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New Contraction Principle in Revised Fuzzy $\mathscr{k}$ −Metric Spaces

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Abstract:

Introduction

Metric spaces play a crucial role in mathematical analysis and topology. In recent years, fuzzy metric spaces have been widely studied due to their applications in various fields. Alexander Sostak introduced the concept of revised fuzzy metric spaces, which extends traditional fuzzy metric spaces by incorporating revised fuzzy sets. In this paper, we introduce a further generalization called revised fuzzy $\mathscr{k}$ −metric spaces, which allows for the involvement of multiple parameters ($\mathscr{k}$), thereby enhancing the flexibility and applicability of the framework.

Objectives

The primary aim of this study is to define and explore the fundamental properties of revised fuzzy $\mathscr{k}$ −metric spaces. We investigate their topological structure and establish significant properties such as first countability and the Hausdorff condition. Additionally, we extend existing results in the literature by proving a fixed-point theorem in this new setting.

Method

We begin by formally defining a revised fuzzy k-metric space and developing its basic properties. Using topological arguments, we demonstrate that the topology induced by a revised fuzzy $\mathscr{k}$ −metric is first countable and that the space satisfies the Hausdorff condition. Finally, we extend the fixed-point theorem established by Muraliraj and Thangathamizh into the context of revised fuzzy $\mathscr{k}$ −metric spaces, using analytical and set-theoretic techniques.

Result

Our findings confirm that revised fuzzy k-metric spaces preserve essential topological characteristics such as first countability and Hausdorff separation. Furthermore, the fixed-point theorem proved in this study generalizes previous results and demonstrates the broader applicability of revised fuzzy $\mathscr{k}$ −metric spaces in fixed-point theory.

Conclusion

This study introduces revised fuzzy $\mathscr{k}$ −metric spaces as a generalization of revised fuzzy metric spaces, providing a more comprehensive framework for analyzing metric structures with multiple parameters. The established topological properties and fixed-point theorem contribute to the further development of fuzzy metric theory, opening new avenues for future research in mathematical analysis and its applications.

Keywords: revised fuzzy $\mathscr{k}$ −metric spaces, revised fuzzy 2-metric spaces, Hausdorff spaces, Contractions, Fixed points

1. Introduction

The idea to revise the concept of a fuzzy metric by means of t-conorms instead of t-norms was first expressed in [6]. In this paper, we have developed further this approach calling fuzzy metrics defined on the base of a t-conorm by t-conorm based fuzzy metrics or by CB-fuzzy metrics for short. The three main issues considered in the paper are the following. Construction of revised fuzzy $\mathscr{k}$ −metrics from ordinary metrics (Section 4), topological structure induced by CB-fuzzy metrics (Section 5), and interrelations between CB-fuzzy metrics and modular metrics (Section 6). Additionally, we make some comments concerning the intuitionistic counterpart of a CB-fuzzy metric (Section 7).

Concerning the construction of CB-fuzzy metrics from ordinary metrics we mainly restrict the case of fuzzy metrics based on Archimedean t-conorms. Just in this situation we can effectively use the tools provided by additive generators of t-conorms. By using additive generators for such CB-fuzzy metrics, we presented a scheme for construction of CB-fuzzy metrics from ordinary metrics and illustrated it with examples for some concrete t-conorms. We guess that the presented construction will provide a scheme allowing to extend some results from the theory of metric spaces to the corresponding results for CB-fuzzy metric spaces. Specifically, this can concern the results in the theory of fixed points.

The motivation in this paper for inventing a new space, which is more general than a revised fuzzy metric space due to Alexander Sostack (2018), is given in this paragraph. In a revised fuzzy metric space, the fuzzy distance of two points is measured by the degree of the nearness of points with respect to a parameter $t \in (0, \infty)$. For instance, we can think of “t” as the time required to travel between two points x and y in a space. There is an interesting situation of the degree of nearness when we measure this degree with respect to different (more than one) parameters. For instance, suppose that we move from India, represented by x, to Serbia, represented by y, by a plane and measure the degree of the nearness of x and y with respect to time and fuel consumption with planes of different fuel efficiency. Then obviously, this degree will be different for distinct planes even for the same time t, as well as for the same plane but for different time intervals.

