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Fixed Point Theorem in Complete Multiplicative S-Metric Space

Manjusha P. Gandhi[1], Anushree A. Aserkar[2], Malabika Adak[3], Nanda S. Thakre[4], Renuka Nafdey[5]

[1]Associate Professor, Department of Applied Mathematics and Humanities, Yeshwantrao Chavan College of Engineering, Nagpur, 441110, India

[1]manjusha_g2@rediffmail.com,[2]aserkar_aaa@rediffmail.com

[2,3,4]Assistant Professor, Department of Applied Mathematics and Humanities, Yeshwantrao Chavan College of Engineering, Nagpur, 441110, India

[5]Assistant Professor, Department of Physics, Ramdeobaba University, Nagpur, India

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Abstract:

The aim of this paper to prove a unique common fixed point theorem for pair of non-continuous mappings in multiplicative S -metric space. The authors have introduced a novel contraction condition to prove this result. This theorem is generalization, improvement and modification of existing results available in the literature. An example has been given to validate the result.

Keywords: Fixed Point, Multiplicative Metric Space, Convergent, Mapping.

1. Introduction

The fixed point theory has started in 1912. It has three main sections: Topological fixed point theory, Metric fixed point theory and Discrete fixed point theory. Fréchet [1] introduced metric spaces in the context of functional analysis.

Further many authors worked on it and initiated various new spaces. Sedghi et al [2] initiated the concept of S-metric space. The authors [3], [4], [5], [6], [7], [8], [9], [10] have extended the study of S-metric space.

Ozavsar et al [11] showed an equivalent of Banach contraction theorem for multiplicative metric space. The researchers Kanchanapally et al [12], Adewale et al [13] and Terentius et al [14] have worked on recently developed multiplicative S-metric space.

The authors are motivated to work on multiplicative S-metric space because very few researchers worked in this area. The purpose of the present study is to establish a novel theorem for multiplicative S-metric space. We have presented a distinguished contraction condition, which is not available in existing literature. Also proved two new lemmas on multiplicative S-metric space.

2. Preliminaries

The sequel include the definitions listed below:

Definition 1.1[1]: Consider a non-empty set M and a function $d: M \times M \rightarrow R$ such that

(i) $d\left(h,i\right)=0$ for all $h,i\in M:d\left(h,i\right)=0$ if and only if $h=i$

(ii) $d\left(h,i\right)=d\left(i,h\right)$ for all $h,i\in M$

(iii) $d\left(h,i\right)\leq d\left(h,j\right)+d\left(j,i\right)$ for all $h,i\in M$.

Here d is named as metric on M , and $\left(M,d\right)$ is metric space.

Definition 1.2[12]: Assume $S:M^3\to[0,\infty)$ is a function, that satisfy the constraints for every $h,i,j,k\in M$, as here under

1) $S(h,i,j)=1$, iff $h=i=j$.

2) $S(h,i,j)\leq S(h,h,k)+S(i,i,k)+S(j,j,k)$

Couple (M,S) is named as S -metric space. Here M is non-empty set.

Definition 1.3 [12]: Let M be a non- empty set. Then the function $S:M^3\to[0,\infty)$ is named as multiplicative S-metric on M , if and only if, the constraints below are true for all $h,i,j,k\in M$

(i) $S(h,i,j)\geq 1$

(ii) $S(h,i,i)=1\Leftrightarrow h=i=j$

(iii) $S(h,i,j)\leq S(h,h,k)S(i,i,k)S(j,j,k)$

Here (M,S) is named as multiplicative S-metric space.

Definition 1.4 [12]: A sequence $\{h_n\}$ in multiplicative S-metric space (M,S) is multiplicative S-converges to $h\in M$ iff for every $\varepsilon>1$, there exists $\eta\in\mathrm{N}$ such that

$S(h_n,h_n,h)<\varepsilon$, for all $n>\eta$.

Definition 1.5[12]: The sequence $\{h_n\}$ of multiplicative S-metric space (M,S), is known as multiplicative S-Cauchy sequence of M if and only if, for every $\varepsilon>1$, there occurs a $\eta\in\mathrm{N}$ such that $S(h_n,h_n,h_m)<\varepsilon$, for each $n,m>\eta$.

