Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

On Modules For Which Every Cosingular Verify The D4-Condition

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Article History:

Received: 14-11-2024

Revised: 22-12-2024

Accepted: 17-01-2025

Abstract

Let R be an associative ring with unity and M an unital left R-module. In this paper we introduce $\boldsymbol{D}$**41**-module which is a generalization of D4-module. A module M is called $\boldsymbol{D}$**41** if $M = N \oplus K$ with $N, K \leq M$, K is cosingular and $f: N \rightarrow K$ is an epimorphism, then $\ker(f)$ is a direct summand of N. Some basic properties of these modules are investigated. It is shown that the class of rings R over which a $\boldsymbol{D}$**41**-module is a D4-module is exactly that of $COSP$-rings. Also, we study the relations between $\boldsymbol{D}$**41**-module and other related modules.

2010 Mathematics Subject Classifications: 16D10, 16D40, 16D60, 16L30.

Keywords : small submodules, cosingular modules, projectivemodules, D4-modules, D41-modules.

1. Introduction

In this paper, we focus on rings that are associative and unitary, and we consider all modules as left modules, unless otherwise specified. We will introduce some important notations. Let M be a module; we denote $N \leq M$ to indicate that N is a submodule of M. The notation $N \leq^{\oplus} M$ signifies that N is a direct summand of M. For a module M, we denote its injective envelope as $E(M)$. The socle and radical of the module M are represented by $\mathrm{Soc}(M)$ and $\mathrm{Rad}(M)$, respectively. We use $N \subseteq M$ to indicate that N is a subset of M.

We say that M satisfies the following conditions:

(D1-Condition): For every submodule $N \leq M$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2$ is small in M.

(D2-Condition): For every submodule $N \leq M$ such that M/N is isomorphic to a direct summand of M, N is also a direct summand of M.

(D3-Condition): If M_1 and M_2 are direct summands of M and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M.

(D4-Condition): If $M = N \oplus K$ with $N, K \leq M$ and $f : N \rightarrow K$ is an epimorphism, then $\ker(f) \leq^{\oplus} N$.

A module that satisfies the Di-condition will be referred to as a Di-module. Every quasi-projective left R-module is a D2-module, every D2-module is a D3-module, and every D3-module is a D4-module. However, there exist examples of D3-modules that are not D2-modules, as well as D2-modules that are not quasi-projective. D1-modules are referred to as lifting modules by Oshiro [16], D2-modules are called direct-projective by W.K. Nicholson in [15], and D3-modules are termed ∩-direct-projective in [2].

In 2016, N. Ding, Y. Ibrahim, M. Yousif, and Y. Zhou [6] defined a module as a $C4$-module if, for any submodules A and B of M, when $M = A \oplus B$ and $f : A \to B$ is a homomorphism with $\ker(f) \leq^{\oplus} A$, then $Im(f) \leq^{\oplus} B$. $D4$-modules were studied by N. Ding and others in 2017 [4].

Motivated by these developments and the findings in [4] and [6], we introduce the concept of cosingular $D4$-modules, which generalizes $D4$-modules and a dual notion of $C41$-modules [3]. A module is defined as a cosingular $D4$-module if, for $M = N \oplus K$ with $N, K \leq M$, K is cosingular and $f : N \to K$ as an epimorphism, it follows that $\ker(f) \leq^{\oplus} N$. We refer to these as $D41$-modules.

In Section 3, we explore some properties of $D41$-modules. We demonstrate that every direct summand of a $D41$-module also inherits this property. We also show that if $M \oplus M$ is a $D41$, then M is a $D2$. It is proved in this section that any $D41$-module satisfying the SSP satisfies also the SIP. Additionally, we show that if M is a cosingular $D41$-module, then M/N is also a $D41$-module and if $\frac{M}{\bar{Z}(M)}$ is a $D41$, then M is a $D41$. It is given some properties of $D41$-modules related to direct finite modules, square free modules, summand square-free modules, summand dual-quare-free modules, Hopfian modules, ...

We begin Section 4 by providing a characterization of $D41$-modules via $COSP$-rings. We also show that a ring R is semiregular iff every finitely presented R-module has a $D41$-cover.

2. Preliminaries

Lemma 2.1. *Lemma [20, Proposition 2.1] Let M, N and $(M_i), i \in I$ be R-modules.*

Then:

a) If $N \leq M$, then $\bar{Z}(N) \leq \bar{Z}(M)$ and $\frac{(N+\bar{Z}(M))}{N} \leq \bar{Z}\left({}^{M}/_{N}\right)$

b) If $f: M \to N$ is a homomorphism, then $f(\bar{Z}(M)) \leq \bar{Z}(N)$

c) $\bar{Z}\left({}^{M}/_{\bar{Z}(M)}\right) = 0$

d) $\bar{Z}(\bigoplus_{i \in I} Mi) = \bigoplus_{i \in I} \bar{Z}(Mi)$

e) $\bar{Z}(\prod_{i \in I} Mi) \leq \prod_{i \in I} \bar{Z}(Mi)$

f) *If $M = N + S$ where S is a small module, then* $\bar{Z}(M) = \bar{Z}(N)$

g) $\bar{Z}(M)$ is the smallest submodule such that $\bar{Z}\left({}^{M}/_{\bar{Z}(M)}\right) = 0$.

Lemma 2.2. *[20, Corollary 2.2] The class of all cosingular modules is closed under submodules, directs sums and directs products.*

Lemma 2.3. *[5, Proposition 2.1] Let M a $D3$-module such that $M = A_1 \oplus A_2$ for submodules A_1 and A_2. If $f : A_1 \to A_2$ is a homomorphism such that $Imf \leq^{\oplus} A_2$, then $kerf \leq^{\oplus} A_1$.*

Proposition 2.4. *Let M be a module, N and K two submodules of M such that K is cosingular. Then the following statements are equivalent*:

1) If $M = N \oplus K$ and $f : N \to K$ an epimorphism, then $kerf \leq^{\oplus} N$.

2) if $M = N \oplus K$ and $f : N \to K$ a homomorphism with $Imf \leq^{\oplus} K$, then

$kerf \leq^{\oplus} N$.

3) If $N \leq K$ and $\frac{M}{K} \cong N \leq^{\oplus} M$, then $K \leq^{\oplus} M$.

4) If $M = N + K$, $N \leq^{\oplus} M$ and $M/N \leq^{\oplus} M/K$, then $N \cap K \leq^{\oplus} M$.

5) If N and K are direct summands of M with $M = N + K$ and $M/N \cong M/K$, then

$N \cap K \leq^{\oplus} M$.

6) If $M = N + K$, $N \leq^{\oplus} M$ and $M/N \cong M/K$, then $K \leq^{\oplus} M$.

