Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

Existence and Uniqueness of Continuous Solutions for Conformable Fractional Integro-Differential Equations in Cone Metric Spaces

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract:

In this paper, by the application of some extensions of Banach's contraction principle in complete cone metric space, we have proved the existence and uniqueness of solutions to fractional order integro-differential equations of Volterra-Fredholm type which are defined in a cone metric space. The fractional order derivative defined in the integro-differential equation is the conformable fractional order derivative. The obtained results are used for solving a couple of fractional order integro-differential equations of Volterra-Fredholm type.

Mathematics Subject Classification: 34B05.

Keywords: Fractional order integro-differential equations, Cone metric space, Contractive mapping, ordered Banach space.

1. Introduction

Numerous scientific and engineering problems involve integral equations. Volterra or Fredholm integral equations can be used to solve a wide range of initial and boundary value problems. More than any other discipline, the potential theory helped in the development of theory of integral equations. Integral equations were also developed using mathematical physics models, including water waves, conformal mapping, diffraction issues, and scattering in quantum mechanics. Integral equations or integro-differential equations describe a wide range of additional applications in science and engineering. Integral equations address a number of topics, including the Volterra population growth model, coexisting biological species, the spread of stocked fish in a new lake, heat transport, and heat radiation. Integral equations with logarithmic kernels arise in many scientific problems. Integral equations are frequently used in electrostatic, low frequency electromagnetic, electromagnetic scattering, and acoustic and elastic wave propagation problems.

In this paper we study the existence and uniqueness of solutions for the conformable fractional order Volterra-Fredholm type integro-differentail equations [1, 2, 3] of the form

$$\frac{d^{\alpha}x(t)}{dt} = f(t) + \int_0^t p\big(t,s,x(s)\big)ds + \int_0^b q\big(t,s,x(s)\big)ds\,, t \in I = [0,b] \quad \ldots \quad (1)$$

$$x(0) = x_0 \quad \ldots \quad (2)$$

where the term $\frac{d^{\alpha}x(t)}{dt}$ represents the conformable fractional order derivative of fractional order $\alpha \in (0,1).$, $f: I \to X, p, q: I \times I \times X \to X$ are continuous functions and x_0 is an element of a real Banach space X, with the norm $\|.\|$. But before investigating the problem, we will take a review of the research work done by the mathematicians in the development of the topic. In [4], using the Schaefer's fixed point theorem, Karouni, A. et. al. have established the existence and uniqueness of the continuous solution to the nonlinear Fredholm integral equation of the form

$$x(t) = f(t) + \int_{a}^{b} g\big(t, s, x(s)\big)ds, \quad -\infty < a \le t \le b < \infty$$

where it is assumed that the function f is continuous over the interval $[a, b]$ and bounded over a measurable set. In [5], Claudia, A. have proved, using the Picards operator theory, the existence and uniqueness to the solution of the Volterra-Fredholm integral equation of the form

$$u(x, t) = g(x, t) + \int_{0}^{t}\int_{\Omega} k\big(t, x, s, y, u(s, y)\big)dyds$$

where $(t, x) \in [0, T] \times \Omega \coloneqq \overline{D},\ T > 0,\ \Omega \in \mathbb{R}^m$ is bounded and closed.

Ahmad et al. in [6] have obtained the solutions of the integro-differential equations with non-local four point and strip multipoint boundary conditions. Wang et al. [7] have established the conditions for the uniqueness and existence of the positive solutions of the fractional integro-differential equation

$$D^{\alpha}u(t) + f\big(t, u(t), Tu(t), Su(t)\big) = 0, \qquad 0 < t < 1$$

under the boundary conditions given by

$$u(0) = u_0, u'(0) = b_1, \dots, u^{(n-3)}(0) = b_{n-3}, u^{(n-2)}(0) = b_{n-2}, u^{(n-1)}(0) = b_{n-1}$$

where $n - 1 < \alpha \le n,\ 0 \le \mu < n - 1,\ n \ge 3,\ bi \ge 0\ (i = 1, 2, \dots, n - 3, n - 2, n - 1)$, D^{α} being the Caputo fractional derivative of order α, f is a continuous function from $[0, 1] \times \mathbb{R}_+^3 \to \mathbb{R}_+$, T and S are defined by

$$(Tx)(t) = \int_{0}^{1} K(t, s)\, x(s)\, ds, \quad (Sx)(t) = \int_{0}^{1} H(t, s)\, x(s)\, ds$$

$$K^* = \sup_{t \in [0,\ 1]} \int_{0}^{t} K(t, s)\ ds\,, \qquad H^* = \sup_{t \in [0,\ 1]} \int_{0}^{t} H(t, s)\ ds$$

where $K \in C(D,\ \mathbb{R}_+),\ H \in C([0, 1] \times\ [0, 1],\ \mathbb{R}_+)$