Definition 1.6 [12]: The multiplicative S-metric space (M,S) is complete iff, each multiplicative S-Cauchy sequence of M is multiplicative S-convergent of M .

3. Main Result

We have proved following two lemmas, which are required to prove the theorems in the present paper.

Lemma1: Consider (M,S) is a multiplicative S -metric space, $\{h_n\}$ and $\{i_n\}$ are S convergent to h,i respectively. Then, we have

$$\limsup_{n\to\infty} S(h_n,j,i_n)\leq S(j,j,h)S(h,h,i).$$

Proof: Let $\lim_{n\to\infty} h_n = h, \lim_{n\to\infty} i_n = i$.

Then for each $\varepsilon > 0$, there exists natural numbers n_1, n_2 such that for every $n \geq n_1$, $S(h_n, h_n, h) < \frac{\varepsilon}{2}$ and for every $n \geq n_2$, $S(i_n, i_n, i) < \frac{\varepsilon}{2}$.

If maximum$\{n_1, n_2\} = n_0$, then by property of multiplicative S-metric space, we have for all $n \geq n_0$

$$S(h_n, j, i_n) \leq S(h_n, h_n, h)S(j, j, h)S(i_n, i_n, h)$$

$$\leq S(h_n, h_n, h)S(j, j, h)S^2(i_n, i_n, i)S(h, h, i)$$

Take upper limit $n \to \infty$.

$$\limsup_{n\to\infty} S(h_n, j, i_n) \leq S(h, h, h)S(j, j, h)S^2(i, i, i)S(h, h, i) = S(j, j, h)S(h, h, i)$$

Hence proved.

Lemma 2: Consider (M, S) is a multiplicative S-metric space, then $S(h, h, i) = S(i, i, h)$.

Proof: By property of multiplicative S-metric space

$$S(h, h, i) \leq S(h, h, h)S(h, h, h)S(i, i, h)$$

i.e. $S(h, h, i) \leq S(i, i, h)$

Now, $S(i, i, h) \leq S(i, i, i)S(i, i, i)S(h, h, i)$

Hence $S(h, h, i) = S(i, i, h)$.

[13] has proved the following theorems.

Theorem 1. [13] Let (M, S) be a complete S-multiplicative metric space and $T : M \to M$ a mapping for which there exist the real number, λ satisfying $0 \leq \lambda < 1$such that for each pair $h, i, j \in M$

$$S(Th, Ti, Tj) \leq [S(h, h, j)]^{\lambda} \qquad (2.1.1)$$

Then, T has a unique fixed point.

Theorem 2. [13] Assume (M, S) is a sequentially compact S-multiplicative metric space and $f : M \to M$ fulfil the constraint

$$S(fh, fi, fj) \leq \phi[S(h, i, j)] \qquad (2.1.2)$$

where ϕ : [0, ∞] → [0,∞] is upper semi-continuous from the right, satisfying ϕ(t) ≥ t for t > 0. Then, f has a unique fixed point.

We have generalized above theorems by proving the following result for pair of mappings which satisfy a new contraction condition.

Theorem 2.1: Consider a complete multiplicative S-metric space (M, S). Assume $\phi : [1,\infty) \to [1,\infty)$, an upper semi- continuous and non-decreasing function, $f, g : M \to M$ are functions such that

(1) $f(M) \subset g(M)$

(2) $S^{\mu}(fh, fi, fj) \leq \phi[S^{\alpha}(gh, gi, gj) S^{\beta}(gj, fh, fi) S^{\gamma}(gj, fi, fj)]$ (2.1.3)

for all $h, i, j \in M$ and $\alpha, \beta, \gamma \geq 0$ and $\mu = \alpha + \beta + \gamma \in [1, \infty)$

Then f and g possesses unique common fixed point.

