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Analyzing the Stability of Euler-Lagrange Additive Functional Equations in Banach Spaces: A Fixed Point and Direct Method Perspective

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Abstract: This study explores the Ulam-Hyers stability of the Euler-Lagrange additive functional equation in the framework of Banach spaces. By employing both direct and fixed point methods, we establish new stability results that contribute to the understanding of functional equations in normed spaces. The direct method provides explicit estimates for stability bounds, while the fixed point approach ensures the existence and uniqueness of stable solutions under suitable conditions. Our findings offer a deeper insight into the interplay between functional equations and stability theory, with potential applications in analysis and mathematical modeling.

Keywords: Euler - Lagrange functional equations, generalized Ulam - Hyers stability, Banach Space, fixed point.

1. Introduction

One of the most influential stability notions was introduced by Ulam in 1940, when he posed a fundamental question regarding the stability of group homomorphisms[1]. This question was later answered affirmatively by Hyers in 1941[2], leading to what is now known as Ulam-Hyers stability. Since then, this stability concept has been extensively studied and extended to various mathematical settings, including functional equations, differential equations, and difference equations. Various types of functional equations, such as Cauchy's functional equation, Jensen's equation, and the Euler-Lagrange additive functional equation, have been studied under Ulam-Hyers stability [3, 4, 5]. In many cases, researchers employ direct analytical methods, fixed point theorems, and iterative techniques to establish stability results. Banach space settings provide a rich framework for these studies, as normed spaces allow rigorous error estimations and convergence analysis.[6, 7, 8].

Differential equations are essential in modeling physical phenomena, engineering systems, and biological processes. Stability analysis is a key aspect of differential equation theory, ensuring that small perturbations in initial conditions or parameters do not lead to drastic changes in solutions [9]. The concept of Ulam-Hyers stability has been extended to differential equations, particularly to ordinary differential equations (ODEs) and partial differential equations (PDEs).For first-order and higher-order differential equations[10, 11], stability is often analyzed using fixed point methods, integral inequalities, and operator theory. The stability of linear and nonlinear differential equations has been extensively studied, leading to significant results that ensure the robustness of solutions under small perturbations. The application of fixed point theorems such as Banach's contraction principle has been particularly useful in proving the existence and stability of solutions. In partial differential equations, researchers have investigated Ulam-Hyers stability in settings involving boundary value problems, heat equations, and wave equations. Such studies are crucial in mathematical physics, where slight variations in initial or boundary conditions should not lead to unbounded or highly unstable behaviors.

Difference equations serve as discrete counterparts to differential equations and have applications in numerical analysis, dynamical systems, and computer science. Ulam-Hyers stability in difference equations ensures that small deviations in the system's structure do not lead to significant instabilities in discrete-time solutions. The stability analysis of difference equations has been widely explored using discrete fixed point methods, iterative approximation techniques, and functional inequalities. Linear and nonlinear difference equations, recurrence relations, and fractional difference equations have all been studied under the Ulam-Hyers stability framework[12, 13]. These results are particularly useful in numerical methods for solving differential equations, where approximations and discretization errors must be controlled. Recently Agilan et.al exploring the stability results in various additive functional equation through various normed spaces such as [14, 15, 16, 17, 18, 19, 20, 21].

In this paper, we introduce new kind of functional Equation and investigate the Ulam-Hyers stability of the Euler-Lagrange additive functional equation in Banach spaces using direct and fixed point methods.

$$\begin{aligned}(s^2+2s)h(px+qy)+(1-2s)h(py+qz)+2sh(pz+qx)-2sph(x-y)\\ -2sqh(y-z)=(s^2p+2sq)h(x)+(p+s^2q)h(y)+(2sp+q)h(z)\end{aligned} \tag{1}$$

with $(s^2+2s),(1-2s),sp,sq\neq 0$ and $p,q,s\in\mathbb{R}$

We aim to:1.Establish new stability results for the Euler-Lagrange equation under different conditions.

2.Compare the effectiveness of direct analytical techniques and fixed point methods.

3.Extend existing stability results to broader classes of functional equations.

By doing so, we contribute to the ongoing research in stability theory, offering insights into its applications in various mathematical and applied fields.

