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# Thermal Stress Analysis of Two Dimensional Thermoelastic Problem for Inhomogeneous Half Plane

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**Abstract:** This paper develops a method for analytically solving plane elasticity and thermo-elasticity problems for inhomogeneous half planes. With the direct integration of the equilibrium equation, the original problems can be reduced to a set of governing harmonic equations with corresponding boundary conditions. Distribution of Young's modulus, shearing modulus and dimensionless stresses are illustrated numerically and shown graphically.

**Keywords**: Thermal stress, isotropic material, inhomogeneous half plane, simple iteration technique, integro-differential equation, Fourier transform

## 1. Introduction

There exist various methods for analysis of thermoelastic problem for arbitrary inhomogeneous solids with boundary conditions in terms of stresses. Several analytical, semi-analytical approaches were developed for solving the heat conduction problems. Explicit analytical solutions are restricted to simple geometries but these are well organized computationally. Exact solutions of the inverse heat conduction problems are significant because they provide closed form solution for heat flux in terms of temperature measurements. In this paper we extend the technique to avoid latter complications to represents solution for an elastic isotropic material. A. Hamoud et al. [2] solved integro-differential equations by using numerical techniques. A. Yasinskyy and O. Ierokhova [3] gives optimization of nonstationary thermal displacements in a given cross section of a half space in the plane strain state. B. Kalynyak et al. [4] studied direct and inverse problems of thermomechanics concerning the optimization and identification of the thermal stressed state of deformed solids. Y. Tokovyy and Ma. Chien-Ching [9, 10,11] gives an explict form solution to the plane elasticity and thermoelasticity problems for anisotropic and inhomogeneous solids and find out analytical solutions to the 2D elasticity and thermoelasticity problems for inhomogeneous planes and half planes.

This method was established by V. Vigak [7]. This method was already applied to solve some direct and inverse boundary value problems [8]. After integrating the differential equilibrium equations, we can determine the relationship between the stress tensor component. With this technique the governing equations are reduced to integro-differential equation for stress tensor component. With application of simple iteration method, derived integral equations has been solved for constructing the solution in explicit form expression with interdependence of elastic moduli

## 2. Preliminary

In this section, we collect some basic definitions that will be important to us in the sequel.

### 2.1 Definition

A Fourier transform of function f(x) is defined as [6]:

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx.$$

### 2.2 Definition

Inverse Fourier transform of function f(x) is defined as [12]:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}dx.$$

## 3. Problem Formulation

Consider an isotropic inhomogeneous half plane $D = \{(x, y) \in [0, \infty) \times (-\infty, \infty)\}$. The problem is governed by the equilibrium equation [8],

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + X = 0,\ \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + Y = 0,\ (x, y) \in D. \tag{1}$$

Strain-compatibility equations [7]:

$$\frac{\partial^2\tau_{xx}}{\partial y^2} + \frac{\partial^2\tau_{yy}}{\partial x^2} = \frac{\partial^2\tau_{xy}}{\partial x\partial y}. \tag{2}$$

Stress- Strain relations [8]:

$$\tau_{xx} = \frac{1}{E^*}\left(\sigma_{xx} - \vartheta\sigma_{yy} - c^* + \alpha^* T(x, y)\right), \tag{3}$$

$$\tau_{xx} = \frac{1}{E^*}\left(\sigma_{xx} - \vartheta\sigma_{yy} - c^* + \alpha^* T(x, y)\right), \tag{4}$$

$$\tau_{xy} = \frac{1}{G}\sigma_{xy.} \tag{5}$$

where $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ are the stress tensor components and $\tau_{xx}, \tau_{xy}, \tau_{yy}$ are the strain tensor components. G, E, ν are shear modulus, modulus of elasticity and Poisson's ratio respectively and α is the coefficient of thermal expansion. X = X (x, y), Y = Y (x, y) are the stress - dimensional projections of body forces in the abscissa and co-ordinate.