The mentioned situation in the previous paragraph brings the inspiration for introducing the notion of revised fuzzy $\mathscr{k}$ −metric spaces, where $\mathscr{k} \in \{1, 2, 3, \dots\}$, which is an extension and generalization of the concept of fuzzy metric spaces due to Alexander Sostack (2018). In a revised fuzzy $\mathscr{k}$ −metric spaces, the fuzzy distance of two points is measured by the degree of nearness with respect to $\mathscr{k}$ −parameter(s). Furthermore, fixed point results for contractive mappings in revised fuzzy $\mathscr{k}$ −metric spaces are proved. These results generalize the fixed-point results of Muraliraj and Thangathamizh (2022) into revised fuzzy $\mathscr{k}$ −metric spaces.

2. Preliminaries

Definition 1[22] (Schweizer and Sklar (1960) A binary operation $\oplus: [0,1]^2 \to [0,1]$ is called a triangular conorm (briefly, t-conorm) if the following conditions are satisfied for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s} \in [0,1]$:

1. $\oplus(\mathfrak{p}, \mathfrak{q}) = \oplus(\mathfrak{q}, \mathfrak{p})$;

2. if $\mathfrak{p} \leq \mathfrak{r}$ and $\mathfrak{q} \leq \mathfrak{s}$, then $\oplus(\mathfrak{p}, \mathfrak{q}) \leq \oplus(\mathfrak{r}, \mathfrak{s})$;

3. $\oplus(\oplus(\mathfrak{p}, \mathfrak{q}), \mathfrak{r}) = \oplus(\mathfrak{p}, \oplus(\mathfrak{q}, \mathfrak{r}))$;

4. $\oplus(\mathfrak{p}, 0) = \mathfrak{p}$. If $\oplus$ is continuous, it is called a continuous t-conorm.

For each t-conorm $\oplus: [0,1]^2 \to [0,1]$ and $\mathfrak{p}, \mathfrak{q} \in [0,1]$, instead of $\oplus(\mathfrak{p}, \mathfrak{q})$ we will use the infix notation $\mathfrak{p} \oplus \mathfrak{q}$. Three typical examples of continuous t-norms are a product t-conorm $\oplus_1$, a minimum t-conorm $\oplus_2$ and a Lukasiewicz t-conorm $\oplus_3$, which are defined for each $\mathfrak{p}, \mathfrak{q} \in [0,1]$ by

$\mathfrak{p} \oplus_1 \mathfrak{q} = max\{\mathfrak{p}, \mathfrak{q}\}$,

$\mathfrak{p} \oplus_2 \mathfrak{q} = \mathfrak{p} + \mathfrak{q} - \mathfrak{p}\mathfrak{q}$,

$\mathfrak{p} \oplus_3 \mathfrak{q} = min\{a + b, 1\}$.

Remark 2 For each t-norm $\oplus: [0,1]^2 \to [0,1]$, the following assertions hold:

1. for each $\mathfrak{p}, \mathfrak{q} \in [0,1]$ with $\mathfrak{p} > q$, there is $\mathfrak{r} \in (0,1)$ such that $\mathfrak{p} \oplus \mathfrak{r} \geq \mathfrak{q}$;

2. for each $\mathfrak{s} \in (0,1)$, there is $\mathfrak{t} \in (0,1)$ such that $\mathfrak{t} \oplus \mathfrak{t} \geq \mathfrak{s}$.

Definition 3[7] An ordered triple $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is called a revised fuzzy metric space if $\mathfrak{M}$ is an arbitrary set, $\oplus$ a continuous t-conorm, $\mathfrak{M}$ is a revised fuzzy set on $\mathfrak{M}^2 \times (0, +\infty)$, and thefollowing conditions are satisfied for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{M}$, $\mathfrak{a}, \mathfrak{b} > 0$

(RF-1) $\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, a) < 1$;

(RF-2) $\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, a) = 0$ if and only if $\mathfrak{p} = \mathfrak{q}$;

(RF-3) $\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, a) = \mathfrak{N}(\mathfrak{q}, \mathfrak{p}, a)$;

(RF-4) $\mathfrak{N}(\mathfrak{p}, r, a) \leq \mathfrak{N}(\mathfrak{p}, \mathfrak{q}, a) \oplus \mathfrak{N}(\mathfrak{q}, r, a)$;

(RF-5) $\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, -): (0, +\infty)^{k} \to [0,1]$is a right continuous mapping.