Hereafter through out this paper, let us consider X and Y to be a normed linear space and a Banach space, respectively.

$$\begin{aligned}H(x,y,z)=(s^2+2s)h(px+qy)+(1-2s)h(py+qz)+2sh(pz+qx)-2sph(x-y)\\ -2sqh(y-z)-(s^2p+2sq)h(x)-(p+s^2q)h(y)-(2sp+q)h(z)\end{aligned}$$

where $(s^2+2s),(1-2s),sp,sq \neq 0$ and $p,q,s \in \mathbb{R}$ for all $x,y,z \in X$.

2 STABILITY RESULTS FOR THE FUNCTIONAL EQUATION (1)

Theorem 2.1 Let $\mathrm{j} \in \{-1,1\}$ and $\Theta: \mathrm{X}^3 \to [0,\infty)$ be a function such that

$$\lim_{\mathrm{n}\to\infty} \frac{\Theta(\mathcal{Q}^{\mathrm{nj}}\mathrm{x},\mathcal{Q}^{\mathrm{nj}}\mathrm{y},\mathcal{Q}^{\mathrm{nj}}\mathrm{z})}{\mathcal{Q}^{\mathrm{nj}}} = 0 \tag{1}$$

for all $x,y,z \in X$. Let $h: X \to Y$ be a function satisfying the inequality

$$\|H(x,y,z)\| \leq \Theta(x,y,z) \tag{2}$$

for all $x,y,z \in X$. Then there exists a unique additive mapping $A: X \to Y$ and satisfying the functional equation (1) such that

$$\|h(x) - A(x)\| \leq \frac{1}{\mathcal{Q}(s^2+2s+1)} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Theta(\mathcal{Q}^{kj}x,\mathcal{Q}^{kj}x,\mathcal{Q}^{kj}x)}{\mathcal{Q}^{kj}} \tag{3}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{n\to\infty} \frac{h(\mathcal{Q}^{nj}x)}{\mathcal{Q}^{nj}} \tag{4}$$

for all $x \in X$. Here $\mathcal{Q} = (p+q)$.

Proof. Assume $j=1$. Replacing (x,y,z) by (x,x,x) in (2), we get

$$\|(s^2+2s+1)h(\mathcal{Q}x) - \mathcal{Q}(s^2+2s+1)h(x)\| \leq \Theta(x,x,x) \tag{5}$$

for all $x \in X$. The above inequality can written as

$$\|\mathcal{Q}(s^2+2s+1)h(x) - (s^2+2s+1)h(\mathcal{Q}x)\| \leq \Theta(x,x,x) \tag{6}$$

for all $x \in X$. bothside divide by $(s^2+2s+1)\mathcal{Q}$ in (6), we have

$$\left\|h(x) - \frac{h(\mathcal{Q}x)}{\mathcal{Q}}\right\| \leq \frac{\Theta(x,x,x)}{\mathcal{Q}(s^2+2s+1)} \tag{7}$$

for all $x \in X$. Now replacing x by $\mathcal{Q}x$ and dividing by $\mathcal{Q}$ in (7), we get

$$\left\|\frac{h(\mathcal{Q}x)}{\mathcal{Q}} - \frac{h(\mathcal{Q}^2x)}{\mathcal{Q}^2}\right\| \leq \frac{\Theta(\mathcal{Q}x,\mathcal{Q}x,\mathcal{Q}x)}{\mathcal{Q}^2(s^2+2s+1)} \tag{8}$$

for all $x \in X$. From (7) and (8), we obtain

$$\left\|h(x) - \frac{h(\mathcal{Q}^2x)}{\mathcal{Q}^2}\right\| \leq \left\|h(x) - \frac{h(\mathcal{Q}x)}{\mathcal{Q}}\right\| + \left\|\frac{h(\mathcal{Q}x)}{\mathcal{Q}} - \frac{h(\mathcal{Q}^2x)}{\mathcal{Q}^2}\right\|$$

$$\leq \frac{1}{\mathcal{Q}(s^2+2s+1)}\left[\Theta(x,x,x) + \frac{\Theta(\mathcal{Q}x,\mathcal{Q}x,\mathcal{Q}x)}{\mathcal{Q}}\right] \tag{9}$$