For plane strains

$$E^* = \frac{1}{1-\vartheta^2}, \vartheta^* = \frac{\vartheta}{1-\vartheta^2}, \alpha^* = \alpha(1-\vartheta),\ c^* = \vartheta c. \tag{6}$$

For plain strain,

$$E^* = E, \vartheta^* = \vartheta, \alpha^* = \alpha, c^* = c. \tag{7}$$

According to Hook's law,

$$Ec = \sigma_{zz} - \vartheta(\sigma_{zz} + \sigma_{zz}) + \alpha ET, \tag{8}$$

c=constant is out of plane strain and $\sigma_{zz}$ is the out of plane strain and $T = T(x, y)$ is the temperature distribution.

We impose tractions at the boundary

$$\sigma_{xx}|_{x=0} = -p_1(y)\ ,\ \sigma_{xy}|_{x=0} = q_1(y). \tag{9}$$

Assume that, as $|\boldsymbol{y}| \to \infty$ the stresses are tending to 0.

The steady state temperature T (x, y) can be found from the following heat conduction equation [5]:

$$\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} = \frac{-q(x,y)}{K}, \tag{10}$$

where k=constant under boundary conditions employed at boundary x=0.

The imposed boundary conditions of the given problems are

$$T(x, y) = T_0(y)\ at\ x = 0,$$

$$\frac{\partial T(x,y)}{\partial x} + a_0 \mathrm{T(x, y)} = \varphi_0(y), \text{ at x=0,}$$

$$\frac{\partial T(x,y)}{\partial x} = b_0, \text{ at x=0.}$$

Where $a_0, b_0$ are constant and $T_0(y), \varphi_0(y)$ are given functions.

## 4. Solution Formulations

From the physical relation of (3, 4, 5) and the equilibrium equation (1) representing (2) as follows,

$$\Delta\left[\frac{\sigma}{E^*} + \alpha^* T\right] = \frac{\sigma_{xx}}{2}\frac{d^2}{dx^2}\left(\frac{1}{G}\right) + \frac{d^2 c^*}{dx^2} + \frac{d}{dx}\left(\frac{1}{G}\right) + \frac{1}{2G}\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial x}\right]. \tag{12}$$

To compute the total stress $\sigma = \sigma_{xx} + \sigma_{yy}$ in terms of $\sigma_{yy.}$

We use the relation,

$$\Delta\sigma_{xx} = \frac{\partial^2 \sigma}{\partial y^2} - \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial x}, \tag{13}$$

$\Delta$ denotes two -dimensional Laplace operator.

To find out solution for problem (1) to (13), selecting one key stress out of three stress components. To find out the two -dimensional stressed state, the equation of continuity for these regions, written for the normal stresses $\sigma_{yy}$ Integrating equation (1) as in [7], express the stresses $\sigma_{xy}$ in terms of $\sigma_{xx}$, $\sigma_{yy.}$

$$4\sigma_{yy} = q_1 - \int_0^\infty \left(\frac{\partial \sigma_{yy}}{\partial y} + Y\right) \mathrm{sgn}(x - \eta)d\eta - \int_{-\infty}^\infty \left(\frac{\partial \sigma_{xx}}{\partial x} + X\right) sgn(y - \xi)d\xi, \tag{14}$$

$$sgnx = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0. \end{cases} \tag{15}$$

To find out the key stresses, using integral Fourier transform [10] for (12) - (13),

$$\left(\frac{d^2}{dx^2}-\omega^2\right)\left[\frac{\sigma}{E^*}+\alpha^*T\right]=-\frac{\sigma_{xx}}{2}\left(\frac{\omega^2}{G}\right)-\omega^2c^*2\pi\delta(\omega)+i\omega X\left(\frac{1}{G}\right)+\frac{1}{2G}\left(i\omega X+\frac{dY}{dy}\right), \tag{16}$$

$$\frac{d^2\sigma_{xx}}{dx^2}-\omega^2\sigma_{xx}=-\omega\sigma-\left(i\omega X-\frac{dY}{dy}\right), \tag{17}$$

$$\sigma_{xx}|_{x=0}=-p_1\ ,\ \sigma_{xy}|_{x=0}=q_1, \tag{18}$$

here, $\delta(\omega)$ is the Dirac delta function.