Proof:

Step 1: Let $h_0 \in M$. Define the sequence $\{i_n\}$ as $i_n = fh_n = gh_{n+1}$

Then

$$S^{\mu}(i_n, i_n, i_{n+1}) \leq \phi[S^{\alpha}(gh_n, gh_n, gh_{n+1}) S^{\beta}(gh_{n+1}, fh_n, fh_n) S^{\gamma}(gh_{n+1}, fh_n, fh_{n+1})]$$

$$= \phi[S^{\alpha}(i_{n-1}, i_{n-1}, w_n) S^{\beta}(i_n, i_n, i_n) S^{\gamma}(i_n, i_n, i_{n+1})]$$

$$\Rightarrow S^{\alpha+\beta+\gamma}(i_n, i_n, i_{n+1}) \leq \phi[S^{\alpha}(i_{n-1}, i_{n-1}, i_n) S^{\gamma}(i_n, i_n, i_{n+1})]$$

$$< S^{\alpha}(i_{n-1}, i_{n-1}, i_n) S^{\gamma}(i_n, i_n, i_{n+1})$$

$$\Rightarrow S^{\alpha+\beta}(i_n, i_n, i_{n+1}) < S^{\alpha}(i_{n-1}, i_{n-1}, i_n) < S^{\alpha+\beta}(i_{n-1}, i_{n-1}, i_n)$$

If $\alpha + \beta = 0$, then contradiction arise. $\therefore \alpha + \beta > 0$.

$$S^{\alpha+\beta}(i_n, i_n, i_{n+1}) < S^{\alpha}(i_{n-1}, i_{n-1}, i_n) < S^{\alpha+\beta}(i_{n-1}, i_{n-1}, i_n)$$

$$S(i_n, i_n, i_{n+1}) < S(i_{n-1}, i_{n-1}, i_n)$$

Similarly, we can show that

$$S(i_{n-1}, i_{n-1}, i_n) < S(i_{n-2}, i_{n-2}, i_{n-1})$$

Hence for all natural numbers n

$$S(i_n, i_n, i_{n+1}) < S(i_{n-1}, i_{n-1}, i_n)$$

Thus $S(i_n, i_n, i_{n+1})$ converges to some $c \geq 1$. Assume $c > 1$.

$$c^{\mu} = \lim_{n\to\infty} S^{\mu}(i_{n+1}, i_{n+1}, i_{n+2})$$

$$= S^{\mu}(fh_{n+1}, fh_{n+1}, fh_{n+2})$$

$$\leq \lim_{n\to\infty} \sup \phi[S^{\alpha}(gh_{n+1}, gh_{n+1}, gh_{n+2}) S^{\beta}(gh_{n+2}, fh_{n+1}, fh_{n+1}) S^{\gamma}(gh_{n+2}, fh_{n+1}, fh_{n+2})]$$

$$= \lim_{n\to\infty} \phi[S^{\alpha}(i_n, i_n, i_{n+1}) S^{\beta}(i_{n+1}, i_{n+1}, i_{n+1}) S^{\gamma}(i_{n+1}, i_{n+1}, i_{n+2})]$$

$$=\lim_{n\to\infty}\phi[S^{\alpha}(i_n,i_n,i_{n+1})S^{\gamma}(i_{n+1},i_{n+1},i_{n+2})]\le c^{\alpha+\gamma}<c^{\alpha+\beta+\gamma}=c^{\mu}$$

Which is a contradiction $\therefore c=1$

$$\therefore \lim_{n\to\infty}S^{\mu}(i_n,i_n,i_{n+1})=1\Rightarrow \lim_{n\to\infty}S(i_n,i_n,i_{n+1})=1. \qquad (2.1.4)$$

Step 2: To Show that $\{i_n\}$ is Cauchy sequence.