Example 4[7] (Induced revised fuzzy metric)

Let (X, d) be a metric space and $\oplus$ be a product t-conorm. Define a revised fuzzy set $\mathfrak{N}$ on $\mathfrak{M}^2 \times (0, +\infty)$ by

$$\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}) = \frac{\mathbb{d}(\mathfrak{p}, \mathfrak{q})}{1 + \mathbb{d}(\mathfrak{p}, \mathfrak{q})}$$

for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{M}$ and $\mathfrak{a} > 0$, where $k, m, n > 0$. Then, $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is a revised fuzzy metric space called the induced revised fuzzy metric.

In the above example, note that

$\lim_{t\to+\infty} \mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}) = 0$ for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{M}$. (1)

As $(\mathfrak{M}, \mathfrak{N}, \oplus)$ represents the degree of the nearness of points $\mathfrak{p}$ and $\mathfrak{q}$ with respect to the parameter $\mathfrak{a}$ and it is a nondecreasing function of $\mathfrak{a}$ for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{M}$; therefore, condition (1) is the most natural condition for the degree of the nearness to be perfect (that is, unity). Notice that this is a specific condition and may not hold in some fuzzy metric spaces, for instance, in stationary revised fuzzy metric spaces. This brings to the following definition:

Definition 5

A revised fuzzy metric space $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is called a natural fuzzy metric space if and only if

$\lim_{t\to+\infty} \mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}) = 0$ for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{M}$.

Definition 6

A 3-tuple $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is said to be a revised fuzzy 2-metric space if $\mathfrak{M}$ is an arbitrary nonempty set, $\oplus$is a continuous t-conorm, and $\mathfrak{M}$ is a revised fuzzy set on$\mathfrak{M}^3 \times (0, +\infty)$ satisfying the following conditions: For all $(\mathfrak{p}, \mathfrak{q}, r \in \mathfrak{M}, a, a_1, a_2, a_3 \in (0, +\infty))$

(RF2M.1) given distinct elements $\mathfrak{p}, \mathfrak{q} \in \mathfrak{M}$ there is an element$\mathfrak{r} \in \mathfrak{M}$ such that $\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, r, a) < 1$ for each $\mathfrak{a} > 0$;

(RF2M.2) $\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, a) = 0$ if at least two of $\mathfrak{p}, \mathfrak{q}, r$ are equal.

(RF2M.3) $\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, a) = \mathfrak{N}(\mathfrak{p}, \mathfrak{r}, \mathfrak{q}, a) = \mathfrak{N}(\mathfrak{r}, \mathfrak{p}, \mathfrak{q}, a)$ for all $\mathfrak{p}, \mathfrak{q}, r \in \mathfrak{M}$ and all $\mathfrak{a} > 0$;

(RF2M.4) $\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, a_1 + a_2 + a_3) \leq \mathfrak{N}(\mathfrak{p}, \mathfrak{r}, \mathfrak{s}, a_1) \oplus \mathfrak{N}(\mathfrak{p}, \mathfrak{s}, \mathfrak{r}, a_2) \oplus \mathfrak{N}(\mathfrak{s}, \mathfrak{q}, \mathfrak{r}, a_3)$;

(RF2M.5)$\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, -) : (0, \infty) \to (0,1]$ is a continuous function.

The pair $(\mathfrak{N}, \oplus)$ (or only $\mathfrak{N}$) is called a revised fuzzy 2-metric on $\mathfrak{M}$.

3. Revised fuzzy k −metric spaces

In this section, we introduce the idea of revised fuzzy k −metric spaces and investigate the properties of such spaces. We begin with the following definition

Definition 6

Let $\mathfrak{M}$ be a nonempty set, $\oplus$a continuous t-conorm, ka positive integer and $\mathfrak{N}$be a revised fuzzy set on $\mathfrak{M}^2 \times (0, +\infty)^k$. An ordered triple $(\mathfrak{M}, \mathfrak{N}, \oplus)$is called a revised fuzzy$k$ −metric space if the following conditions are satisfied for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \mathfrak{M}$, $\mathfrak{a}, \mathfrak{b} > 0$ and$a_1, a_2, \ldots, a_k > 0$:

(RF-k1) $\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, a_1, a_2, \ldots, a_k) < 1$;

(RF-k2) $\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, a_1, a_2, \ldots, a_k) = 0$ if and only if $\mathfrak{p} = \mathfrak{q}$;

(RF-k3) $\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, a_1, a_2, \ldots, a_k)$is symmetric.