7) If $M = N \oplus N' = K \oplus K' = N + K = N + K'$, where N' and K' are submodules of M, then $N \cap K \leq^{\oplus} M$.

8) If N and K are direct summands of M with $M = N + K$ and $N \cong K$, then $N \cap K \leq^{\oplus} M$.

Proof: The proof follows by the same method as in [5, Theorem 2.2].

3. Some properties of $D41$-modules

Throughout this section, we shall investigate some general properties of $D41$-modules.

Definition 3.1. *Let M be an R-module. We say that M is a $D41$ if, $M = N \oplus K$ for $N, K \leq M$ such that K is cosingular and $f\colon N \to K$ is an epimorphism, then $ker(f) \leq^{\oplus} N$. The ring R is a right (left) $D41$-ring if the right R-module R_R (left ${}_RR$) is $D41$.*

Example 3.2.

1) Every $D4$-module has $D41$. Particularly, each projective module has $D41$.

2) Each hereditary module is $D41$ as any submodule of such module is projective.

3) Every semisimple module is a $D41$-module.

4) Each module with the summand intersection property (SIP), has $D41$.

Next, we give an example of a $D41$-module that is not a $D4$-module.

Example 3.3. *Consider the module $M = \mathbb{Q} \oplus \mathbb{Z}$, where $\mathbb{Q}$ is the rational as a $\mathbb{Z}$ -module and $\mathbb{Z}$ is the integers as a $\mathbb{Z}$ -module.*

We decompose $M = \mathbb{Q} \oplus \mathbb{Z}$. $\mathbb{Z}$ is cosingular as a $\mathbb{Z}$ -module. Now, consider an epimorphism $f : \mathbb{Q} \to \mathbb{Z}$. A natural choice for such an epimorphism is the inclusion map:$f : \mathbb{Q} \to \mathbb{Z}$, with *$f(q) = \lfloor q \rfloor$ for $q \in \mathbb{Q}$, where the notation $\lfloor q \rfloor$ represents the floor function of a real number q. It is defined as the greatest integer less than or equal to q. Here, the kernel of this map is $ker(f) = \mathbb{Q}$, which is clearly a direct summand of $\mathbb{Q}$ since $\mathbb{Q} = \mathbb{Q} \oplus \{0\}$. Thus, the $D41$-condition holds for this map.*

The $D4$-condition requires that for every decomposition $M = N \oplus K$ and for any epimorphism $f : N \to K$, the kernel $ker(f)$ must be a direct summand of N, without assuming that K is cosingular.

In this case, the map $f : \mathbb{Q} \to \mathbb{Z}$ (in the previous step) satisfies the $D4$-condition, because $ker(f) = \mathbb{Q}$ and it is a direct summand of $N = \mathbb{Q}$. However, if $K = \mathbb{Z}$ which is not cosingular, the kernel might fail to be a direct summand of N. Therefore, the $D4$-condition does not automatically hold unless we impose the cosingularity condition on K, which is present in the $D41$-condition.

Thus, $M = \mathbb{Q} \oplus \mathbb{Z}$ is an example of a $D41$-module that is not a $D4$-module. The cosingularity of K is what ensures the kernel is a direct summand in the $D41$-case, but this condition is not guaranteed in the $D4$-case.

Proposition 3.4. *Let M be a $D41$-module. Then any direct summand of M is a $D41$-module.*

Proof:

Let M be a $D41$-module and $N \leq^{\oplus} M$. We want to demonstrate that if $N = K \oplus K'$ such that K' is cosingular and $f : K \to K'$ is an epimorphism, then $kerf \leq^{\oplus} K$.

Consider $M = N \oplus N' = K \oplus K' \oplus N'$, where $N' \leq M$. Define the canonical projection $\pi : K \oplus N' \rightarrow K$. It follows that the composition $f \circ \pi : K \oplus N' \rightarrow K'$ is also an epimorphism, with $ker(f \circ \pi) = kerf \oplus N'$.

Since $M = (K \oplus N') \oplus K'$ is a $D41$-module, we conclude that $kerf \oplus N' \leq^{\oplus} K \oplus N' \leq^{\oplus} M$. Therefore, it follows that $kerf \leq^{\oplus} N$. Consequently, N is a $D41$-module.

Proposition 3.5. *If $M \oplus M$ is a $D41$-module with M cosingular, then M is a $D2$-module.*

Proof: We need to demonstrate that if $N, K \leq M$ with ${}^{M}/_{K} \cong N \leq^{\oplus} M$, then $K \leq^{\oplus} M$.

To show this, express M as $M = N \oplus L$ for some $L \leq M$. Then we have $M \oplus M \cong M \oplus N \oplus L \cong M \oplus (M/K) \oplus L$. From this, we see that $M \oplus (M/K)$ is a $D41$-module, which implies that the natural epimorphism $M \longrightarrow M/K$ splits. Consequently, it follows that $K \leq^{\oplus} M$.

Proposition 3.6. *If $M_1 \oplus M_2$ is a $D41$-module and there is an epimorphism $f : M_1 \rightarrow M_2$ with M_2 is cosingular, then M_2 is a $D2$-module.*

Proof: Since $M_1 \oplus M_2$ is a $D41$-module and $f : M_1 \rightarrow M_2$ is an epimorphism, we have $\ker f \leq^{\oplus} M_1$. We can express M_1 as $M_1 = N \oplus \ker f$, where $N \leq M_1$. Thus, $N \cong {}^{M_1}/_{\ker(f)} \cong M_2$.

Now, we have $M_2 \oplus M_2 \cong (N \oplus M_2) \leq^{\oplus} (M_1 \oplus M_2)$. Since $M_2 \oplus M_2$ is a $D41$-module (by Proposition 3.4), it follows that M_2 is a $D2$-module (by Proposition 3.5).

Recall that a module M has the summand intersection property (SIP) if the intersection of any two direct summands of M is a direct summand of M.

Lemma 3.7. *[9, Theorem 1.2] A module M has the summand intersection property (SIP) if and only if, for every decomposition $M = A \oplus B$ and every $\gamma : A \rightarrow B$, the kernel of γ is a direct summand of A*

Corollary 3.8. *The following conditions are equivalent for a $D41$-module M. 1) M has the SIP property*

2) For every decomposition $M = N \oplus N$ and for every epimorphism $f : N \rightarrow K$ such that K is cosingular and $Im(f) \leq^{\oplus} K$, then $ker(f) \leq^{\oplus} N$.

Proposition 3.9. *If every submodule of a $D41$-module M is a $D41$-module, then M has the summand intersection property.*

Proof: Let M be $D41$-module such that $M = N \oplus K$ with K cosingular and $\lambda : N \rightarrow K$. Then $N \oplus Im(\lambda)$ is a submodule of M. By hypothesis, $N \oplus Im(\lambda)$ is a $D41$-module. So, the epimorphism $\alpha : N \rightarrow Im(\lambda)$ splits. Thus $ker(\alpha)$ is a direct summand of N. consequently, M has the summand intersection property by lemma 3.7.

