 \\ 0 \end{bmatrix} [2 \quad 1]\right)$

$$= \zeta([0][1][2 \quad 1])$$

$$= \zeta([0 \quad 0]) = 0.5$$

$max\{\zeta(p), \zeta(q), \zeta(r)\} = max\{0.5, 0.9, 0.7\} = 0.9$

$\zeta(p\alpha q\beta r) \leq max\{\zeta(p), \zeta(q), \zeta(r)\}$

Definition 4.4:[18] A fuzzy ideal ζ of a Γ- Semigroup □ is called a fuzzy Semiprime ideal if

$\zeta(x) \geq \inf_{\gamma\in\Gamma} \zeta(x\gamma x)$.

Definition 4.5: A fuzzy ideal ζ of a ternary Γ- Semigroup □ is called a fuzzy Semiprime ideal if

$$\zeta(x) \geq \inf_{\alpha,\beta\in\Gamma} \zeta(x\alpha x\beta x) .$$

6.Lemma: [19] Let □ be a Γ-Semigroup and ∅≠I⊆ □. Then I is a prime ideal (Semiprime ideal) of □ iff ζ_I is a fuzzy prime ideal (respectively fuzzy Semiprime ideal) of □, where ζ_I is the characteristic function of I.

Definition 4.6: [35] Let □ be Γ- Semigroup, ζ be a fuzzy subset of □ and x∈ □ then the fuzzy subset $< x, \zeta >$:□→ [0,1] defined by $< x, \zeta > (y) = \inf_{\gamma\in\Gamma} \zeta(x\gamma y)$ is called the extension of ζ by x.

Note: For a fuzzy subset ζ of ℛ(Ω of ℒ) a fuzzy subset ζ* (Ω°) of □ by $\zeta^*(a) = \inf(\gamma \in \Gamma)\, \zeta([\gamma, a])$;

$\Omega^o(a) = \inf_{\gamma\in\Gamma} \Omega([a,\gamma])$ $and\ \eta *'([\alpha,a]) = \inf_{s\in□} \eta(s\alpha a)$; $\eta o'([a,\alpha]) = \inf_{s\in□} \eta(a\alpha s)$.

7. Lemma: Let ζ be a non-empty fuzzy subset of commutative Γ-Semigroup □, then for all x∈□

$$(i) \ < x, \zeta >^{*'} \subseteq < [\alpha, x], \zeta^{*'} > \ for\ all\ \alpha\epsilon\Gamma$$

$$(ii) \ < x, \zeta >^{*'} = \inf_{\alpha\epsilon\Gamma} < [x, \alpha], \zeta^{*'} > \ for\ all\ \alpha\epsilon\Gamma$$

(i) Let [β,y] ∈ □ then

$$< x, \zeta >^{*'}([\beta, y]) = inf_{s\in S} < x, \zeta > (s\beta y)$$

$$= inf_{s\in S}\, inf_{\gamma\in\Gamma} \zeta(x\gamma s\beta y)$$

Again

$$< [\alpha, x], \zeta^{*'} > ([\beta, y]) = \zeta^*([\alpha, x][\beta, y])$$

$$= \zeta^{*'}([\alpha, x\beta y])$$

$$= inf_{s\in S} \zeta(s\alpha x\beta y)$$

$$Since\ inf_{\gamma\in\Gamma} inf_{s\in S} \zeta(x\gamma s\beta y) \leq inf_{s\in S} \zeta(x\alpha s\beta y)$$

$< x,\zeta >^{*'} ([\beta,y]) \leq < [\alpha,x],\zeta^{*'} > ([\beta,y])$

$Hence, < x,\zeta >^{*} \subseteq < [\alpha,x],\zeta^{*'} >$

(ii) Let [β,y]∈R then

$inf < [\alpha,x],\zeta^{*'} > ([\beta,y])$

$= inf_{\alpha\in\Gamma}\zeta^{*'}([\alpha,x]\,[\beta,y])$

$= inf_{\alpha\in\Gamma}\zeta^{*'}([\alpha,x\beta y])$

$= inf_{\alpha\in\Gamma}\, inf_{s\in S}\, \zeta([s\alpha x\beta y])$

$= inf_{s\in S} < x,\zeta > (s\beta y)$

$=< x,\zeta >^{*'} ([\beta,y])$

$Thus\ < x,\zeta >^{*'} = inf_{\alpha\in\Gamma} < [\alpha,x],\zeta^{*'} >$

5. Results

1. let σ be a non-empty fuzzy subset of the right operator semi-groups of a Γ-Semigroup □.