If $\{i_n\}$ is not Cauchy sequence, there exists $\varepsilon>0$, then one can find sub-sequences $\{i_{m(l)}\}$ and $\{i_{n(l)}\}$ of $\{i_n\}$ and increasing sequences of integers $\{m_l\}$ and $\{n_l\}$ such that $\{n_l\}$ is the smallest index such that $n(l)>m(l)>l, S^{\mu}(i_{m(l)},i_{m(l)},i_{n(l)})\ge\varepsilon$ .Then

$$S^{\mu}(i_{m(l)},i_{m(l)},i_{n(l)-1})<\varepsilon \qquad (2.1.5)$$

Now

$\varepsilon<S^{\mu}(i_{m(l)},i_{m(l)},i_{n(l)})$

$S^{\mu}(i_{n(l)},i_{n(l)},i_{m(l)})$ --By Lemma 2

$\le S^{2\mu}(i_{n(l)},i_{n(l)},i_{n(l)-1})S^{\mu}(i_{m(l)},i_{m(l)},i_{n(l)-1})$ --By property of S -metric space.

$<\varepsilon S^{2\mu}(i_{n(l)},i_{n(l)},i_{n(l)-1})=\varepsilon$

$\varepsilon<S^{\mu}(i_{m(l)},i_{m(l)},i_{n(l)})<\varepsilon$ --Using (2.1.4)

Which is a contradiction

$$\therefore \lim_{k\to\infty}S^{\mu}(i_{m(l)},i_{m(l)},i_{n(l)})=\varepsilon. \qquad (2.1.6)$$

Then

$S^{\mu}(i_{m(l)},i_{m(l)},i_{n(l)})$

$\le S^{2\mu}(i_{m(l)},i_{m(l)},i_{m(l)-1})S^{\mu}(i_{n(l)},i_{n(l)},i_{m(l)-1})$

$$\le S^{2\mu}(i_{m(l)},i_{m(l)},i_{m(l)-1})S^{2\mu}(i_{n(l)},i_{nl)},i_{n(l)-1})S^{\mu}(i_{m(l)-1},i_{m(l)-1},i_{n(l)-1}) \qquad (2.1.7)$$

Also

$S^{\mu}(i_{m(l)-1},i_{m(l)-1},i_{n(l)-1})$
$\le S^{2\mu}(i_{m(l)-1},i_{m(l)-1},i_{m(l)})S^{\mu}(i_{n(l)-1},i_{n(l)-1},i_{m(l)})$

$=S^{2\mu}(i_{m(l)-1},i_{m(l)-1},i_{m(l)})S^{\mu}(i_{m(l)},i_{m(l)},i_{n(l)-1})$, using Lemma 2 (2.1.8)

Letting $l\to\infty$ in (2.1.8) and using (2.1.4), (2.1.5), (2.1.6),(2.1.7)

$$\lim_{k\to\infty} S^{\mu}(i_{m(l)-1}, i_{m(l)-1}, i_{n(l)-1}) = \varepsilon \tag{2.1.9}$$

Now, again

$$\varepsilon = S^{\mu}(i_{m(l)}, i_{m(l)}, i_{n(l)})$$

$$= S^{\mu}(f(h_{m(l)}), f(h_{m(l)}), f(h_{n(l)}))$$

$$\leq \phi[S^{\alpha}(g(h_{m(l)}), g(h_{m(l)}), g(h_{n(l)}))S^{\beta}(g(h_{n(l)}), f(h_{m(l)}), f(h_{n(l)}))S^{\gamma}(g(h_{n(l)}), f(h_{m(l)}), f(h_{n(l)}))]$$

$$= \phi[S^{\alpha}(i_{m(l)-1}, i_{m(l)-1}, i_{n(l)-1})S^{\beta}(i_{n(l)-1}, i_{m(l)}, i_{m(l)})S^{\gamma}(i_{n(l)-1}, i_{m(l)}, i_{n(l)})$$

$$< S^{\alpha}(i_{m(l)-1}, i_{m(l)-1}, i_{n(l)-1})S^{\beta}(i_{m(l)}, i_{m(l)}, i_{n(l)-1})S^{\gamma}(i_{n(l)-1}, i_{m(l)}, i_{n(l)}) \tag{2.1.10}$$

Letting $l \to \infty$ in (2.1.10) and using (2.1.5),(2.1.9) and Lemma 1,we have $\varepsilon < \varepsilon$,which is a contradiction.

This indicates that $\{i_n\}$ is a Cauchy sequence in complete multiplicative S metric space (M, S).