Definition 3.10. *Let M be a left R-module. We say that M satisfies the condition $(\star)$ if $M = N \oplus K$ for some submodules N and K of M with K being cosingular, and $f : N \rightarrow K$ is an R-epimorphism while $id : K \rightarrow K$ is the identity map, then there exists an R-homomorphism $f' : K \longrightarrow N$ such that $f \circ f' = id$.*

It is clear that every $D41$-module satisfies condition $(\star)$.

Remark 3.11. *Given that Morita equivalence preserves summands, epimorphisms, cosingularity, and isomorphisms, it follows that if* R *and* S *are Morita-equivalent rings related by a category equivalence* ϕ: *Mod-R* $\rightarrow$ *Mod-S, then a left* R*-module* M_R *satisfies condition* $(\star)$ *if and only if* $\phi(M_R)_S$ *also satisfies condition* $(\star)$.

Proposition 3.12. *Let* $M \oplus M$ *be a cosingular* $D41$*-module. If* M *is a dual-Rickart module, then* $End_R(M)$ *is a Von Neumann regular ring.*

Proof: Let $M \oplus M$ be a $D41$-module and $f\colon M \rightarrow M$ an endomorphism.

We have $\frac{M}{\ker(f)} \cong Im(f)$. Since $Im(f)$ is a direct summand of M (due to the

dual-Rickart condition), we can express M as $Im(f) \oplus K \cong M/ker(f) \oplus K$ for some $K \leq M$. Therefore, we obtain:

$$M \oplus M = M \oplus Im(f) \oplus K \cong M \oplus M/ker(f) \oplus K.$$

Since $M \oplus M$ is a $D41$-module, it follows that $M \oplus M/ker(f)$ is also a $D41$-module, which implies that the epimorphism $g : M \rightarrow M/ker(f)$ splits. As a result, $ker(f)$ is a direct summand of M. Hence, $\mathrm{End}_R(M)$ is a Von Neumann regular ring.

Proposition 3.13. *The following statements are equivalent for a ring* R *and any finitely generated module* M:

1) Every R*-module is Rickart ;*

2) Every R*-module has the* SIP *property ;*

3) Every submodule of a cosingular and finitely generated R*-module is a summand ;*

4) Every R*-module is a* $D41$*-module.*

Proof:

1) ⇒ 2). Every Rickart module has the SIP property (see [12, Proposition 2.16]).

2) ⇒ 3). Let N and K be finitely generated with $N \leq M$. By hypothesis, $M \oplus N$ has the SIP property. Thus, $M \cap N$ is a direct summand of $M \oplus N$, leading to $N \leq^{\oplus} M$.

3) ⇒ 4). Let N and K be direct summands of a finitely generated R-module M with $N + K = M$ and K cosingular. Then, $N \cap K$ is finitely generated, which implies that $N \cap K \leq^{\oplus} M$. Hence, M is a $D4$-module, i.e., a $D41$-module.

4) ⇒ 1). Let M be a finitely generated R-module and f an endomorphism of M. Since

$M + (M/ker(f))$ is finitely generated and, by hypothesis, is a $D41$-module, it follows that $ker(f)$ is a direct summand of M. Therefore, M is a dual-Rickart module.

Recall that a module M has the summand sum property (SSP) if the sum of any two direct summands of M is also a direct summand of M.

Proposition 3.14. *Let* M *be a* $D41$*-module. If* M *possesses the* SSP, *then* M *also has the* SIP.

Proof: Let M be a $D41$-module with the SSP property. Consider direct summands N and K of M. Since M has the SSP, then $N + K$ is a direct summand of M. By proposition 3.4, $P = N + K$ is a $D41$-module. Applying the isomorphisms theorems, we have $K \cong \frac{P}{N} \cong \frac{N+K}{N} \cong N/N \cap K$.

Since P is a $D41$-module, then the epimorphism $f : K \to N/(N \cap K)$ splits. So, $N \cap K \leq^{\oplus} K$ and hence a direct summand of M proving that M has the SIP.

Proposition 3.15. *Let M be a module such that $M/\bar{Z}(M)$ is a $D41$-module. Then M is a $D41$-module.*

Proof: Let N be a cosingular submodule of M containing $\bar{Z}(M)$ such that $\frac{M}{N} \cong K$, for K a direct summand of M with $K \leq N$. Since $M/\bar{Z}(M)$ is a D41-module, there exists a direct summand L of $M/\bar{Z}(M)$ such that $\left(\frac{M}{\bar{Z}(M)}\right)\Big/\left(\frac{N}{\bar{Z}(M)}\right) \cong L$ and $L \leq \frac{N}{\bar{Z}(M)}$. Applying the isomorphism theorem, we have $\frac{M}{N} \cong \left(\frac{M}{\bar{Z}(M)}\right)\Big/\left(\frac{N}{\bar{Z}(M)}\right) \cong L \cong K \leq^{\oplus} M$.

As $\frac{N}{\bar{Z}(M)} \leq^{\oplus} M/N$, thus $N \leq^{\oplus} M$. Hence, M is a D41-module.

The converse of the previous proposition holds (see 3.17).

Corollary 3.16. *Let $M = M_1 \oplus M_2$ a direct sum of submodules M_1 and M_2 such that $\bar{Z}(M_1) = M_1$ and M_2 a cosingular $D41$-module. Then M is a $D41$-module.*

Proof: Since $\bar{Z}(M_2) = 0$, then $\bar{Z}(M) = M_1$. Thus, $\frac{M}{\bar{Z}(M)} \cong M_2$ which is a $D41$-module. Hence M is a $D41$-module by the proposition 3.15.

Proposition 3.17. *Let M be a $D41$-module. Then, for every submodule N of M, M/N is a $D41$-module.*

Proof: Let M be a $D41$-module and N a submodule of M and let $K = P/N$ be a co-singular submodule of M/N with P a submodule of M containing N such that $(M/N)/(P/N) \cong L$, $L \leq^{\oplus} M/N$ and $L \leq P/N$. By the second theorem of isomorphism, we have: $M/P \cong (M/N)/(P/N) \cong L$. As M is a $D41$-module, then P is a direct summand of M. Hence $K = P/N$ is a direct summand of M/N. Thus M/N is a $D41$-module.

Recall that a module is called directly finite if it is not isomorphic to a proper summand of itself. A module is called square-free if it contains no nonzero submodules isomorphic to a square $N \oplus N$.

Proposition 3.18. *Let $M = N \oplus K$ be a $D41$-module, with epimorphisms $f : N \to K$ and $h : K \to N$ such that $ker(f)$ is cosingular.*

1) If K is cosingular and directly finite, then $N \cong K$.