for all $x \in X$. In general for any positive integer n , we get

$$\left\|h(x) - \frac{h(\mathcal{Q}^nx)}{\mathcal{Q}^n}\right\| \leq \frac{1}{\mathcal{Q}(s^2+2s+1)} \sum_{k=0}^{n-1} \frac{\Theta(\mathcal{Q}^kx,\mathcal{Q}^kx,\mathcal{Q}^kx)}{\mathcal{Q}^k} \tag{10}$$

$$\leq \frac{1}{\mathcal{Q}(s^2+2s+1)} \sum_{k=0}^{\infty} \frac{\Theta(\mathcal{Q}^kx,\mathcal{Q}^kx,\mathcal{Q}^kx)}{\mathcal{Q}^k}$$

for all $x \in X$. In order to prove the convergence of the sequence

$$\left\{\frac{h(Q^n x)}{Q^n}\right\},$$

replace x by $Q^m x$ and dividing by Q^m in (10), for any $m, n > 0$, we deduce

$$\left\|\frac{h(Q^m x)}{Q^m} - \frac{h(Q^{n+m} x)}{Q^{(n+m)}}\right\| = \frac{1}{Q^m} \left\|h(Q^m x) - \frac{h(Q^n \cdot Q^m x)}{Q^n}\right\|$$

$$\leq \frac{1}{Q(s^2+2s+1)} \sum_{k=0}^{n-1} \frac{\Theta(Q^{k+m}x, Q^{k+m}x, Q^{k+m}x)}{Q^{k+m}}$$

$$\leq \frac{1}{Q(s^2+2s+1)} \sum_{k=0}^{\infty} \frac{\Theta(Q^{k+m}x, Q^{k+m}x, Q^{k+m}x)}{Q^{k+m}}$$

$$\to 0 \quad as \quad m \to \infty$$

for all $x \in X$. Hence the sequence $\left\{\frac{h(Q^n x)}{Q^n}\right\}$ is Cauchy sequence. Since Y is complete, there exists a mapping $A: X \to Y$ such that

$$A(x) = \lim_{n \to \infty} \frac{h(Q^n x)}{Q^n} \quad \forall \quad x \in X.$$

Letting $n \to \infty$ in (10) we see that (3) holds for all $x \in X$. To prove that A satisfies (1), replacing (x, y, z) by $(Q^n x, Q^n y, Q^n z)$ and dividing by Q^n in (2), we obtain

$$\frac{1}{Q^n} \|H(Q^n x, Q^n y, Q^n z)\| \leq \frac{1}{Q^n} \Theta(Q^n x, Q^n y, Q^n z)$$

for all $x, y, z \in X$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$(s^2 + 2s)A(px + qy) + (1 - 2s)A(py + qz) + 2sA(pz + qx) - 2spA(x - y)$$
$$-2sqA(y - z) = (s^2 p + 2sq)A(x) + (p + s^2 q)A(y) + (2sp + q)A(z)$$

Hence A satisfies (1) for all $x, y, z \in X$. To prove A is unique, we let $B(x)$ be another mapping satisfying (1) and (3), then

$$\|A(x) - B(x)\| = \frac{1}{Q^n} \|A(Q^n x) - B(Q^n x)\|$$

$$\leq \frac{1}{Q^n} \{\|A(Q^n x) - h(Q^n x)\| + \|h(Q^n x) - B(Q^n x)\|\}$$

$$\leq \frac{2}{(s^2+2s+1)} \sum_{k=0}^{\infty} \frac{\Theta(Q^{k+n}x, Q^{k+n}x, Q^{k+n}x)}{Q^{(k+n)}}$$

$$\to 0 \quad as \quad n \to \infty$$

for all $x \in X$. Hence A is unique.