Step 3: To show that f and g have common fixed point.

Consider $\lim_{n\to\infty} fh_n = \lim_{n\to\infty} gh_{n+1} = p$.

Let $gu = p$.

Now

$$S^{\mu}(fu, fu, fh_n) \leq \phi[S^{\alpha}(gu, gu, gh_n)S^{\beta}(gh_n, fu, fu)S^{\gamma}(gh_n, fu, fh_n)]$$

$$\Rightarrow \lim_{\sup n\to\infty} S^{\mu}(fu, fu, i_n) \leq \phi[S^{\alpha}(gu, gu, i_{n-1})S^{\beta}(i_{n-1}, fu, fu)S^{\gamma}(i_{n-1}, fu, i_n)]$$

$$\Rightarrow \lim_{\sup n\to\infty} S^{\mu}(fu, fu, p) \leq \phi[S^{\alpha}(gu, gu, p)S^{\beta}(p, fu, fu)S^{\gamma}(p, fu, p)]$$

$$\leq \phi[S^{\alpha}(p, p, p)S^{\beta}(fu, fu, p)S^{\gamma}(p, p, p)S^{\gamma}(fu, fu, p)S^{\gamma}(p, p, p)]$$

$$= \phi[S^{\beta+\gamma}(fu, fu, p)] < S^{\beta+\gamma}(fu, fu, p)$$

$\Rightarrow S^{\alpha}(fu, fu, p)] < 1$, which is a contradiction.

$\therefore fu = p$.

Hence $gu = p \Rightarrow fu = p$

$\therefore gu = u \Rightarrow fu = u$

Hence if g has fixed point u then f will also have fixed point u .i.e. f and g have common fixed point.

Step 4: To prove that f and g have fixed point

Since (M,S) is complete multiplicative metric space, there exists $u \in M$, such that

$S(u,u,i_n)] = 1$.

Assume $S(u,u,f(u)) > 1$. Then

$$S(i_{n+1}, i_{n+1}, f(u)) = S^{\mu}(fh_{n+1}, fh_{n+1}, fu)$$

$$\le \phi[S^{\alpha}(gh_{n+1}, gh_{n+1}, gu) S^{\beta}(gu, fh_{n+1}, fh_{n+1}) S^{\gamma}(gu, fh_{n+1}, fu)$$

$$= \lim_{\sup n \to \infty} \phi[S^{\alpha}(i_n, i_n, gu) S^{\beta}(gu, i_{n+1}, i_{n+1}) S^{\gamma}(gu, i_{n+1}, fu)$$

$$S^{\mu}(u,u,fu) \le \phi[S^{\alpha}(u,u,gu) S^{\beta}(gu,u,u) S^{\gamma}(gu,u,fu)$$

$$= \phi[S^{\alpha}(u,u,fu) S^{\beta}(fu,u,u) S^{\gamma}(fu,u,fu)] \qquad \therefore gu = fu$$

$$= \phi[S^{\alpha}(u,u,fu) S^{\beta}(fu,u,u) S^{\gamma}(fu,fu,fu) S^{\gamma}(u,u,fu) S^{\gamma}(fu,fu,fu)],$$

using Lemma 2 and triangular inequality of multiplicative S metric space.

$$S^{\mu}(u,u,fu) \le \phi[S^{\alpha+\beta+\gamma}(u,u,fu)] < S^{\mu}(u,u,fu)$$

which is a contradiction $\therefore$ $fu = u$ i.e. u is fixed point of f, which implies $gu = u$, using step3.

Step 5: To prove uniqueness of fixed point.

Let u and v be two distinct common fixed point of f and g.

i. e. $fu = gu = u$ and $fv = gv = v$.