2) If N is square-free and K is cosingular, then $N \cong K$.

Proof:

1) Assume K is directly finite and cosingular. Since M is a $D41$-module, the epi-morphism f splits, allowing us to express K as $K \cong L$, where $L \leq^{\oplus} N$. Write $N = L \oplus K'$ for some submodule $K' \leq M$. Since direct summands of a $D41$-module remain $D41$-modules, $L \oplus K$ is also a $D41$-module. Let $\pi : N \to L$ be the natural projection. The composition $\pi \circ h : K \to L$ is an epimorphism that splits by Lemma 2.4. Therefore, we can express K as $K = ker(\pi \circ h) \oplus P$ for some submodule $P \leq M$. Given $K \cong L \cong K/ker(\pi \circ h) \cong P$ and that K is directly finite, we have $ker(\pi \circ h) = 0$. Thus, h is an isomorphism since $ker h \leq ker(\pi \circ h)$, leading to the conclusion that $N \cong K$.

2) Assume N is square-free. Since M is a $D41$-module, the epimorphism f splits, allowing us to write $N = \ker f \oplus L$ for a submodule L isomorphic to K. Consider the epimorphism $\pi \circ h : K \to kerf$ with the natural projection $\pi : N \to kerf$. The direct summands of $D41$-modules remain $D41$-modules, so $\ker f \oplus K$ is a $D41$-module. By Lemma 2.4, the epimorphism $\pi \circ h$ also splits, yielding $K = ker(\pi \circ h) \oplus Q$ for some $Q \leq M$ with $kerf \cong Q$. Let $\phi : K \to Q$ be the natural projection. As a direct summand of M, $L \oplus Q$ is a $D41$-module, so the epimorphism $\phi \circ \psi : L \to Q$ splits by Lemma 2.4, where $\psi : L \to K$ is an isomorphism. Hence, we have $L = ker(\phi \circ \psi) \oplus L'$ where $Q \cong L' \leq M$. This leads to the expression, $N = kerf \oplus L = kerf \oplus ker(\phi \circ \psi) \oplus L'$ with $L' \cong Q \cong kerf$. Since N is square-free, it follows that $kerf = 0$, thus $N \cong K$.

Recall that two direct summands N and K of a module M are called perspective in [13] exactly when they have a common (direct sum) complement L, i.e. $M = N \oplus L = K \oplus L$.

Proposition 3.19. *The following conditions for a module M are equivalent*:

1) M is a $D41$-module.

2) If N and K are perspective direct summands of M with common direct sum complement is cosingular and $N + K = M$, then $N \cap K$ is a direct summand of M.

3) If N and K are perspective direct summands of M with common direct sum complement is cosingular and $N + K$ is a direct summand of M, then $N \cap K$ is also a direct summand of M.

Proof:

1) ⇒ 2) Let N and K be perspective direct summands with a common direct sum complement L and L is cosingular, and suppose $N + K = M$. Define the projection $\pi : M \to L$ with $ker\pi = K$. Consider the restricted map $\pi|_N : N \to L$. Since $N + K = M$, it follows that $\pi(N) = L \leq^{\oplus} L$, which implies $ker\pi|_N \leq^{\oplus} N$. Thus, since $ker\pi|_N = N \cap K \leq^{\oplus} N \leq^{\oplus} M$, we have $N \cap K \leq^{\oplus} M$.

2) ⇒ 3) Let N and K be perspective direct summands of M such that $N + K \leq^{\oplus} M$. There exist L and $P \leq^{\oplus} M$ such that $M = N \oplus L = K \oplus L = (N + K) \oplus P$ with L is cosingular. By modularity, we can express $N + K = N \oplus (L \cap (N + K))$ and

$N + K = K \oplus (L \cap (N + K))$. The modules $N \oplus P$ and $K \oplus P$ are then perspective direct summands of M, satisfying $(N \oplus P) + (K \oplus P) = M$. By the hypothesis, this implies that $(N \oplus P) \cap (K \oplus P) = (N \cap K) \oplus P \leq^{\oplus} M$. Thus, we conclude that $N \cap K \leq^{\oplus} M$.

3) ⇒ 1) Let $M = N \oplus K$ and let $f : N \to K$ be an epimorphism with K is cosingular. Define the graph submodule $G = \{a + f(a) : a \in A\}$ within M. Clearly, $M = G + K$ and $G \cap K = 0$. This shows that $M = G \oplus K = N \oplus K$, meaning N and G are perspective direct summands of M. Since f is an epimorphism, we have $N + G = M$. By the hypothesis, it follows that $N \cap G \leq^{\oplus} M$. It can be readily demonstrated that $N \cap G = \ker f$, leading to the conclusion that $\ker f \leq^{\oplus} M$. Therefore, we find that $\ker f \leq^{\oplus} N$.

Corollary 3.20. *The following conditions for a module M are equivalent*:

1) M is a $D41$-module.

2) If $M = N + K$ for any perspective direct summands N and K of M with common direct sum complement is cosingular, then there exists a submodule $K' \subseteq K$ such that $M = N \oplus K'$.

Proof:

1) ⇒ 2) Let $M = N + K$, where N and K are perspective direct summands of M with common direct sum complement is cosingular. By Proposition 3.19, we can express M as $M = (N \cap K) \oplus L$ for some submodule L. According to the modular law, we have $K = (N \cap K) \oplus (L \cap K)$. Thus, we can write $M = N \oplus (L \cap K)$, leading us to define $K^{'} := L \cap K$.

2) ⇒ 1) Now let $M = N + K$, where N and K are perspective direct summands of M with common direct sum complement is cosingular. By the assumption, there exists a submodule $K^{'} \subseteq K$ such that $M = N \oplus K^{'}$. This means we can express K as $K = (N \cap K) \oplus K^{'}$. Consequently, $N \cap K$ is a direct summand of K, and therefore also a direct summand of M.

Recall that a module M is a $C4$-module If $M = N \oplus K$, then every monomorphism $f: N \rightarrow K$ splits [6]. We next give a relationship between $C4$-modules and $D41$-modules under certain conditions and having perspective direct summand. A module M is referred to as a DSF-module (dual square-free module) if it does not contain any proper submodules N and K such that $M = N + K$ and $M/N \cong M/K$ [4]. Additionally, M is called summand-dual-square-free (SDSF) if the submodules N and K are direct summands of M.

Proposition 3.21. *The following conditions on a module M are equivalent*:

1) M is a $D41$- and summand-square-free module.

2) M is a $C4$- and summand-dual-square-free module.