(RF-k4) for any$j \in \{1, 2, 3, \ldots, k\}$, we have

$$\mathfrak{N}(\mathfrak{p}, r, a_1, a_2, \ldots, a_{j-1}, a + b, a_{j+1}, \ldots, a_k) \leq \begin{Bmatrix} \mathfrak{N}(\mathfrak{q}, r, a_1, a_2, \ldots, a_{j-1}, a, a_{j+1}, \ldots, a_{j-1}, a_k) \\ \oplus\, \mathfrak{N}(\mathfrak{q}, r, a_1, a_2, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_{j-1}, a_k) \end{Bmatrix}$$

(RF-k5) $\mathfrak{N}(\mathfrak{p},\mathfrak{q},-)\colon (0,+\infty)^{\mathscr{k}} \to [0,1]$is a right continuous mapping.

Remark 7 For $\mathscr{k}=1$, the revised fuzzy$\mathscr{k}$ −metric space reduces into the revised fuzzy metric space in the sense of Alexander Sostak.

Example 1

Let $(\mathfrak{M},\mathbb{d})$be a metric space, $\oplus$ the product (maximum) t-conorm, $\mathfrak{u}>0$ and $\mathscr{k}$ be a positive integer. Define a revised fuzzy set $\mathfrak{N}$on $\mathfrak{M}^2 \times (0,\infty)^{\mathscr{k}}$by

$$\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1,a_2,\ldots,a_{\mathscr{k}}) = \frac{\mathbb{d}(\mathfrak{p},\mathfrak{q})}{\mathfrak{u}(a_1,a_2,\ldots,a_{\mathscr{k}}) + \mathbb{d}(\mathfrak{p},\mathfrak{q})}$$

for all all $\mathfrak{p},\mathfrak{q} \in \mathfrak{M}$, $\mathfrak{a},\mathfrak{b}>0$ and $a_1,a_2,\ldots,a_{\mathscr{k}}>0$. Then, $(\mathfrak{M},\mathfrak{N},\oplus)$ is a revised fuzzy $\mathscr{k}$ −metric space.

From the application point of view, one should define the revised fuzzy $\mathscr{k}$ −metric with care to the physical nature of quantities. For instance, if one considers the degree of the nearness of two points $\mathfrak{p}$ *and* $\mathfrak{q}$ in a space with respect to time and fuel consumed in moving from $\mathfrak{p}$ *to* $\mathfrak{q}$, one cannot use the formulae for the degree of the nearness as given in the above examples due to the different dimensions of these quantities. In the following example, one such case is presented.

Example 2

Let $(\mathfrak{M},\mathbb{d})$ be a metric space, $\oplus$ the product (maximum) t-conorm, $\mathfrak{u}>0$ and $\mathscr{k}$ be a positive integer. Define a revised fuzzy set $\mathfrak{N}$ on $\mathfrak{M}^2 \times (0,+\infty)^{\mathscr{k}}$ by

$$\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1,a_2,\ldots,a_{\mathscr{k}}) = 1-\mathfrak{u}\left[\mathfrak{u}+\left(\sum_{j=1}^{\mathscr{k}}\frac{1}{a_j}\right)\mathbb{d}(\mathfrak{p},\mathfrak{q})\right]^{-1}$$

for all all $\mathfrak{p},\mathfrak{q} \in \mathfrak{M}$, $\mathfrak{a},\mathfrak{b}>0$ and $a_1,a_2,\ldots,a_{\mathscr{k}}>0$. Then, $(\mathfrak{M},\mathfrak{N},\oplus)$is a revised fuzzy $\mathscr{k}$ −metric space.

Example 3

Let $\mathfrak{M}=\mathcal{R}^{\mathscr{k}}$, where $\mathscr{k}$ is a positive integer, $\oplus$ the product t-conorm. Define a revised fuzzy set $\mathfrak{N}$ on $\mathfrak{M}^2 \times (0,+\infty)^{\mathscr{k}}$ by

$$\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1,a_2,\ldots,a_{\mathscr{k}}) = 1-\mathfrak{u}\left[\mathfrak{u}+\left(\sum_{j=1}^{\mathscr{k}}\frac{|\mathfrak{q}_j-\mathfrak{p}_j|}{a_j}\right)\right]^{-1}$$

for all $\mathfrak{p},\mathfrak{q} \in \mathfrak{M}$ and $a_1,a_2,\ldots,a_{\mathscr{k}}>0$. Then, $(\mathfrak{M},\mathfrak{N},\oplus)$ is a revised fuzzy $\mathscr{k}$ −metric space.