Now

$$S^{\mu}(u,u,v) = S^{\mu}(fu, fu, fv)$$

$$\le \phi[S^{\alpha}(gu,gu,gv) S^{\beta}(gv,fu,fu) S^{\gamma}(gv,fu,fv)]$$

$$= \phi[S^{\alpha}(u,u,v) S^{\beta}(v,u,u) S^{\gamma}(v,u,v)]$$

$\le \phi[S^{\alpha}(u,u,v) S^{\beta}(u,u,v) S^{\gamma}(v,v,v) S^{\gamma}(u,u,v) S^{\gamma}(v,v,v)]$, using lemma 2 and property of multiplicative S-metric space. Thus

$$S^{\alpha+\beta+\gamma}(u,u,v) < S^{\alpha+\beta}(u,u,v) \Rightarrow S^{\gamma}(u,u,v) < 1$$

Which is a contradiction $\therefore u = v$.

Hence f and g have unique common fixed point.

Example: Let $M = R$ and S is multiplicative S-metric space on M, defined by

$S(h,i,j) = a^{|h+i-2j|}, a > 1$.

Then (i) $S(h,i,j) \ge 1$ (ii) $S(h,i,j) = 1$, when $h = i = j$.

Now $S(h,h,u)S(i,i,u)S(j,j,u) = a^{|h+h-2u|}a^{|i+i-2u|}a^{|j+j-2u|} = a^{2\{|h-u|+|i-u|+|j-u|\}}$

Also $S(h,i,j) = a^{|h+i-2j|} = a^{|(h-u)+(i-u)-2(j-u)|}$

Obviously $S(h,i,j) < S(h,h,u)S(i,i,u)S(j,j,u)$

Hence all three conditions of multiplicative S-metric space are fulfilled. Now assume $f,g : M \to M$ defined by

$$f(h) = \frac{h+1}{2}, g(h) = 5h-4, \phi(t) = \frac{t}{7}, \alpha = \beta = \frac{\mu}{6}, \gamma = \frac{2\mu}{3}.$$

L.H.S. = $S^{\mu}(f(2), f(3), f(4)) = S^{\mu}\left(\frac{3}{2}, 2, \frac{5}{2}\right) = a^{-\frac{3}{2}\mu}$

R.H.S. = $\phi[S^{\alpha}(g(2), g(3), g(4))S^{\beta}(g(4), f(2), f(3))S^{\gamma}(g(4), f(3), f(4))]$

$$< \frac{1}{7}[S^{\alpha}(6,11,16)S^{\beta}(16,\frac{3}{2},2)S^{\gamma}(16,2,\frac{5}{2})] = \frac{1}{7}a^{\frac{101\mu}{12}}$$

Thus contraction condition is satisfied. Also $f(1) = g(1) = 1$.

Corollary 1: Consider a complete multiplicative S-metric space (M,S). Assume $\phi : [1,\infty) \to [1,\infty)$, an upper semi- continuous and non-decreasing function. Let $f : M \to M$ be function such that

(1) $f(M) \subset M$

(2) $S^{\mu}(fh, fi, fj) \le \phi[S^{\alpha}(h,i,j)S^{\beta}(j, fh, fi)S^{\gamma}(j, fi, fj)]$

for all $h,i,j \in M$ and $\alpha, \beta, \gamma \ge 0$ and $\mu = \alpha + \beta + \gamma \in [1,\infty)$

Then f possesses unique fixed point.

Remark 1: If we put $\alpha = 1, \beta = 0, \gamma = 0$ in corollary 1, we get theorem 1[13]

Corollary 2: Consider a complete multiplicative S-metric space (M,S). Assume $\phi : [1,\infty) \to [1,\infty)$, an upper semi- continuous and non-decreasing function. Let $f,g : M \to M$ be functions such that

(1) $f(M) \subset g(M)$

(2) $S^{\mu}(f^p h, f^p i, f^p j) \le \phi[S^{\alpha}(g^q h, g^q i, g^q j)S^{\beta}(g^q j, f^p i, f^p k)S^{\gamma}(g^q j, f^p i, f^p j)]$

for all $h,i,j \in M$, $\alpha, \beta, \gamma \ge 0$ and $\mu = \alpha + \beta + \gamma \in [1,\infty)$, $p,q \in N$

Then f and g have unique common fixed point.

Proof: From theorem 2.1, f^p and g^q will have unique common fixed point u.