Proof:

1) ⇒ 2) Clearly, M is a $C4$-module by [6]. Next, we show that M is summand-dual-square-free. Suppose M is not summand-dual-square-free; then there exist two non-zero proper direct summands N and K of M such that $N + K = M$ with K cosingular and $M/N \cong M/K$. Since M is a $D41$-module, we have $N \cap K \leq^{\oplus} M$. We can write $M = (N \cap K) \oplus L$, leading to $N = (N \cap K) \oplus (N \cap L)$ and $K = (N \cap K) \oplus (K \cap L)$. This gives us $N \cap L \cong N/(N \cap K) \cong M/K \cong M/N \cong K/(N \cap K) \cong K \cap L$ with $(N \cap L) \cap (K \cap L) = (N \cap K) \cap L = 0$, meaning both $N \cap L$ and $K \cap L$ are direct summands of M. Since M is summand-square-free, we have $N \cap L = K \cap L = 0$. Thus, $N = (N \cap K) = K$, leading to the contradiction $M = N + K = N = K$. Hence, M is summand-dual-square-free.
2) ⇒ 1) Clearly, M is a $D41$-module by [4]. Now, we demonstrate that M is summand-square-free. Assume M is not summand-square-free, and let N and K be non-zero direct summands of M with $N \cong K$ and $N \cap K = 0$. Since M is a C4-module, we have $N \oplus K \leq^{\oplus} M$. We can write $M = N \oplus K \oplus L$ for some submodule $L \leq M$.

Now, we have $M/(N \oplus L) \cong K \cong N \cong M/(K \oplus L)$ with

$M = N \oplus K \oplus L = (N \oplus L) + (K \oplus L)$, where both $N \oplus L$ and $K \oplus L$ are direct summands of M. Since M is summand-dual-square-free, it follows that

$M = N \oplus L = K \oplus L$, implying $N = K = 0$, which leads to a contradiction. Therefore, M is summand-square-free.

Proposition 3.22. *Let $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i. If for every submodule N of M, we have $N = \oplus_{i \in I} (N \cap M_i)$, then M is a $D41$-module if and only if each M_i (for $i \in I$) is also a $D41$-module.*

Proof: Assume that each M_i is a $D41$-module for every $i \in$ I.

Let $M = N \oplus L = K \oplus L$ with $N + K = M$ and K cosingular. According to the hypothesis, we can express $N = \oplus_{i\in I} (N \cap M_i)$, $K = \oplus_{i\in I} (K \cap M_i)$ and $L = \oplus_{i\in I} (L \cap M_i)$

Since $M = N \oplus L = K \oplus L$, we have:

$$M = \oplus_{i\in I} [(N \cap M_i) \oplus (L \cap M_i)] = \oplus_{i\in I} [(K \cap M_i) \oplus (L \cap M_i)]$$

Thus, it follows that $M_i = (N \cap M_i) \oplus (L \cap M_i) = (K \cap M_i) \oplus (L \cap M_i)$ for every $i \in I$. Additionally, since $M = N + K$, we have : $M = \oplus_{i\in I} [(N \cap M_i) + (K \cap M_i)]$, which implies $M_i = (N \cap M_i) + (K \cap M_i)$. Given that $N \cap M_i$ and $K \cap M_i$ are perspective direct summands of M_i with $(N \cap M_i) + (K \cap M_i) = M_i$, it follows that $(N \cap M_i) \cap (K \cap M_i) \leq^{\oplus} M_i$ for every $i \in I$. Consequently, we have $N \cap K = \oplus_{i\in I} (N \cap M_i) \cap \oplus_{i\in I} (K \cap M_i) = \oplus_{i\in I} [(N \cap M_i) \cap (K \cap M_i)] \leq^{\oplus} M$, which demonstrates that M is a $D41$-module. The converse is straightforward, as a direct summand of a $D41$-module is itself a $D41$-module by proposition 3.4.

Recall that a module M is called Hopfian if every surjective R-homomorphism $f : M \to M$ is an automorphism.

Definition 3.23. *An R-module M is termed cosingular Hopfian if every cosingular epimorphism* $\xi : M \to M$ *is an automorphism.*

Proposition 3.24. *Every indecomposable cosingular Hopfian module is a* $D41$*-module.*

Proof:

Let M be an indecomposable cosingular Hopfian left R-module, and let $f : M \to N$ be a cosingular epimorphism where N is a direct summand of M. In this case, N must either be 0 or M. In the first scenario, f trivially splits. In the second scenario, where $N = M$, f is an automorphism and thus also splits.

A left R-module M is said to be generalized Hopfian if every surjective R-endomorphism f of M is superfluous, meaning $ker(f) \ll M$.

Proposition 3.25. *Every generalized Hopfian* $D41$*-module has a* $D41$*-cover.*

Proof: The proof is straightforward.

Proposition 3.26. *Let M be a lifting module and consider the following conditions*:

1) M is Hopfian ;

2) M is generalized Hopfian ; 3) M is Dedekind finite.

Then, 1) ⇒ 2) ⇒ 3) *(by [8, Corollary 1.4]).*

If in addition, $M \oplus M$ is a $D41$-module, then we have the equivalence.

Proof:

For 3) ⇒ 1): Since $M \oplus M$ is a $D41$-module, then M is a $D2$-module. If $f : M \to M$ is an epimorphism, then there exists an endomorphism g of M such that $fg = 1$. Given that End(M) is a directly finite ring, we can conclude that $gf = 1$. Therefore, f is an isomorphism, which implies that M is Hopfian.

In this section we introduce the concept of t-cosingular $D4$-module. The notion of t-cosingular module was introduced by Y. Talebi and A. R. M. Hamzekolaee in 2013 (see [17]). A module M is called t-cosingular if $\overline{Z_t}(M) = 0$ where $\overline{Z_t}(M) = Rej(M,TS) = \{\ker f \mid f: M \to L,\ L \in TS\}$ with **TS** the class of t-small modules.

Definition 3.27. *Let M be an R-module. We say that M is a t-cosingular $D4$-module if, $M = N \oplus K$ for $N,\ K \leq M$ such that K is t-cosingular and $f: N \to K$ is an epimorphism, then $ker(f) \leq^{\oplus} N$. The ring R is a right (left) t-cosingular $D4$-ring if the right R-module R_R (left ${}_RR$) is t-cosingular $D4$.*

Remark 3.28. *It has been proved in [17, Remark 1] that $\overline{Z_t}(M) \subseteq \bar{Z}(M)$. From this, we can say that every $D41$-module is also a t-cosingular $D4$-module.*

Proposition 3.29. *Any direct summand of a t-cosingular $D4$-module is again a t-cosingular $D4$-module.*

Proof:

Let M be a t-cosingular $D4$-module, and suppose $N \leq^{\oplus} M$. We aim to show that if $N = K \oplus K'$, where K' is t-cosingular and $f: K \to K'$ is an epimorphism, then $\ker f \leq^{\oplus} K$.