This key stress $\sigma_{xx}$ should satisfy the boundary condition

$$\frac{\partial\sigma_{xx}}{\partial y}|_{x=0}=-i\omega\overline{q_1}+X(0). \tag{19}$$

Here, ω denotes integral transform parameter $i=\sqrt{-1}$. Solving (16) -(17), particular solution for $\bar{\sigma}_{xx}$ from equation (17) obtained in the form, we obtain the expression as,

$$\bar{\sigma}_{xx}=-\overline{p_1}exp(-|\omega|x)+\frac{|\omega|}{2}\int_0^\infty\sigma(\xi)\left[exp(-|\omega||x-\xi|)-exp(-|\omega||x+\xi|)\right]d\xi-\frac{1}{2|\omega|}\int_0^\infty\left(i\bar{X}(\xi)-\frac{d\bar{Y}}{d\xi}\right)\left[exp(-|\omega||x-\xi|)-exp(-|\omega||x+\xi|)\right]d\xi, \tag{20}$$

which satisfy integral conditions,

$$\omega^2\int_0^\infty\sigma exp(-|\omega|x)d\xi=-|\omega|\bar{p}-i\omega\bar{q}-X(0)+\int_0^\infty\left(i\bar{X}(\xi)-\frac{d\bar{Y}}{d\xi}\right)exp(-|\omega|x)d\xi. \tag{21}$$

Analogously, we construct solution from equation (17) to equation (16) in the form,

$$\bar{\sigma}=E^*\left[Aexp(-|\omega|x-\alpha^*\bar{T}-\frac{\pi}{|\omega|}\int_0^\infty c^*(\xi)exp(-|\omega||x-\xi|)d\xi+\frac{1}{2\omega}\int_0^\infty\left(\frac{X(\xi)}{G(\xi)}+\frac{1}{2G(\xi)}\left(i\omega\bar{X}(\xi)-\frac{d\bar{Y}}{d\xi}\right)-\frac{1}{4\omega}\frac{\sigma_{xx}}{G(\xi)}\right)exp(-|\omega||x-\xi|)\right]d\xi, \tag{22}$$

where, A is the constant of integration. Substitution of the expression (20) into (22) yields expression of the form,

$$\bar{\sigma}=E^*\left[\psi-\alpha^*\bar{T}-\phi_1-\frac{1}{8}\int_0^\infty\bar{\sigma}(\xi_1)N(x,\xi_1)\right]d\xi_1\ . \tag{23}$$

Where, $N(x,\xi_1)=\int_0^\infty\frac{1}{G(\xi_1)}exp(-|\omega||x-\xi|)d\xi_1;$

$$\psi=\frac{\pi}{\omega}\int_0^\infty c^*exp(-|\omega||x-\xi|)d\xi;$$

$$\phi_1=\frac{1}{2\omega}\int_0^\omega\frac{X(\xi)}{G(\xi)}+\frac{1}{G(\xi)}\left((i\omega X(\xi)+\frac{d\bar{Y}}{d\xi}\right).$$

Following the solution technique [7], construct the solution to the integral equation (23) as the limit,

$$\bar{\sigma}=\lim_{n\to\infty}\bar{\sigma}_n.$$

We can solve (23), by simple iteration method [7] as follows,

$$\bar{\sigma}_n = E^* \left[\psi - \alpha^* \bar{T} - \phi_1 - \tfrac{1}{8}\int_0^\infty \bar{\sigma}_{n-1}(\xi_1) N(x,\xi_1)\right] d\xi_1; \qquad (24)$$

$$\bar{\sigma}_0 = 0, n = 1,2,3\ldots$$

For n = 1 iteration can be calculated easily, let us represent the expression (24) in the form for the n th iteration,

$$\bar{\sigma}_n = \bar{\sigma}_1 + \bar{F}_{n-1.}$$

By the substitution of (20) into (23), we obtain,

$$\bar{\sigma} = E^* \left[\psi + p_1 + Aexp(-|s|x - \alpha^* \bar{T} - \phi_2 - \tfrac{1}{8}\int_0^\infty \bar{\sigma}(\xi_1) N(x,\xi_1)\right] d\xi_1, \qquad (25)$$

$$Here, N(x,\xi_1) = \int_0^\infty \frac{1}{G(\xi_1)}\left[[exp(-|\omega|(|\xi - \xi_1| + |x - \xi_1|) - exp(-|\omega||\xi + \xi_1||y + \xi_1|)]\right].$$