Corollary 2.2 Let Θ and s be nonnegative real numbers. Let a function $h: X \to Y$ satisfies the inequality

$$\|H(x, y, z)\| \leq \begin{cases} \Theta, & \\ \Theta\{||x||^s + ||y||^s + ||z||^s\}, & s \neq 1; \\ \Theta||x||^s ||y||^s ||z||^s, & 3s \neq 1; \\ \Theta\{||x||^s ||y||^s ||z||^s + \{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\}\}, & 3s \neq 1; \end{cases} \tag{11}$$

for all $x, y, z \in X$. Then there exists a unique additive function $A: X \to Y$ such that

$$\|h(x) - A(x)\| \leq \begin{cases} \frac{\Theta}{(s^2+2s+1)|\mathcal{Q}-1|}, \\ \frac{3\,\Theta||x||^s}{(s^2+2s+1)|\mathcal{Q}-\mathcal{Q}^s|}, \\ \frac{\Theta||x||^{3s}}{(s^2+2s+1)|\mathcal{Q}-\mathcal{Q}^{3s}|}, \\ \frac{4\,\Theta||x||^{3s}}{(s^2+2s+1)|\mathcal{Q}-\mathcal{Q}^{3s}|} \end{cases} \tag{12}$$

for all $x \in X$.

3 STABILITY RESULTS:FIXED POINT METHOD

Theorem 3.1 [22](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T: X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either

(B_1) $d(T^n x, T^{n+1} x) = \infty \quad \forall\ n \geq 0$,

(B_2) there exists a natural number n_0 such that:

(i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;

(ii)The sequence $(T^n x)$ is convergent to a fixed point y^* of T;

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X: d(T^{n_0} x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L}\ d(y, Ty)$ for all $y \in Y$.

Theorem 3.2 Let $h: V \to B$ be a mapping for which there exists functions $\alpha, \beta, \gamma: V^3 \to [0, \infty)$ with the condition

$$\lim_{k\to\infty} \frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k z)}{\mu_i^k} = 0, \tag{13}$$

where

$$\mu_i = \begin{cases} \mathcal{Q}, & i = 0\,, \\ \frac{1}{\mathcal{Q}}, & i = 1 \end{cases}$$

satisfying the functional inequality

$$\|H(x, y, z)\| \leq \alpha(x, y, z) \tag{14}$$

for all $x, y, z \in V$. If there exists an $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \frac{1}{(s^2+2s+1)}\Theta\left(\frac{x}{\mathcal{Q}}\right),$$

one has the property

$$\gamma(x) = L\,\mu_i\,\gamma\left(\frac{x}{\mu_i}\right) \tag{15}$$

for all $x \in V$. Then there exists a unique additive function $A: V \to B$ satisfying the functional equation (1) and

$$\|h(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) \tag{16}$$

holds for all $x \in X$.

Proof. Consider the set $X = \{p/p: V \to B, p(0) = 0\}$ and introduce the generalized metric on X,

$$d(p,q) = \inf\{K \in (0,\infty): \| p(x) - q(x) \| \leq K\gamma(x), x \in V\}.$$

It is easy to see that (X,d) is complete.

Define $T: X \to X$ by

$$Tp(x) = \frac{1}{\mu_i}p(\mu_i x), \forall \quad x \in V.$$

Now $p,q \in X$,

$$d(p,q) \leq K \Rightarrow \| p(x) - q(x) \| \leq K\gamma(x), x \in V.$$

$$\Rightarrow \left\|\frac{1}{\mu_i}p(\mu_i x) - \frac{1}{\mu_i}q(\mu_i x)\right\| \leq \frac{1}{\mu_i}K\gamma(\mu_i x), x \in V,$$

$$\Rightarrow \left\|\frac{1}{\mu_i}p(\mu_i x) - \frac{1}{\mu_i}q(\mu_i x)\right\| \leq LK\gamma(x), x \in V,$$

$$\Rightarrow \| Tp(x) - Tq(x) \| \leq LK\gamma(x), x \in V,$$

$$\Rightarrow d(Tp,Tq) \leq LK.$$

This implies

$$d(Tp,Tq) \leq Ld(p,q),$$

for all $p,q \in X$. i.e., T is a strictly contractive mapping on X with Lipschitz constant L.