then $\forall\, x \in$ □$, < [\beta,x],\sigma >^{*} \geq < x,\sigma^{*} > \ \forall\beta \in \Gamma$

Let p∈□ then

$< [\beta,x],\sigma >^{*} (p) = inf_{\alpha\in\Gamma} < [\beta,x],\sigma > ([\gamma,p])$

$= inf_{\alpha\in\Gamma}\, \sigma([\beta,x]\,[\gamma,p])$

$= inf_{\alpha\in\sigma}\, \sigma([\beta,x\gamma p]\,)$

$Again\ < x,\sigma^{*} > (p) = inf_{\gamma\in\Gamma}\sigma^{*}(x\gamma p)$

$= inf_{\gamma\in\Gamma}\, inf_{\beta\in\Gamma}\, \sigma[(\beta,x\gamma p)]$

$= inf_{\beta\in\Gamma}\, inf_{\gamma\in\Gamma}\, \sigma[(\beta,x\gamma p)]$

$\because inf_{\gamma\in\Gamma}\, \sigma[(\beta,x\gamma p)] \geq inf_{\beta\in\Gamma}\, inf_{\gamma\in\Gamma}\, \sigma[(\beta,x\gamma p)]$

We have

$< [\beta,x],\sigma >^{*} (p) \geq < x,\sigma^{*} > (p)$

Consequently

$< [\beta,x],\sigma >^{*} \supseteq < x,\sigma^{*} >$

2. let $\{A_\alpha\}_{\alpha\in A}$ be a family of ideals of a Γ-semigroup □. Then ⟦$(\cap_{\alpha\in\wedge} A_\alpha)^{*'} = \cap_{\alpha\in\wedge}\ A_\alpha^{*'}$

Let $[\alpha,x](\cap_{\alpha\in\wedge} A_\alpha)^{*'}$

Then $s\alpha x\ \in \cap_{\alpha\in\wedge} A_\alpha\ \ \forall\, s \in S$

Here $s \in S$

$s\alpha x \in A_\alpha \ \forall \alpha \in \Lambda$

$[\alpha, x] \cap_{\alpha\in\Lambda} A_\alpha^{*'}$

Here $(\cap_{\alpha\in\Lambda} A_\alpha)^{*'} \subseteq \cap_{\alpha\in\Lambda} A_\alpha^{*'} - (1)$

We can deduce that $\cap_{\alpha\in\Lambda} A_\alpha^{*'} \subseteq (\cap_{\alpha\in\Lambda} A_\alpha)^{*'}$

$[\alpha, x] \in \cap_{\alpha\in\Lambda} A_\alpha^{*'}$

Then

$s\alpha x \in A_\alpha \ \forall \alpha \in \Lambda$

$s\alpha x \in \cap_{\alpha\in\Lambda} A_\alpha \ \forall s \in S$

$[\alpha, x] \in (\cap_{\alpha\in\Lambda} A_\alpha)^{*'}$

$\cap_{\alpha\in\Lambda} A_\alpha^{*'} \subseteq (\cap_{\alpha\in\Lambda} A_\alpha)^{*'} - (2)$

From (1) & (2) we get

$\cap_{\alpha\in\Lambda} A_\alpha^{*'} = (\cap_{\alpha\in\Lambda} A_\alpha)^{*'}$

3. Let □ be a Γ-semigroups, R its right operator Semigroup and $\xi = inf\{\xi_i : i \in I\}$ a non-empty family of the fuzzy subset of □. Then $\xi^{*'} = inf\left\{\xi_{p^{*'}} : i \in I\right\}$ fuzzy ideal extension of Γ-semigroup.

Let [α,x]∈R then

$\xi^{*'}[\alpha, x] = inf\{\xi_i : i \in I\}^{*'}[\alpha, x]$

$= inf_{s\in S}(inf\{\xi_i : i \in I\}(s\alpha x))$

$= inf_{s\in S}\, inf_{i\in I} \xi_i(s\alpha x)$

Now

$inf \left\{\xi_i^{*'} \ i \in I\right\}[\alpha, x] = inf_{i\in I}\left(\xi_i^{*'}[\alpha, x]\right)$

$= inf_{i\in I} inf_{s\in S} \xi_i^{*'}(s\alpha x)$

$= inf_{i\in I} inf_{s\in S} \xi_i(s\alpha x)$

$= inf_{s\in S} inf_{i\in I} \xi_i(s\alpha x)$

$\therefore \xi^{*'}$ is the fuzzy ideal extension of Γ-semigroup.