The authors in [8, 9] have obtained the results stating the existence and uniqueness of the solutions of the fractional integro-differential equations under different boundary conditions. In [10], Bragdi, A. et al. have obtained the solution of the BVP given by

$$D^{\alpha}\big(D^{\beta}\big)u(t) = f\big(t, u(t), \phi u(t), \psi u(t)\big)$$

under the boundary conditions given by $u(1) = u(0) = u'(0) = 0$ where it is assumed that $1 < \alpha \leq 2,\ 0 < \beta \leq 1,\ f : I \times \mathbb{R}^3 \to \mathbb{R},\ I = [0,\ 1]$, the function f is continuous and

$$\phi(u)(t) = \int_0^t \gamma(t,s)u(s)\ ds, \quad \psi(u)(t) = \int_0^t \lambda(t,s)u(s)\ ds$$

$$\gamma, \lambda :\ I \times I \to [0,1), \qquad sup\int_0^1 \lambda(t,s)\ ds < \infty, \qquad sup\int_0^1 \gamma(t,s)\ ds < \infty.$$

Ibnelazyz, L. et al.[11] have explored the existence and uniqueness for a nonlinear fractional integro-differential equations with integral and anti-periodic boundary conditions where the existence is proved by means of Krasnoselskii's fixed point theorem and the uniqueness of solutions is established via the Banach's contraction principle. in [12], Kamble, R., and Kukarni, P. have proved the existence and uniqueness of solutions for the following equation

$$D^\alpha D^\beta x(\tau) = f\big(t, x(\tau), \phi x(\tau), \psi x(\tau)\big), \tau \epsilon [0,1], x(0) = x(1) = 0$$

where $0 < \alpha \leq 1$, $0 < \beta \leq 1$, D^α, D^β are the Caputo fractional derivatives of order α , β,

$\lambda, \delta: [0,1] \times [0,1] \to [0,+\infty)$, $f: [0,1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuous function, and

$$\phi x(\tau) = \int_0^\tau \lambda(\tau,s)x(s)ds\,, \psi x(\tau) = \int_0^\tau \delta(\tau,s)x(s)ds$$

$$\phi^* = \sup_{t\in[0,1]} \left| \int_0^t \lambda(\tau,s)ds \right| < \infty,\ \psi^* = \sup_{t\in[0,1]} \left| \int_0^t \delta(\tau,s)ds \right| < \infty,$$

2. Some Preliminary Concepts in Fixed Point Theory and Results

In this section, we take a look on the basic concepts in fixed point theory.

Definition 2.1: Let X be a Banach space over the set of real numbers. A subset E of X is called a *cone* if

1) E is closed, non-empty with $E \neq \{0\}$
2) $cx + dy \in E$ whenever $c, d \in \mathbb{R}$ and $x,\ y \in E$
3) $x = 0$ whenever both $x, -x \in E$

Definition 2.2: For a given cone $E \subset X$, X being a Banach space over the set of real numbers, the relation $\leq$ defined on E by $x \leq y \iff y - x \in E$ is a partial ordering relation. The cone E is said to be *normal* if there is a real number $k > 0$ such that $\|x\| \leq k\|y\|\ \forall\ x, y \in E$. The least positive number k satisfying above is called the normal constant of E.

Definition 2.3: Let X be a non empty real Banach space, E be a normal cone in X with $\leq$ as the partial ordering relation defined on E. Then the mapping $d: X \times X \to E$ satisfying

1) $0 \leq d(x,y)\ \forall\ x, y \in E$
2) $d(x,y) = d(y,x)\ \forall\ x, y\ \in E$

3) $d(x,y) \leq d(x,z) + d(z,y)) \ \forall \ x, y\ , z \in E$

is called a *cone metric* on X and the space (X,d) is called *cone metric space.*

Definition 2.4: Let X be an ordered space. A function $\varphi: X \to X$ is said to a *comparison function* if for every $x, y \in X, x \leq y \Longrightarrow \varphi(x) \leq \varphi(y), \varphi(x) \leq x$ and $\lim_{n\to\infty} \|\varphi^n(x)\| = 0$ for every $x \in X$.

A more detailed theory and examples on cone metric spaces and the different versions of the fixed points theorems can be obtained in the books by the authors Smart D. and O'Regan D [13, 14]. We will take a small review of the research work on cone metric spaces. In [15], the authors have proved the following result.