In the present paper, we restrict ourselves to only mathematical properties of revised fuzzy $\mathscr{k}$ −metric space.

Definition 12

A revised fuzzy $\mathscr{k}$ −metric space $(\mathfrak{M},\mathfrak{N},\oplus)$ is called $\mathcal{T}$ −natural revised fuzzy $\mathscr{k}$ −metric space if there exists $\mathcal{T} \in \{1,2,\ldots,\mathscr{k}\}$ such that

$\lim_{a_{k}\to+\infty} \mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1,a_2,\dots,a_k)=0$, $\mathfrak{p},\mathfrak{q}\in\mathfrak{M}$ and $a_1,a_2,\dots,a_k>0$.

For the rest of this paper, for a given revised fuzzy k −metric space$(\mathfrak{M},\mathfrak{N},\oplus)$, $\mathfrak{p},\mathfrak{q}\in\mathfrak{M}$ and $a_1,a_2,\dots,a_k>0$, for simplicity, we write $\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1^k)$ instead $\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1,a_2,\dots,a_k)$.

Next, we discuss some properties revised fuzzy k −metric space and establish the topology of such spaces.

Proposition 13

Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a revised fuzzy k −metric space, $a,a_1,a_2,\dots,a_k$. Suppose that $a_{\mathcal{T}}<a$ for some $\mathcal{T}\in\{1,2,\dots,k\}$. Then, $\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1^k)\geq\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1,a_2,..,a_{\mathcal{T}-1},a_{\mathcal{T}},a_{\mathcal{T}+1},..,a_k)$ for all $\mathfrak{p},\mathfrak{q}\in\mathfrak{M}$.

Remark 14

In a revised fuzzy k −metric space$(\mathfrak{M},\mathfrak{N},\oplus)$, if $\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1^k)<\varepsilon$, where $\mathfrak{p},\mathfrak{q}\in\mathfrak{M}$, $a_1,a_2,\dots,a_k>0$ and $0<\varepsilon<1$, then for each $\mathcal{T}\in\{1,2,\dots,k\}$, we can find $a\in(0,a_k)$ such that $\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1,a_2,..,a_{\mathcal{T}-1},a_{\mathcal{T}},a_{\mathcal{T}+1},..,a_k)<\varepsilon$.

Definition 15

Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a revised fuzzy k −metric space. An open ball with center $\mathfrak{p}\in\mathfrak{M}$ and radius $\varepsilon\in(0,1)$ with respect to parameters$a_1,a_2,\dots,a_k>0$, denoted by $\mathfrak{B}(\mathfrak{p},\varepsilon;a_1,a_2,\dots,a_k)$, is defined by

$\mathfrak{B}(\mathfrak{p},\varepsilon;a_1,a_2,\dots,a_k)=\{\mathfrak{q}\in\mathfrak{M}:\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1^k)<\varepsilon\}$.

Definition 16

Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a revised fuzzy k −metric space. A subset $\mathcal{X}$ of $\mathfrak{M}$ is called an open set if and only if there is an open ball $\mathfrak{B}$ such that $\mathfrak{B}\subseteq\mathcal{X}$. A subset $\mathcal{Y}$ of $\mathfrak{M}$ is called a closed set if and only if its complement is an open set.

Theorem 17

Every open ball in a revised fuzzy k −metric space is an open set.

Proof

Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a revised fuzzy k −metric space, $\mathfrak{p}\in\mathfrak{M}$, $a_1,a_2,\dots,a_k>0$ and $\varepsilon\in(0,1)$.

Assume that $\mathfrak{q}\in\mathfrak{B}(\mathfrak{p},\varepsilon;a_1,a_2,\dots,a_k)$. Then, we have $\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1^k)<\varepsilon$.

Therefore, we can find $\mathcal{T}\in\{1,2,\dots,k\}$ and $a\in(0,a_k)$such that

$\varepsilon_0:=\mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1,a_2,..,a_{\mathcal{T}-1},a_{\mathcal{T}},a_{\mathcal{T}+1},..,a_k)<\varepsilon$.

Then, we can find $\delta\in(0,1)$ such that $\varepsilon_0<\delta<\varepsilon$.

By Remark 2, there is $\varepsilon_1\in(0,1)$ such that $\varepsilon_0\oplus\varepsilon_1\leq\delta$.