4. Suppose ξ:S→[0,1] the fuzzy subset of semigroup and then

(i) $\left(\xi^i\right)_* = (\xi_*)i \quad \forall \ i \in N$

(ii) $\left(\xi^{(i)}\right)_* = (\xi_*)^i$

Proof: (i) if $i = 1$

Then $(\xi^1)_* = (\xi_*)^1$

$\rightarrow \xi_* = \xi_*$

$\therefore\ i = 1$ the results are true

We assume that it is true for $i > 1$

Let $x \epsilon \left(\xi^{i+1}\right)_*$

Then $\xi^{i+1}(x) = \xi^{i+1}(0) = \xi(0)$

But $\xi^{i+1}(x) = V\{\xi^i(y)\xi(z) \ / y, z \in S$

$x = yz\}\, \xi(0)$

Thus $\xi^i(y) = \xi(0) = \xi(z)$ for some $y, z \in S$

$\because \xi^i(y) \leq \xi^i(0) = \xi(0)$

ξ^i is finite-valued $\forall\, i \geq 1$

Thus $x = yz$ for some $y \in \left(\xi^i\right)_* = (\xi_*)^i$ and $z \in \xi_*$

Hence $x \in (\xi_*)^i\, \xi_* = (\xi_*)^{i+1}$

Thus $\left(\xi^{(i+1)}\right)_* \subseteq (\xi_*)^{(i+1)}$

Now let $x = \sum_{(j=1)}^{r} x_{j_1}, x_{j_2} \dots x_{j_1} \in (\xi_*)^{i+1}$

Where $x_{jk} \in \xi_* \forall\, k = 1, 2, \dots i + 1$

$j = 1, 2, \dots r$

Then $\xi^{i+1}(x) \geq \wedge_{j=1}^{r} \xi^{i+1}\left(x_{j1}\, x_{j2} \dots x_{j+1}\right)$

$\geq \wedge_{j=1}^{r}(\wedge_{k=1}^{i+1} \xi\left(x_{jk}\right))$

$= \xi(0)$

Thus $\xi^{i+1}(x) = \xi(0) = \xi^{i+1}(0)$

Hence $x \in \left(\xi^{i+1}\right)_*$

Thus $(\xi_*)^{i+1} \subseteq \left(\xi^{i+1}\right)_*$

Hence $(\llbracket \xi^{i+1})\rrbracket_* = (\xi_*)^{i+1}$

Clearly, the result is true for $i = 1$

Assume that it is true for $i \geq 1$

Let $x \in \left(\xi^{i+1}\right)_*$

Then $\xi^{i+1}(x) = \xi^{i+1}(0) = \xi(0)$

But $\xi^{i+1}(x) = V\{\wedge_{k=1}^{n} \xi^{i}(y_k) \wedge \xi(z_k)) \,/\, y_k, z_k \in S, 1 \le k \le n, n \in N; \sum_{k=1}^{n} y_k z_k = x\} = \xi(0)$

$\because \xi^{i}$ is finite-valued $\forall\, i \geq 1$

This implies that $\xi^{i}(y_k) = \xi(0) = \xi(z_k)$

$\forall k, 1 \le k \le n$

Thus $z_k \in \xi_k \quad \forall\ k, 1 \le k \le n$

Also $\xi(0) \geq \xi^{(i)}(0) \geq \xi^{(i)}(y_k) = \xi(0)$

So $\xi^{(i)}(0) = \xi^{(i)}(y_k)$

i.e. $y_k \in \left(\xi^{i}\right)_{*} \quad \forall k, i \le k \le n$

hence $x = \sum_{k=1}^{n} y_k z_k \in (\xi_*)^{(i+1)} where\ x_{jk} \in \xi_* \forall\ k = 1,2, \ldots$

$$\xi^{(i+1)}(x) \geq \wedge_{j=1}^{r} \xi^{(i+1)}\left(x_{j1}\, x_{j2} \ldots x_{j+1}\right)$$