Lemma 2.1: Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. Let $f: X \to X$ be a function such that there exists a comparison function $\phi: P \longrightarrow P$ such that

$$d(f(x), f(y)) \leq \phi(d(x,y) \ \forall \ x, y \in X$$

Then the contraction mapping f has a unique fixed point in X.

In [16], the authors Ilic, D, and Rakocevic, V. has proved the following result.

Lemma 2.2: Let (X,d) be a complete cone metric space and let P be a normal cone. Let $f: X \to X$, f^2 be continuous, $g: f\,(X) \to X$ be such that $gf(X) \subseteq f^2(X)$, and $f(g(x)) = g(f(x))$ whenever both sides are defined. Furthermore, let there exists $\lambda \in (0,1)$ such that $d(gx, gy) \leq \lambda\, u$ for every $x, y \in f\,(X)$. Then f and g have a common unique fixed point u in X.

3. Fractional Derivative and Conformable Fractional Derivative

For many centuries, the derivative of non-integer order has been an interesting area of study. Riemann-Liouville, Caputo, Hadamard, Grunwald-Letnikov, Marchaud, and Riesz were among the fractional derivative types that were introduced [17, 18, 19, 20, 21, 22]. None of these fractional derivatives satisfy the properties of the classical integer order derivatives. We are aware that the derivative of an integer order constant is zero, and we expect fractional derivatives to be no different. This is not the case for most fractional derivatives, except for the Caputo fractional derivative. Furthermore, fractional derivatives do not meet the requirements of classical derivatives, including the Mean value theorems of Rolle and Lagrange, the Product rule, the Quotient rule, and the Chain rule. Many authors, such as R. Khalil [23], Abdljawad et al. [24, 25, 26] etc. have made important contributions in this direction. They have established some of the properties that the previous fractional derivatives did not meet and defined a few new fractional derivatives. All of these conformable derivatives are extensions of the traditional limit form formulation. The following definitions are revised from these references.

Definition 3.1: Let : $[0,\infty) \to \mathbb{R}$, be any function of real variable t. Then the *conformable fractional derivative* of u of fractional order $\alpha, \alpha \in (0,1]$ at $t > 0$, denoted $D^\alpha u(t)$, is defined by the limit

$$D^\alpha u(t) = \lim_{h\to 0} \frac{u\left(t + ht^{(1-\alpha)}\right) - u(t)}{h}$$

provided the limit exists.

For $n < \alpha \leq n+1, n \in N,$ the conformable fractional derivative is defined by

$$D^{\alpha}(u)(t) = \lim_{h\to 0}\frac{u^{\lceil\alpha\rceil-1}\left(t+ht^{(\lceil\alpha\rceil-\alpha)}\right)-u^{\lceil\alpha\rceil-1}(t)}{h}$$

where ⌈α⌉ is the smallest integer greater than or equal to α provided the limit exist.

Definition 3.2: Let $\alpha \in (0,1]$ and $t > 0$. The α conformable fractional integral is given by

$$I^{\alpha}u(t) = \int_0^t \frac{u(s)}{s^{1-\alpha}}ds.$$

Definition 3.3: Let $B = C(I,X)$ be the Banach space of all continuous functions defined on the set $I = [0,\ b]$ into X, where the norm of an $x \in B$ is defined by $||x||_{\infty} = \sup\{||x(t)||: t\} \in I$

Define a metric $d: B \times B \longrightarrow \mathbb{R}$ by $d(x,y) = (||x-y||_{\infty}, a||x-y||_{\infty})\ \forall x,y \in B,\ 0 < a < 1$.Then it can be verified that $(B,\ d)$ is a cone metric space. The function $x \in B$ is called as a *solution* of the initial value problem (1)—(2) if it satisfies the condition

$$x(t) = u_0 + \int_0^t f(t).\,t^{\alpha-1}dt + \int_0^t t^{\alpha-1}\ \left[\int_0^t k\big(t,s,x(s)\big)ds\right]dt + \int_0^t t^{\alpha-1}\left[\int_0^b h\big(t,s,x(s)\big)ds\right]dt$$

4. Main Result

Now we have enough material to prove the main result.