Now, we will claim that $\mathfrak{B}(\mathfrak{p},\varepsilon;a_1,a_2,\dots,a_k)\subseteq\mathfrak{B}(\mathfrak{q},1-\varepsilon_1;a_1,a_2,..,a_{\mathcal{T}-1},a_{\mathcal{T}},a_{\mathcal{T}+1},..,a_k)$.

Assume that $\mathfrak{r}\in\mathfrak{B}(\mathfrak{q},1-\varepsilon_1;a_1,a_2,..,a_{\mathcal{T}-1},a_{\mathcal{T}},a_{\mathcal{T}+1},..,a_k)$.

Then, $\mathfrak{N}(\mathfrak{q},\mathfrak{r},a_1,a_2,..,a_{\mathcal{T}-1},a_{\mathcal{T}},a_{\mathcal{T}+1},..,a_k)<1-\varepsilon_1$.

$$\mathfrak{N}(\mathfrak{p},\mathfrak{r},a_1^{\Bbbk}) \leq \mathfrak{N}(\mathfrak{p},\mathfrak{q},a_1,a_2,..,a_{\mathcal{T}-1},a_{\mathcal{T}},a_{\mathcal{T}+1},..,a_{\Bbbk}) \oplus \mathfrak{N}(\mathfrak{q},\mathfrak{r},a_1,a_2,..,a_{\mathcal{T}-1},a_{\mathcal{T}},a_{\mathcal{T}+1},..,a_{\Bbbk})$$
$$\leq \varepsilon_0 \oplus \varepsilon_1 \leq \delta < \varepsilon,$$

which proves the result.

From the above theorem, we can directly get the following result:

Theorem 18

Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a revised fuzzy $\Bbbk$ −metric space and

$\tau = \{ \mathcal{X} \subseteq \mathfrak{M} : a \in \mathfrak{M}$ if and only if there exist $a_1, a_2, \ldots, a_{\Bbbk} > 0$ and $\varepsilon \in (0,1)$ such that $\mathfrak{B}(\mathfrak{p},\varepsilon; a_1, a_2, \ldots, a_{\Bbbk}) \subseteq \mathcal{X}\}$.

Then, τ is a topology on $\mathfrak{M}$.

Remark 19

Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a revised fuzzy $\Bbbk$ −metric space and $a \in \mathfrak{M}$. Since

$$\mathfrak{B}_{\mathfrak{p}} = \left\{\mathfrak{B}\left(\mathfrak{p},\frac{1}{n}; a_1, a_2, \ldots, a_{\Bbbk}\right) : n \in \mathcal{N}\right\}$$

Where $a_1 = a_2 = \ldots = a_{\Bbbk} = \frac{1}{n}$, is a local base at a point a, the topology τ given in Theorem 18 is first countable.

Theorem 20

Every revised fuzzy $\Bbbk$ −metric space is Hausdorff.

Definition 21

Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a revised fuzzy $\Bbbk$ −metric space. A sequence $\{\mathfrak{p}_n\}$ in $\mathfrak{M}$ is said to be convergent and converges to $\mathfrak{p} \in \mathfrak{M}$ if and only if for every real $\epsilon \in (0,1)$, there exists $n_0 \in \mathcal{N}$ such that $\mathfrak{N}(\mathfrak{p}_n,\mathfrak{p},a_1^{\Bbbk}) < \epsilon$ for all $n \in n_0$ and $a_1, a_2, \ldots, a_{\Bbbk} > 0$.

The proof of the following lemma is straightforward, so we will omit the proof.

Lemma 22

Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a revised fuzzy $\Bbbk$ −metric space. A sequence $\{\mathfrak{p}_n\}$ in $\mathfrak{M}$ converges to $\mathfrak{p} \in \mathfrak{M}$ if and only if $\lim_{n\to+\infty} \mathfrak{N}(\mathfrak{p}_n,\mathfrak{p},a_1^{\Bbbk}) = 0$ for all $a_1, a_2, \ldots, a_{\Bbbk} > 0$.

Definition 23

Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a revised fuzzy $\Bbbk$ −metric space and $\{\mathfrak{p}_n\}$ be a sequence in $\mathfrak{M}$.

1. $\{\mathfrak{p}_n\}$ is called an $\mathfrak{N}$ −Cauchy sequence if for every $\epsilon \in (0,1)$, there exists $n_0 \in \mathcal{N}$ such that $\mathfrak{N}(\mathfrak{p}_n,\mathfrak{p}_m,a_1^{\Bbbk}) < \epsilon$ for all $n,m > n_0$ and $a_1, a_2, \ldots, a_{\Bbbk} > 0$.