Assume $M = N \oplus N' = K \oplus K' \oplus N'$, where $N' \leq M$. Define the canonical projection $\pi : K \oplus N' \to K$. Then the composition $f \circ \pi : K \oplus N' \to K'$ is an epimorphism, and its kernel is given by $ker(f \circ \pi) = kerf \oplus N'$.

Since $M = (K \oplus N') \oplus K'$ is a t-cosingular $D4$-module, we deduce that

$kerf \oplus N' \leq^{\oplus} K \oplus N' \leq^{\oplus} M$. Thus, $\ker f \leq^{\oplus} N$. Consequently, N is a t-cosingular $D4$-module.

Proposition 3.30. *Let M be a t-cosingular $D4$-module. Then, for every submodule N of M, the quotient module M/N is also a t-cosingular $D4$-module.*

Proof:

Let M be a t-cosingular $D4$-module, and let N be a submodule of M. Consider a t-cosingular submodule $K = P/N$ of M/N, where P is a submodule of M containing N. Assume that $(M/N)/(P/N) \cong L$, where L is a direct summand of M/N and $L \leq P/N$.

By the second isomorphism theorem, we have: $M/P \cong (M/N)/(P/N) \cong L$.

Since M is a t-cosingular $D4$-module, P is a direct summand of M. Consequently, $K = P/N$ is a direct summand of M/N.

Therefore, M/N is a t-cosingular $D4$-module.

Recall that a module M has C^* if every submodule N of M contains a direct summand K of M such that N/K is cosingular [19] and a module M is *lifting* if every submodule N of M contains a direct summand K of M such that $N/K \ll M/K$. We now define a module having t-C^* if every submodule N of M contains a direct summand K of M such that N/K is t-cosingular. We recall a module is t-*lifting* if every submodule N of M contains a direct summand K of M such that N/K is t-small in M/K.

Proposition 3.31. *Every t-cosingular module (and consequently, every t-small module) satisfies the property t-C^*.*

Proof: Since a submodule of a t-cosingular module is t-cosingular, the Proposition follows easily.

Proposition 3.32. *For an R-module M, the following statements are equivalent*:

1) M satisfies t-C^*.

2) For every submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2$ is t-cosingular.

3) For every submodule N of M, N can be decomposed as $N = N_1 \oplus N_2$, where N_1 is a direct summand of M, and N_2 is t-cosingular.

Proof:

1) ⇒ 2) Let $N \leq M$. By definition, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and N/M_1 is t-cosingular. This gives $N = M_1 \oplus (N \cap M_2)$, and since $N \cap M_2 \cong N/M_1$, it follows that $N \cap M_2$ is t-cosingular.

2) ⇒ 3) Assume $M = M_1 \oplus M_2$ with $M_1 \leq N$. Then $N = M_1 \oplus (N \cap M_2)$. Define $N_1 = M_1$ and $N_2 = N \cap M_2$, satisfying the required decomposition.

3) ⇒ 1) Suppose $N \leq M$. By assumption, $N = N_1 \oplus N_2$, where N_1 is a direct summand of M, and N_2 is t-cosingular. Since $N/N_1 \cong N_2$ and N_2 is t-cosingular, N/N_1 is also t-cosingular. Hence, M satisfies t-C^*.

Proposition 3.33. *The following statements are equivalent for a ring R*:

1) Every right R-module satisfies t-C^*.

2) Every injective right R-module satisfies t-C^*.

3) Every right R-module can be expressed as a direct sum of an injective module and a t-cosingular module.

Proof:

1) ⇔ 2) This equivalence is evident, as any submodule of a module that satisfies t-C^* also satisfies t-C^*.

2) ⇒ 3) Let M be an injective module that satisfies t-C^*. By Proposition 3.32, every submodule of M can be expressed as a direct sum of an injective module and a t-cosingular module.

3) ⇒ 1) If every submodule of M can be expressed as a direct sum of an injective module and a t-cosingular module, then M satisfies t-C^*. This follows from the fact that injective submodules are direct summands, as established in Proposition 3.32.

Proposition 3.34. *Let $M = M_1 \oplus M_2$, where M_1 is semisimple and M_2 satisfies t-C^*. Then M also satisfies t-C^*.*

Proof: Let $M = M_1 \oplus M_2$, where M_1 is semisimple and M_2 satisfies t-C^*. Let $N \leq M$. Then M_1 can be decomposed as $M_1 = (N \cap M_1) \oplus M'$ for some submodule $M' \leq M_1$. Consequently, $M = (N \cap M_1) \oplus M' \oplus M_2$ and $N = (N \cap M_1) \oplus A$, where A = $N \cap (M' \oplus M_2)$.

Since $(M_2 \oplus M')/M'$ satisfies t-C^*, it follows that $(A+M')/M' = K/M' \oplus L/M'$ for some submodules K and L containing M', where K/M' is a direct summand of $(M_2 \oplus M')/M'$ and L/M' is t-cosingular. Thus, K is a direct summand of M.

Moreover, since $K = M' \oplus (K \cap A)$, $K \cap A$ is also a direct summand of M.

Therefore, $(N \cap M \oplus (K \cap A)$ is a direct summand of M.

Finally, $N/ \ ((N \cap M_1) \oplus (K \cap A)) \cong A/(K \cap A) \cong (A+K)/K \cong (A+M')/K \cong L/M'$ is t-cosingular. Hence, M satisfies t-C^*.

Proposition 3.35. *Every t-lifting module satisfies the t-C^* property.*

Proof It's obvious.

4. Rings over which certain modules have $D41$

In this section, we provide a characterization of certain classes of rings in relation to $D41$-modules. We start with a characterization of the class of rings R for which every $D41$-module is also a $D4$-module.

Recall that a ring is cosingular projective ($COSP$) if every cosingular module over R is projective (see [18, 3]).

Theorem 4.1. *The following conditions are equivalent for a semisimple ring R:*

1) R is a left cosingular projective-ring ;

2) Every $D41$-R-module is a $D4$-module ;

Proof:

1) ⇒ 2) it's obvious.

2) ⇒ 1) Since R is a semisimple ring then every module of R is projective. In particular every cosingular module is projective. Thus, R is a cosingular projective-ring.

Remark 4.2. *Let R be a ring which is not a left cosingular projective-ring. From Theorem 4.1, it follows that R has a $D41$-module that is not a cosingular projective-module.*

Theorem 4.3. *The following statements are equivalent*:

1) R is a left cosingular semisimple ring.

2) Every left R-module is cosingular projective.

3) Every left R-module is cosingular direct-projective. 4) Every left R-module is a $D41$-module.

5) Every submodule of a $D41$-module module is a $D41$-module. 6) Every direct sum of $D41$-modules is a $D41$-module.