From (7), we have

$$\left\|h(x) - \frac{h(Qx)}{Q}\right\| \leq \frac{\Theta(x,x,x)}{Q(s^2+2s+1)} \tag{17}$$

where $\beta(x) = \frac{\Theta(x,x,x)}{Q(s^2+2s+1)}$

for all $x \in V$. Using (15) for the case $i = 0$, it reduces to

$$\left\|\frac{1}{Q}h(Qx) - h(x)\right\| \leq \frac{1}{Q}\gamma(x)$$

for all $x \in V$.

$$i.e., \qquad d(Th,h) \leq \frac{1}{Q} = L = L^{1-0} = L^{1-i} < \infty.$$

Again replacing $x = \frac{x}{Q}$ in (17), we get

$$\left\| h(x) - Qf\left(\frac{x}{Q}\right) \right\| \leq \frac{1}{(s^2+2s+1)} \Theta\left(\frac{x}{Q}\right).$$

for all $x \in V$. Using (15) for the case $i = 1$, it reduces to

$$\left\| h(x) - Qf\left(\frac{x}{Q}\right) \right\| \leq \gamma(x)$$

for all $x \in V$.

$$i.e., \qquad d(h, Th) \leq 1 = L^0 = L^{1-1} = L^{1-i} < \infty.$$

In the above cases, we arrive

$$d(h, Th) \leq L^{1-i}.$$

Therefore $(B_2(i))$ holds.

By $(B_2(ii))$, it follows that there exists a fixed point A of T in X such that

$$A(x) = \lim_{k\to\infty} \frac{h(\mu_i^k x)}{\mu_i^k}, \qquad \forall\ x \in V. \tag{18}$$

Claim that $A: V \to B$ is additive. Replacing (x, y, z) by $\left(\mu_i^k x, \mu_i^k y, \mu_i^k z\right)$ in (14) and dividing by μ_i^k, it follows from (13) and (18), A satisfies (1) for all $x, y, z \in X$.

By $(B_2(iii))$, A is the unique fixed point of T in the set $Y = \{h \in X: d(Th, A) < \infty\}$, using the fixed point alternative result A is the unique function such that

$$\|h(x) - A(x)\| \leq K\gamma(x)$$

for all $x \in V$ and $K > 0$. Finally by $(B_2(iv))$, we obtain

$$d(h, A) \leq \frac{1}{1-L} d(g, Tg)$$

implying

$$d(h, A) \leq \frac{L^{1-i}}{1-L}.$$

Hence we conclude that

$$\| h(x) - A(x) \| \leq \frac{L^{1-i}}{1-L} \gamma(x).$$

for all $x \in V$.

Corollary 3.3 Let $\mathrm{h}: \mathrm{V} \to \mathrm{B}$ be a mapping and there exist real numbers Θ and s such that

$$\|\mathrm{H}(\mathrm{x},\mathrm{y},\mathrm{z})\| \leq \begin{cases} \Theta, & \\ \Theta\{||\mathrm{x}||^{\mathrm{s}} + ||\mathrm{y}||^{\mathrm{s}} + ||\mathrm{z}||^{\mathrm{s}}\}, & \mathrm{s} \neq 1; \\ \Theta||\mathrm{x}||^{\mathrm{s}}||\mathrm{y}||^{\mathrm{s}}||\mathrm{z}||^{\mathrm{s}}, & 3\mathrm{s} \neq 1; \\ \Theta\left\{||\mathrm{x}||^{\mathrm{s}}||\mathrm{y}||^{\mathrm{s}}||\mathrm{z}||^{\mathrm{s}} + \{||\mathrm{x}||^{3\mathrm{s}} + ||\mathrm{y}||^{3\mathrm{s}} + ||\mathrm{z}||^{3\mathrm{s}}\}\right\}, & 3\mathrm{s} \neq 1; \end{cases} \tag{19}$$

for all $x, y, z \in X$, then there exists a unique additive function $A: V \to B$ such that

$$\|h(x) - A(x)\| \leq \begin{cases} \frac{\Theta}{(s^2+2s+1)|\mathcal{Q}-1|}, \\ \frac{3\,\Theta||x||^s}{(s^2+2s+1)|\mathcal{Q}-\mathcal{Q}^s|}, \\ \frac{\Theta||x||^{3s}}{(s^2+2s+1)|\mathcal{Q}-\mathcal{Q}^{3s}|}, \\ \frac{4\,\Theta||x||^{3s}}{(s^2+2s+1)|\mathcal{Q}-\mathcal{Q}^{3s}|} \end{cases} \tag{20}$$

for all $x \in X$.