## 5. Examples

To find the exact solution, consider an example of inhomogeneity,

Let $X = Y = T = 0$ ,

$E = E_0 f(x),\ f(x) = exp(x^4),\ E_0 = constant,\ b > 0 = constant$

From (20), we obtain required thermal stresses $\sigma_{xx}$, as follows

$$\bar{\sigma}_{xx} = -\overline{p_1} exp(-|\omega|x) + \frac{|\omega|}{2}\int_0^\infty \sigma(\xi)\,[exp(-|\omega||x - \xi|) - exp(-|\omega||x + \xi|)]d\xi.$$

From (25)

$$\bar{\sigma} = E^* \left[\psi + p_1 + Aexp(-|s|x - \alpha^* \bar{T} - \phi_2 - \frac{1}{8}\int_0^\infty \bar{\sigma}(\xi_1) N(x,\xi_1)\right] d\xi_{1\,.}$$

Consider the Poisson's ratio μ=const.

By the relation between Young's and Shear Modulus latter can represent in the form $E = E_0 f(x)$where, $G_0 = E_0\left(\frac{1}{1+\mu}\right)$.

We can see that figure-1 indicates the distribution of μ(x) and figure-2 indicates the distribution of normal stress $\sigma_{yy}, \sigma_{xy}$ respectively at α= 1, 1.5, 2 respectively. we conclude that $\sigma_{xx,,}\sigma_{xy,}\ \sigma$are independent on G and depend on Poisson's ratio ν which vary with x-coordinate.

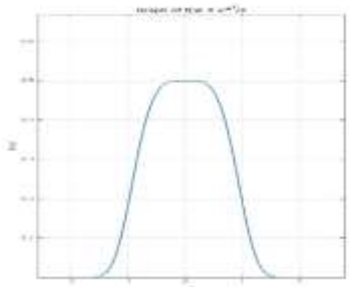

**Figure 1: Distribution of $\mu(x)$ $for$ $\alpha = 1$**

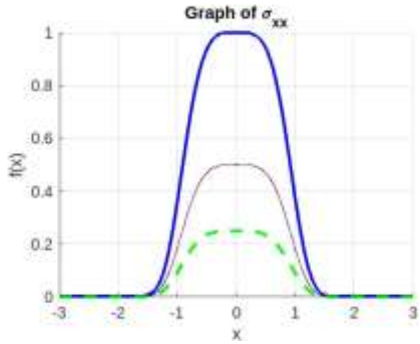


**Figure 2: Distribution of $\sigma_{xx}$ $for$ $\alpha = 1, 1.5, 2$**

## 6. Conclusion

In this manuscript, we studied an numerical approach to solve the two-dimensional problems of elasticity and thermoelasticity in terms of stresses for isotropic material in an inhomogeneous strip which is infinite. This approach is placed on the direct integration of differential equilibrium equations. This technique permits to construct analytical solution for interdependence between the elastic modulie of an isotropic material. We reduce the governing integro-differential equations with variable coefficients in accordance with compatibility and equilibrium equations. The calculation for constructing the solution can be also applied to solve optimization problems, comparable inverse thermoelasticity problems in terms of stresses. In this method we can easily calculate the stressed state in an infinite strip, as compare to solving such problem in terms of displacement. With the help of simple iteration method, we have solved these governing equations. This method gives exact analytical solutions if the shear modulus is reciprocal of linear function in Cartesian coordinate system for corresponding problems. Direct integration method is very useful technique to solve the boundary value problems. Since, application of this method depend on the direct integration of the equilibrium equations for efficient analysis of inhomogeneous solids.

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