7) Every direct sum of two cyclic $D41$-modules is a $D41$-module.

Proof:

The implications 1) ⇒ 2) ⇒ 3) ⇒ 4) ⇒ 5) ⇒ 6) ⇒ 7) are obvious.

6) ⇒ 1) Let V be a simple R-module and $f: R \to V$ an R-epimorphism. Since $R \oplus V$ is a $D41$-module, by Proposition 2.4, V is isomorphic to a direct summand of R, implying that V is projective. Therefore, R is semisimple.

A module M is termed regular if every cyclic submodule of M is a direct summand of M. This is equivalent to stating that every finitely generated submodule of M is also a direct summand of M (see [21, p. 67]).

According to [11], a module M is defined as d-Rickart if for every endomorphism φ of M, the image $Im(\varphi)$ is a direct summand of M.

Next, we provide a characterization in terms of $D41$-modules for a left semi-hereditary ring to be von Neumann regular.

Proposition 4.4. *The following conditions are equivalent for a left semi-hereditary ring* R:

1) Every finitely generated R*-module is a* $D4$*-module ;*

2) Every finitely generated R*-module is a* $D41$*-module ;*

3) Every finitely generated projective R*-module is a d-Rickart module ;*

4) Every finitely generated projective R*-module is a regular module ;*

5) R *is a von Neumann regular ring.*

Proof: This is immediate.

Proposition 4.5. *Let* A *be a class of cosingular* R*-module and closed under iso-morphisms and directs summand. The following conditions are equivalent*:

1) All $A \in A$ *is* A*-projective.*

2) Every left R*-module in* A *is a* $D2$*-module.*

3) Every left R*-module in* A *is a cosingular direct-projective module.*

4) Every left R*-module in* A *is a* $D4$*module.*

5) Every left R*-module in* A *satisfies (*$D41$*).*

Proof:

The implications 1) ⇒ 2) ⇒ 3) ⇒ 4) ⇒ 5) are obvious.

The implication 5) ⇒ 1) is demonstrated in [14, Proposition 3.6], and we restate it here for clarity.

Let $N \in \mathrm{A}$ and consider the epimorphism $R^{(I)} \to N$. This implies that $R^{(I)} \oplus N$ is an A-$D41$ module. As noted in [14, Proposition 2.6], N is isomorphic to a direct summand of $R^{(I)}$. Therefore, N is a projective module.

Recall that a submodule A of an R-module B is termed a pure submodule if, for any left R-module X, the natural homomorphism $A \otimes X \to B \otimes X$ is injective. A module M is called pure-injective if every homomorphism from a pure submodule A of any module B can be extended to a homomorphism from B into M. It is established that both direct summands and direct products of pure-injective modules are also pure-injective.

Proposition 4.6. *The following conditions are equivalent for a ring* R *:*

1) R *is a semisimple Artinian ring.*

2) Every R*-module is a* $D41$*-module.*

3) Every direct sum of two cyclic $D41$*-modules is a* $D41$*-module.*

4) Every 2*-generated* R*-module is a* $D41$*-module.*

5) Every factor module of an injective R-module is a $D41$-module.

6) R is a von Neumann regular ring in which every injective module is a $D41$-module.

7) Every pure-injective R-module is a $D41$-module.

Proof:

1) ⇒ i) for $i = 2,\ldots,7$ is straightforward. The implications 2) ⇒ 3) and 4) ⇒ 3) are also clear.

To show 3) ⇒ 1), let S be a simple R-module and consider the R-epimorphism $f : R \to S$. Since $R \oplus S$ is a $D41$-module, Lemma 2.4 implies that S is isomorphic to a direct summand of R, which indicates that S is projective. Thus, R must be semisimple Artinian.

For 5) ⇒ 1), if N is a right ideal of R, then $E(R) \oplus E(R)/N$ is a $D41$-module by 5). Therefore, the canonical homomorphism $\eta : E(R) \to E(R)/N$ splits by Lemma 2.4, which means that N is a direct summand of $E(R)$ and hence a direct summand of R. This shows that R is semisimple Artinian.

Now, for *6 ⟹ 1*), it suffices to demonstrate that R is right Noetherian. For a countable family of injective R-modules $\{M_i : i = 1, 2, \ldots\}$, the module $(\prod_{i=1}^{\infty} M_i)/\bigoplus_{i=1}^{\infty} M_i$ is pure-injective by [7, Theorem 38.1 and Corollary 42.2]. Since R is Von Neumann regular, this module is also injective by [22, 37.6]. Therefore, by 6), $(\prod_{i=1}^{\infty} M_i) \oplus [\frac{\prod_{i=1}^{\infty} M_i}{\bigoplus_{i=1}^{\infty} M_i}]$ is a D41-module. This implies that the natural epimorphism $\prod_{i=1}^{\infty} M_i \to \frac{\prod_{i=1}^{\infty} M_i}{\bigoplus_{i=1}^{\infty} M_i}$ splits, leading to $\bigoplus_{i=1}^{\infty} M_i \leq^{\oplus} \prod_{i=1}^{\infty} M_i$. Thus $\bigoplus_{i=1}^{\infty} M_i$ is injective, demonstrating that R is right Noetherian.

Finally, for *7*) ⟹ *2*), consider a countable family of pure-injective R-modules $\{M_i : i = 1, 2, \ldots\}$. The module $(\prod_{i=1}^{\infty} M_i)/\bigoplus_{i=1}^{\infty} M_i$ is pure-injective by [7, Theorem 38.1 and Corollary 42.2], and hence it is a D41-module according to *7*). Consequently, the natural epimorphism $\prod_{i=1}^{\infty} M_i \to \frac{\prod_{i=1}^{\infty} M_i}{\bigoplus_{i=1}^{\infty} M_i}$ splits, leading to $\bigoplus_{i=1}^{\infty} M_i \leq^{\oplus} \prod_{i=1}^{\infty} M_i$. Therefore $\bigoplus_{i=1}^{\infty} M_i$ is pure-injective [22, 53.7], this means every R-modules is pure-injective, thus every R-module is a D41-module by *7*).

Recall that a ring is called perfect if $R/Rad(R)$ is semisimple and $Rad(R)$ is T-nilpotent (see [10, Definition 23.18]). An R-module M is termed finitely presented (or finitely related) if there exists an exact sequence $0 \to K \to F \to M \to 0$ of R-modules, where F is a free module and both F and K are finitely generated. A ring R is referred to as semiregular if every finitely presented R-module possesses a projective cover (see [15, Theorem 2.9]).

Proposition 4.7. *The following statements are equivalent for a ring R:*

1) R is semiregular.

2) Every finitely presented R-module has a $D41$-cover.

Proof:

1) ⇒ 2): This is evident.