Proof. Let us set

$$\alpha(x, y, z) = \begin{cases} \Theta, \\ \Theta\{||x||^s + ||y||^s + ||z||^s\}, \\ \Theta\ \ ||x||^s\ ||y||^s\ ||z||^s, \\ \Theta\{||x||^s||y||^s||z||^s + (||x||^{3s} + ||y||^{3s} + ||z||^{3s})\} \end{cases}$$

for all $x, y, z \in X$. Now

$$\frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k z)}{\mu_i^k} = \begin{cases} \frac{\Theta}{\mu_i^k}, \\ \frac{\Theta}{\mu_i^k}\{||\mu_i^k x||^s + ||\mu_i^k y||^s + ||\mu_i^k z||^s\}, \\ \frac{\Theta}{\mu_i^n}||\mu_i^n x||^s\ ||\mu_i^n y||^s\ ||\mu_i^n z||^s, \\ \frac{\Theta}{\mu_i^k}\{||\mu_i^k x||^s\ ||\mu_i^k y||^s\ ||\mu_i^k z||^s\{||\mu_i^k x||^{3s} + ||\mu_i^k y||^{3s} + ||\mu_i^k z||^{3s}\}\} \end{cases}$$

$$= \begin{cases} \to 0\ as\ \ k \to \infty, \\ \to 0\ as\ \ k \to \infty, \\ \to 0\ as\ \ k \to \infty, \\ \to 0\ as\ \ k \to \infty. \end{cases}$$

i.e., (13) is holds. But, we have

$$\gamma(x) = \frac{1}{(s^2+2s+1)}\left[\Theta\left(\frac{x}{\mathcal{Q}}, \frac{x}{\mathcal{Q}}, \frac{x}{\mathcal{Q}}\right)\right].$$

Hence

$$\gamma(x) = \frac{1}{(s^2+2s+1)}\left[\Theta\left(\frac{x}{\mathcal{Q}}, \frac{x}{\mathcal{Q}}, \frac{x}{\mathcal{Q}}\right)\right] = \begin{cases} \frac{\Theta}{(s^2+2s+1)\mathcal{Q}}, \\ \frac{3\Theta}{(s^2+2s+1)\mathcal{Q}^s}||x||^s, \\ \frac{\Theta}{(s^2+2s+1)\mathcal{Q}^{3s}}||x||^{3s}, \\ \frac{4\Theta}{(s^2+2s+1)\mathcal{Q}^{3s}}||x||^{3s}. \end{cases}$$

Also,

$$\frac{1}{\mu_i}\gamma(\mu_i x) = \begin{cases} \frac{\Theta}{\mu_i\cdot(s^2+2s+1)Q}, \\ \frac{3\Theta}{\mu_i\cdot(s^2+2s+1)Q^s}||\mu_i x||^s, \\ \frac{\Theta}{\mu_i\cdot(s^2+2s+1)Q^{3s}}||\mu_i x||^{3s}, \\ \frac{\Theta}{\mu_i\cdot(s^2+2s+1)Q^{3s}}||\mu_i x||^{3s}. \end{cases}$$

$$= \begin{cases} \mu_i^{-1}\frac{\Theta}{(s^2+2s+1)}, \\ \mu_i^{s-1}\frac{3\Theta}{(s^2+2s+1)Q^s}||x||^s, \\ \mu_i^{3s-1}\frac{\Theta}{(s^2+2s+1)Q^{3s}}||x||^{3s}, \\ \mu_i^{3s-1}\frac{4\Theta}{(s^2+2s+1)Q^{3s}}||x||^{3s}. \end{cases}$$

$$= \begin{cases} \mu_i^{-1}\gamma(x), \\ \mu_i^{s-1}\gamma(x), \\ \mu_i^{3s-1}\gamma(x), \\ \mu_i^{3s-1}\gamma(x). \end{cases}$$

we prove the following cases for conditions using (16)

Case:1 $L = Q^{-1}$ if $i = 0$

$$\|h(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{(Q^{-1})^{1-0}}{1-(Q)^{-1}}\cdot\frac{\Theta}{(s^2+2s+1)Q} = \frac{\Theta}{(s^2+2s+1)(Q-1)}.$$