Theorem 4.1: The initial value problem (1)—(2) has a unique solution x in I if the following conditions are satisfied

1) there exist continuous functions $p_1, p_2: I \times I \to \mathbb{R}_+$ and a comparison function $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$(\|k(t,s,u)-k(t,s,v)\|, a\|k(t,s,u)-k(t,s,v)\|) \leq p_1(t,s)\phi\big(d(u,v)\big)$$
$$(\|h(t,s,u)-h(t,s,v)\|, a\|h(t,s,u)-h(t,s,v)\|) \leq p_2(t,s)\phi\big(d(u,v)\big)$$

where the metric $d: B \times B \longrightarrow \mathbb{R}$ is defined by $d(x,y) = (||x-y||_{\infty}, a||x-y||_{\infty})\ \forall x,y \in B$

2)

$$\int_0^b t^{\alpha-1}\left[\int_0^b [p_1(t,s)+p_2(t,s)]\,ds\right]dt \leq 1$$

Proof: We define the operator F by

$$Fx(t) = x_0 + \int_0^t f(t).\,t^{\alpha-1}dt + \int_0^t t^{\alpha-1}\left[\int_0^t k\big(t,s,x(s)\big)ds\right]dt + \int_0^t t^{\alpha-1}\left[\int_0^b h\big(t,s,x(s)\big)ds\right]dt$$

$$Fy(t) = y_0 + \int_0^t f(t).\,t^{\alpha-1}dt + \int_0^t t^{\alpha-1}\left[\int_0^t k\big(t,s,y(s)\big)ds\right]dt + \int_0^t t^{\alpha-1}\left[\int_0^b h\big(t,s,y(s)\big)ds\right]dt$$

Using the conditions 1) and 2), for all $x,y \in B$, we have

$(\|Fx(t)-Fy(t)\|, a\|Fx(t)-Fy(t)\|)$

$$\begin{aligned}
= \Bigg(\|x_0 - y_0\| &+ \Bigg\| \int_0^t t^{\alpha-1} \Bigg[\int_0^t k\big(t,s,x(s)\big)ds \Bigg] dt + \int_0^t t^{\alpha-1} \Bigg[\int_0^b h\big(t,s,x(s)\big)ds \Bigg] dt \\
&- \int_0^t t^{\alpha-1} \Bigg[\int_0^t k\big(t,s,y(s)\big)ds \Bigg] dt - \int_0^t t^{\alpha-1} \Bigg[\int_0^b h\big(t,s,y(s)\big)ds \Bigg] dt \Bigg\|, a\|x_0 - y_0\| \\
&+ a \Bigg\| \int_0^t t^{\alpha-1} \Bigg[\int_0^t k\big(t,s,x(s)\big)ds \Bigg] dt + \int_0^t t^{\alpha-1} \Bigg[\int_0^b h\big(t,s,x(s)\big)ds \Bigg] dt \\
&- \int_0^t t^{\alpha-1} \Bigg[\int_0^t k\big(t,s,y(s)\big)ds \Bigg] dt - \int_0^t t^{\alpha-1} \Bigg[\int_0^b h\big(t,s,y(s)\big)ds \Bigg] dt \Bigg\| \Bigg)
\end{aligned}$$

$$\begin{aligned}
\leq \Bigg(\|x_0 - y_0\| &+ \int_0^t t^{\alpha-1} \Bigg[\int_0^t \big\|k\big(t,s,x(s)\big) - k\big(t,s,y(s)\big)\big\| ds \Bigg] dt \\
&+ \int_0^t t^{\alpha-1} \Bigg[\int_0^b \big\|h\big(t,s,x(s)\big) - h\big(t,s,y(s)\big)\big\| ds \Bigg] dt,\ a\|x_0 - y_0\| \\
&+ a \int_0^t t^{\alpha-1} \Bigg[\int_0^t \big\|k\big(t,s,x(s)\big) - k\big(t,s,y(s)\big)\big\| ds \Bigg] dt \\
&+ a \int_0^t t^{\alpha-1} \Bigg[\int_0^b \big\|h\big(t,s,x(s)\big) - h\big(t,s,y(s)\big)\big\| ds \Bigg] dt \Bigg)
\end{aligned}$$

$$\begin{aligned}
\leq \Bigg(\int_0^t t^{\alpha-1} \Bigg[\int_0^t \big\|k\big(t,s,x(s)\big) - k\big(t,s,y(s)\big)\big\| ds \Bigg] dt,\ \ a \int_0^t t^{\alpha-1} \Bigg[\int_0^t \big\|k\big(t,s,x(s)\big) \\
- k\big(t,s,y(s)\big)\big\| ds \Bigg] dt \Bigg)
\end{aligned}$$

$$\begin{aligned}
+ \Bigg(\int_0^t t^{\alpha-1} \Bigg[\int_0^b \big\|h\big(t,s,x(s)\big) - h\big(t,s,y(s)\big)\big\| ds \Bigg] dt,\ a \int_0^t t^{\alpha-1} \Bigg[\int_0^b \big\|h\big(t,s,x(s)\big) \\
- h\big(t,s,y(s)\big)\big\| ds \Bigg] dt \Bigg)
\end{aligned}$$