1) ⇒ 2): Let M be a finitely presented R-module and $g: F \to M$ be a homomorphism from a free R-module F to M. According to 2), $F \oplus M$ has a $D41$-cover, and by [4, Theorem 3.2], M has a projective cover. Therefore, R is a semiregular ring.

Proposition 4.8. *The following statements are equivalent*:

1) R is left perfect ;

2) Every flat left R-module is quasi-projective ;

3) Every flat left R-module is a $D41$-module ;

4) Every left R-module has a $D41$-cover.

5) Every countably generated R-module has a $D41$-cover.

Proof: The implications 1) ⇒ 2) ⇒ 3) and 1) ⇒ 4) ⇒ 5) are straightforward.

3) ⇒ 1). Let M be a flat left R-module, N a free left R-module, and $g : N \to M$ an R-epimorphism. Since $K = M \oplus N$ is flat and, by assumption, a $D41$-module, then g splits, which implies that M is projective. Consequently, by [24, Lemma 10], we conclude that R is a left perfect ring.

5) ⇒ 1). By [4, Corollary 3.4], $R^{\mathbb{N}}$ is a lifting module. So, by [1, Theorem 1.2.17], R is a right perfect ring.

Definition 4.9. *A ring R is called strongly left $D41$ if $({}_RR)^n$ satisfies the $D41$ condition for every positive integer n ; that is, every finitely generated free left R-module is a $D41$-module. A left R-module M is called strongly left $D41$ if M^n is a $D41$-module for every positive integer n.*

Proposition 4.10. *The following conditions are equivalent for a ring R:*

1) R is Artinian semisimple ;

2) Every (finitely generated) left R-module is a $D41$-module ;

3) Every (finitely generated) left R-module is a strongly $D41$-module ; 4) Every 2-generated left R-module is a strongly $D41$-module ;

5) The class of all left $D41$-modules is closed under (finite) direct sums ;

6) The class of all strongly left $D41$-modules is closed under (finite) direct sums.

Proof:

The implications 1) ⇔ 2) ⇔ 5) follow from [23, Theorem 9].

6) ⇒ 4, 3) ⇒ 4) and 1) ⇒ 3) ⇒ 2) are obvious.

6) ⇒ 1) Let M be a simple R-module. This implies that M is a strongly $D41$-module. Given the hypothesis, $R \oplus M$ is also strongly $D41$. We note that M is isomorphic to R/J for some maximal left ideal J of R. Thus, there exists an epi-morphism $R \to M \to 0$. By Proposition 3.5, we conclude that M is a $D41$-module. Therefore, R is semisimple.

4) ⇒ 1) Since every simple R-module M is a strongly $D41$-module and $R \oplus M$ is 2-generated, the claim can be proved using an argument similar to that in the proof of 6) ⇒ 1).

Proposition 4.11. *The following conditions are equivalent for a ring R:*

1) R is a semisimple Artinian ring.

2) Every left R-module has a $D41$-cover.

3) Every 2-generated left R-module has a $D41$-cover. 4) Every left R-module has a $D41$-envelope.

5) Every 2-generated left R-module has a $D41$-envelope.

Proof: The implications 1) ⇒ 2) ⇒ 3) and 1) ⇒ 4) ⇒ 5) are straightforward.

3) ⇒ 1). Let S be a simple left R-module, and denote $\psi : {}_RR \to S$ as an epimorphism. By condition 3), $M = {}_RR \oplus S$ has a $D41$-cover, which we can denote as $\lambda : C \to M$, where C is a $D41$-module. Let

$\delta_1 : S \to M$ and $\delta_2 : {}_RR \to M$ be the inclusion maps for $i = 1, 2$. Notably, both S and ${}_RR$ are D41-modules, and there exist homomorphisms $\gamma_1 : S \to C$ and $\gamma_2 : {}_RR \to C$ such that $\lambda \circ \gamma_i = \delta_i$.

Clearly, we have $\mathrm{id}_M = \delta_1 \oplus \delta_2 = \lambda \circ (\gamma_1 \oplus \gamma_2)$. This shows that M is isomorphic to a direct summand of C, implying that M is a D41-module. Consequently, we conclude that $ker(\psi)$ is a direct summand of ${}_RR$ by [24, Proposition 4]. Therefore, S is a projective module, leading us to conclude that R is semisimple.

5) ⇒ 1). Let S be a simple right R-module, and let $\psi : {}_RR \to S$ be an epimorphism. By condition 5), $M =_R R \oplus S$ has a D41-envelope, denoted by $\alpha : M \to K$, where K is a D41-module. Since both S and R are D41-modules, there exist homo-morphisms $f_1 : K \to S$ *and* $f_2 : K \to R$ such that $f_i\alpha = \pi_i$, where $\pi_1 : M \to S$ and $\pi_2 : M \to R$ are the projection maps.

Additionally, there exists a homomorphism $\phi : K \to M$ such that $\pi_i\phi = f_i$ for $i = 1, 2$. This implies that $\phi\alpha = \mathrm{id}_M$, making α a split monomorphism. Thus, $S \oplus E(S)$ is isomorphic to a direct summand of K, indicating that $S \oplus R$ is also a D41-module. From this, we conclude that $ker(\psi)$ is a direct summand of ${}_RR$. Therefore, S is a projective module, leading us to the conclusion that R is semisimple.

Proposition 4.12. *The following conditions are equivalent for a ring R*:

1) R is hereditary (or semihereditary).

2) Every submodule (or finitely generated submodule) of a projective R-module is a D41-module.

3) Every submodule (or finitely generated submodule) of a projective R-module is a strongly D41-module.

Proof: The implications 1) ⇒ 3) ⇒ 2) and 1) ⇒ 2) are evident.

2) ⇒ 1). Let K be a projective submodule of an R-module M. Consider a free R-module F and an epimorphism $g : F \to K$. Then $F \oplus K$ is a projective submodule of $F \oplus M$, which implies that $F \oplus K$ is a D41-module. Consequently, K is a pure direct-projective module, leading to the conclusion that the epimorphism $g : F \to K \to 0$ splits. Thus, K is projective, which implies that R is hereditary.

Proposition 4.13. *For a ring R, the following conditions are equivalent*:

1) R is a semisimple ring.

2) Every D41-module over R is projective.

3) Every cosingular pure direct-projective R-module is projective. 4) Every quasi-pure-injective R-module is projective.

5) Every pure-injective R-module is projective.

Proof:

1) ⇒ 2). Since R is a semisimple ring, every R-module M is projective [22, Proposition 20.7].

The implications 2) ⇒ 3) ⇒ 4) ⇒ 5) are straightforward.

4) ⇒ 1). If every pure-injective left R-module is projective, it follows that every injective module is projective. Thus, condition 1) holds according to [22, Proposition 20.7].

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