$$\geq \wedge_{j=1}^{r} \wedge_{k=1}^{i+1} \xi\left(x_{jk}\right)$$

$$= \xi(0)$$

Thus $\xi^{(i+1)}(x) = \xi(0) = \xi^{(i+1)}(0)$

Hence $x \in \left(\xi^{(i+1)}\right)_{*}$

Thus $(\xi_*)^{i+1} \subseteq \left(\xi^{(i+1)}\right)_{*}$

Hence $\left(\xi^{(i+1)}\right)_{*} = (\xi_*)^{i+1}$

Definition 6.1: Let $\xi \in F^S$ define ξ^n and ξ^n as follows when $x \in N, n > 1$; $\xi^1 = \xi\ and\ \xi^n = \xi^1 \circ \xi^{n-1}$

$$\xi^{(1)} = \xi$$

$$\xi^{(n)} = \xi^{(1)} \xi^{(n-1)}$$

Definition 6.2: Let $\xi_1, \xi_2\, F^S$. Define $\xi_1 \xi_2 \in F^S\ \forall\ x \in S$

$$(\xi_1 \xi_2)(x) = V\{\wedge_{i=1}^{n}\left(\xi(y_i) \wedge \xi(z_i)\right) |\, y_i z_i \in S \text{ and } 1 \le i \le n, n \in N, \sum_{i=1}^{n} y_i z_i = x\}$$

5. If S is commutative then $\xi_1 \xi_2 = \xi_2 \xi_1$

$$(\xi_1 \xi_2)(x) = \vee\{\wedge_{i=1}^{n}(\xi_1(y_i) \wedge \xi_2(z_i) | y_i, z_i \in S \quad 1 \le i \le n, n \in N, \sum_{i=1}^{n} [\![y_i z_i = x\}$$

$$= \vee\{\wedge_{i=1}^{n}(\xi_2(z_i) \wedge \xi_1(y_i) \,|\, y_i, z_i \in S \quad 1 \le i \le n, n \in N, \sum_{i=1}^{n} [\![z_i y_i = x\}$$

$$= (\xi_2 \xi_1)(x)$$

$$\xi_1 \xi_2 = \xi_2 \xi_1$$

$\xi_* = \{x \in R | \xi(x) = \xi(0)\}$

6. Let $\xi \in F^S$ and $k \in N, if\ x_1, x_2, \dots, x_k \in S$

Then 1. $\xi_*^k(x_1, x_2, \dots, x_k) \geq \wedge_{i=1}^{k} \xi_*(x_i)$

2. $\xi_*^{(k)}(x_1, x_2, \dots, x_k) \geq \wedge_{i=1}^{k} \xi_*(x_i)$

Proof: 1. Clearly, the result is true for $k = 1$

Assume that it is true for $k > 1$

Now $\xi_*^{k+1}(x_1, x_2, \dots, x_{k+1}) = (\xi_*^k \circ \xi_*)(x_1, x_2, \dots, x_{k+1})$

$$\geq \xi_*^k(x_1, x_2, \dots, x_k) \wedge \xi_*(x_{k+1})$$

$$\geq \xi_*^k(0,0, \dots, 0) \wedge \xi_*(x_{k+1})$$

$$\xi_*^k(x_1, x_2, \dots, x_k) \geq \wedge_{i=1}^{k} \xi_*(x_i)$$

2. this proof follows the same as lemma 1.

6. Discussion

A fuzzy ternary gamma Semigroup is a mathematical structure that combines concepts from fuzzy set theory, ternary algebra, and gamma Semigroup. We discussed that fuzzy ternary gamma semigroups can be used to model fuzzy inference Systems. Fuzzy ternary gamma semigroups can be applied to computer science, particularly in studying fuzzy automata and fuzzy languages. Fuzzy ternary gamma semigroup may have applications in cryptography, especially in developing fuzzy cryptographic protocols. New problems are explored in studying fuzzy ternary gamma semigroups, a relatively new research area.

❖ **Clarify Key Terms:** While "star zeta" and "gamma semigroup" are central concepts, a brief clarification of their role in fuzzy semigroups could help readers unfamiliar with the topic.

❖ **Strengthen Logical Progression**: Ensure the discussion flows from definitions to properties, followed by implications and the collaborative nature of the work.

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