$$\leq \int\limits_0^t t^{\alpha-1} \Bigg[\int\limits_0^t p_1(t,s)\phi(d(x,y))ds \Bigg] dt + \int\limits_0^t t^{\alpha-1} \Bigg[\int\limits_0^b p_2(t,s)\phi(d(x,y))ds \Bigg] dt$$

$$\begin{aligned}
\leq \int_0^t t^{\alpha-1} &\Bigg[\int_0^t p_1(t,s)\phi(\|x-y\|_\infty, a\|x-y\|_\infty)ds \Bigg] dt \\
&+ \int_0^t t^{\alpha-1} \Bigg[\int_0^b p_2(t,s)\phi(\|x-y\|_\infty, a\|x-y\|_\infty)ds \Bigg] dt
\end{aligned}$$

$$\begin{aligned}
\leq \int_0^b t^{\alpha-1} &\Bigg[\int_0^b p_1(t,s)\phi(\|x-y\|_\infty, a\|x-y\|_\infty)ds \Bigg] dt \\
&+ \int_0^b t^{\alpha-1} \Bigg[\int_0^b p_2(t,s)\phi(\|x-y\|_\infty, a\|x-y\|_\infty)ds \Bigg] dt
\end{aligned}$$

$$\leq \phi(\|x-y\|_\infty, a\|x-y\|_\infty)\int_0^b t^{\alpha-1}\left[\int_0^b [p_1(t,s)+p_2(t,s)]\,ds\right]dt$$

$$\leq \phi(\|x-y\|_\infty, a\|x-y\|_\infty)$$

This implies that $d\big(F(x),F(y)\big) \leq \phi d(x,y)\ \forall\, x,y \in B$. Now by Lemma 2.1, the operator F has a unique fixed point in B. This means that the initial value problem (1)—(2) has a unique solution x in I. This completes the proof of theorem. □

5. Application of the Result

In order to support the result proved, now we will present an example. In the initial value problem (1)—(2), let $k(t,s,x) = t^2s^2 + \frac{xs^2}{2},\ h(t,s,x) = t^2s^2 + \frac{t^2s^2x}{2}, s,t \in I = [0,1], x \in (C[0,1],\mathbb{R}), 0 < \alpha < 1$. We define the metric $d(x,y) = (\|x-y\|_\infty, a\|x-y\|_\infty)$ on $(C[0,1],\mathbb{R})$ and $a \geq 0$. Then it is clear that $(C[0,1],\mathbb{R}\ )$ is a complete cone metric space.

Now we have

$$(|k(t,s,x(s)) - k(t,s,y(s))|, a|k(t,s,x(s)) - k(t,s,y(s))|)$$

$$= \left(\left|t^2s^2 + \frac{xs^2}{2} - t^2s^2 - \frac{ys^2}{2}\right|, a\left|t^2s^2 + \frac{xs^2}{2} - t^2s^2 - \frac{ys^2}{2}\right|\right)$$

$$= \left(\left|\frac{xs^2}{2} - \frac{ys^2}{2}\right|, a\left|\frac{xs^2}{2} - \frac{ys^2}{2}\right|\right)$$

$$= \frac{s^2}{2}(|x-y|, a|x-y|)$$

$$\leq \frac{s^2}{2}(\|x-y\|_\infty, a\|x-y\|_\infty)$$

$$= {p_1}^*\phi^*(\|x-y\|_\infty, a\|x-y\|_\infty)$$

where ${p_1}^* = s^2$, which is a continuous function from $[0,1]\times[0,1]$ into $\mathbb{R}_+$ and a comparison function $\phi^*(x,y) = \frac{1}{2}(x,y)$. Similarly we can prove that

$$|h(t,s,x(s)) - h(t,s,y(s))|, a|h(t,s,x(s)) - h(t,s,y(s))| \leq {p_2}^*\phi^*(\|x-y\|_\infty, a\|x-y\|_\infty)$$

where ${p_2}^* = s^2t^2$ which is a continuous function of $[0,1]\times[0,1]$into $\mathbb{R}_+$.

Also, we note that $if\ \alpha = \frac{1}{2}$ then

$$\int_0^1 t^{\alpha-1}\int_0^1 \big({p_1}^*(t,s) + {p_2}^*(t,s)\big)dsdt = \int_0^1 t^{\alpha-1}\int_0^1 (s^2 + s^2t^2)dsdt < 1$$

With these choices of functions, all the hypothesis in theorem 4.1 are satisfied. Hence the existence and uniqueness of the solution is verified.

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