Case:2 $L = Q$ if $i = 1$

$$\|h(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{(Q)^{1-1}}{1-Q}\cdot\frac{\Theta}{(s^2+2s+1)Q} = \frac{\Theta}{(s^2+2s+1)(1-Q)}.$$

Case:1 $L = Q^{s-1}$ if $i = 0$

$$\|h(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{(Q^{s-1})^{1-0}}{1-Q^{s-1}}\frac{3\Theta}{(s^2+2s+1)Q^s}||x||^s$$

$$= \frac{Q^s}{Q-Q^s}\frac{3\Theta}{(s^2+2s+1)Q^s}||x||^s$$

$$= \frac{3\Theta||x||^s}{(s^2+2s+1)(Q-Q^s)}.$$

Case:2 $L = \frac{1}{Q^{s-1}}$ if $i = 1$

$$\|h(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{\left(\frac{1}{Q^{s-1}}\right)^{1-1}}{1-\frac{1}{(s^2+2s+1)Q^{s-1}}}\frac{3\Theta}{(s^2+2s+1)Q^s}||x||^s$$

$$= \frac{Q^s}{Q^s-Q}\frac{3\Theta}{(s^2+2s+1)Q^s}||x||^s$$

$$= \frac{3\Theta||x||^s}{(s^2+2s+1)(Q^s-Q)}.$$

Case:1 $L = Q^{3s-1}$ if $i = 0$

$$\|h(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{\left(Q^{3s-1}\right)^{1-0}}{1-Q^{3s-1}} \frac{\Theta}{(s^2+2s+1)Q^{3s}} ||x||^{3s}$$

$$= \frac{Q^{3s}}{Q-Q^{3s}} \frac{\Theta}{(s^2+2s+1)Q^{3s}} ||x||^{3s}$$

$$= \frac{\Theta||x||^{3s}}{(s^2+2s+1)(Q-Q^{3s})}.$$

Case:2 $L = \frac{1}{Q^{3s-1}}$ if $i = 1$

$$\|h(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{\left(\frac{1}{Q^{3s-1}}\right)^{1-1}}{1-\frac{1}{Q^{3s-1}}} \frac{\Theta}{(s^2+2s+1)Q^{3s}} ||x||^{3s}$$

$$= \frac{Q^{3s}}{Q^{3s}-Q} \frac{\Theta}{(s^2+2s+1)Q^{3s}} ||x||^{3s}$$

$$= \frac{\Theta||x||^{3s}}{(s^2+2s+1)(Q^{3s}-Q)}.$$

Case:1 $L = Q^{3s-1}$ if $i = 0$

$$\|h(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{\left(Q^{3s-1}\right)^{1-0}}{1-Q^{3s-1}} \frac{4\Theta}{(s^2+2s+1)Q^{3s}} ||x||^{3s}$$

$$= \frac{Q^{3s}}{Q-Q^{3s}} \frac{4\Theta}{(s^2+2s+1)Q^{3s}} ||x||^{3s}$$

$$= \frac{4\Theta||x||^{3s}}{(s^2+2s+1)(Q-Q^{3s})}.$$

Case:2 $L = \frac{1}{Q^{3s-1}}$ if $i = 1$

$$\|h(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{\left(\frac{1}{Q^{3s-1}}\right)^{1-1}}{1-\frac{1}{Q^{3s-1}}} \frac{4\Theta}{(s^2+2s+1)Q^{3s}} ||x||^{3s}$$

$$= \frac{Q^{3s}}{Q^{3s}-Q} \frac{4\Theta}{(s^2+2s+1)Q^{3s}} ||x||^{3s}$$

$$= \frac{4\Theta||x||^{3s}}{(s^2+2s+1)(Q^{3s}-Q)}.$$

4 Conclusion

The research article titled "Analyzing the Stability of Euler-Lagrange Additive Functional Equations in Banach Spaces: A Fixed Point and Direct Method Perspective" investigates the stability of specific functional equations within Banach spaces. The study employs both fixed point and direct methods to establish conditions under which these equations exhibit stability. The findings contribute to a deeper understanding of the behavior of Euler-Lagrange additive functional equations in the context of Banach spaces, offering valuable insights for further research in this area.

Conflict of interest. The authors declare that they have no competing interests.

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