Now $f(u) = f[f^p(u)] = f^{p+1}(u) = f^p[f(u)]$

And $g(u) = g[g^q(u)] = g^{q+1}(u) = g^q[g(u)]$.

Thus $f(u)$ and $g(u)$ are also fixed point of f^p and g^q respectively. Due to the uniqueness of fixed point $f(u) = g(u) = u$ .i.e. f and g have unique common fixed point.

Corollary 3: Consider a complete multiplicative S-metric space (M, S). Assume $\phi : [1,\infty) \to [1,\infty)$, an upper semi- continuous and non-decreasing function. Let $f, g : M \to M$ be functions such that

(1) $f(M) \subset g(M)$

(2) $S^{\mu}(fh, fi, fj) \le [S^{\alpha}(gh, gi, gj) S^{\beta}(gj, fh, fi) S^{\gamma}(gj, fi, fj)]^{\lambda}$

For all $h, i, j \in M$, $\alpha, \beta, \gamma \ge 0$ and $\mu = \alpha + \beta + \gamma \in [1,\infty)$, $l \in [0,1)$

Then f and g possesses unique common fixed point.

Proof: Put $\phi(t) = t^{\lambda}$ in theorem 2.1, one can get desired result.

Remark 2: If we put $\alpha = 1, \beta = 0, \gamma = 0$ and consider g as identity mapping, then we get theorem 2[13].

Corollary 4: Let (M, S) be a complete multiplicative S-metric space. Assume $\phi : [1,\infty) \to [1,\infty)$, an upper semi- continuous and non-decreasing function. Let $f, g : M \to M$ be functions such that,

(1) $f(M) \subset g(M)$

(2) (i) $S(fh, fi, fj) \le \phi[S(gh, gi, gj)]$

(ii) $S(fh, fi, fj) \le \phi[S(gj, fi, fh)]$

(iii) $S(fh, fi, fj) \le \phi[S(gj, fi, fj)]$,for all $h, i, j \in M$

Then f and g have unique common fixed point.

Proof: The result can be proved easily by substituting the following values in condition (2.1.3) of theorem 2.1.

(i) $\alpha = 1, \beta = 0 = \gamma$

(ii) $\alpha = 0 = \gamma, \beta = 1$

(iii) $\alpha = \beta = 0, \gamma = 1$

Corollary 5: Consider a complete multiplicative S-metric space (M, S). Assume $\phi : [1,\infty) \to [1,\infty)$, an upper semi- continuous and non-decreasing function. Let $f, g : M \to M$ be functions such that

(1) $f(M) \subset g(M)$

(2) (i) $S^2(fh, fi, fj) \le S(gh, gi, gj) S(gj, fh, fi)$

(ii) $S^2(fh, fi, fj) \le S(gh, gi, gj) S(gj, fi, fj)$

(iii) $S^2(fh, fi, fj) \le S(gj, fh, fi) S(gj, fi, fj)$,for all $h, k, j \in X$

Then f and g have unique common fixed point.

Proof: The result can be proved easily by substituting the following values in condition (2.1.3) of theorem 2.1.

(i) $\alpha = 1, \beta = 1, \gamma = 0$

(ii) $\alpha = 1 = \gamma, \beta = 0$

(iii) $\alpha = 0, \beta = \gamma = 1$

Corollary 6: Consider a complete multiplicative S-metric space (M, S). Assume $\phi : [1, \infty) \to [1, \infty)$, an upper semi- continuous and non-decreasing function. Let $f, g : M \to M$ be functions such that

(1) $f(M) \subset g(M)$

(2) $S^3(fh, fi, fj) \le \phi[S(gh, gi, gj)S(gj, fh, fi)S(gj, fi, fj)]$, for all $x, y, z \in M$.

Then f and g have unique common fixed point.

Proof: Put $\alpha = \beta = \gamma = 1$ in condition (2.1.3) of theorem 2.1, we will get the desired result.

Discussion: A fixed-point theorem has been formulated for a novel category of contractive condition for pair of mappings within multiplicative S-metric space. It has proven that this theorem is generalization of results in existing literature. The various corollaries are also discussed.

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