2. $\{\mathfrak{p}_n\}$ is called a $\mathbb{G}$ −Cauchy sequence if $\lim_{n\to+\infty} \mathfrak{N}(\mathfrak{p}_n,\mathfrak{p}_{n+x},a_1^{\Bbbk}) = 0$ for all $a_1, a_2, \ldots, a_{\Bbbk} > 0$ and $x > 0$. Note that the above definitions of Cauchy sequences are different (for the case $\Bbbk = 1$.

Definition 24 Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a revised fuzzy $\Bbbk$ −metric space.

1. $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is said to be $\mathfrak{N}$ −complete if every $\mathfrak{N}$ −Cauchy sequence in $\mathfrak{M}$ converges to some $\mathfrak{p} \in \mathfrak{M}$.

2. $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is said to be $\mathbb{G}$ −complete if every G-Cauchy sequence in $\mathfrak{M}$ converges to some $\mathfrak{p} \in \mathfrak{M}$.

4. Fixed point theorems

In this section, we prove many fixed-point results in revised fuzzy k −metric space. For simplicity, for a given revised fuzzy k −metric space$(\mathfrak{M}, \mathfrak{N}, \oplus)$, $\mathcal{T} \in \{1, 2, \dots, k\}$, $b > 0$, $\mathfrak{p}, \mathfrak{q} \in \mathfrak{M}$ and $a_1, a_2, \dots, a_k > 0$, we write $\mathfrak{N}_{\mathcal{T}}^{b}(\mathfrak{p}, \mathfrak{q}, a_1^k)$ instead $\mathfrak{N}\left(\mathfrak{p}, \mathfrak{q}, a_1, a_2, .., a_{\mathcal{T}-1}, \frac{a_{\mathcal{T}}}{b}, a_{\mathcal{T}+1}, .., a_k\right)$.

Theorem 26

Let $(\mathfrak{M}, \mathfrak{N}, \oplus)$ be a $\mathbb{G}$ −complete revised fuzzy k −metric space and $\mathfrak{T}: \mathfrak{M} \to \mathfrak{M}$ be a mapping satisfying the following condition:

$$\mathfrak{N}_{\mathcal{T}}^{\frac{1}{\lambda}}(\mathfrak{T}\mathfrak{p}, \mathfrak{T}\mathfrak{q}, a_1^k) \leq \mathfrak{N}(\mathfrak{p}, \mathfrak{q}, a_1^k) \tag{2}$$

for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{M}$ and $a_1, a_2, \dots, a_k > 0$, where $\mathcal{T} \in \{1, 2, \dots, k\}$ and $\lambda \in (0,1)$ is a constant. Suppose that $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is an $\mathcal{T}$ −natural revised fuzzy k −metric space. Then, $\mathfrak{T}$ has a unique fixed point.

Proof

First, we will show that if a fixed point of $\mathfrak{T}$ exists, then it is unique. Suppose that $\mathfrak{x}$ and $\mathfrak{y}$ are fixed points of $\mathfrak{T}$. By (2), we have

$$\mathfrak{N}(\mathfrak{x}, \mathfrak{y}, a_1^k) = \mathfrak{N}(\mathfrak{T}\mathfrak{x}, \mathfrak{T}\mathfrak{y}, a_1^k) \leq \mathfrak{N}\left(\mathfrak{x}, \mathfrak{y}, a_1, a_2, .., a_{\mathcal{T}-1}, \frac{a_{\mathcal{T}}}{\lambda}, a_{\mathcal{T}+1}, .., a_k\right) = \mathfrak{N}_{\mathcal{T}}^{\lambda}(\mathfrak{x}, \mathfrak{y}, a_1^k)$$

By repeating this process, we obtain

$$\mathfrak{N}(\mathfrak{x}, \mathfrak{y}, a_1^k) \leq \mathfrak{N}_{\mathcal{T}}^{\lambda^n}(\mathfrak{x}, \mathfrak{y}, a_1^k) \tag{3}$$

for all $n \in \mathcal{N}$. Note that, if $\{\mathfrak{p}_n\}$ be any sequence such that $\mathfrak{p}_n > 0$ and $\lim_{n\to\infty} \mathfrak{p}_n = 0$, then since $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is $\mathcal{T}$ −natural, we have $\lim_{n\to+\infty} \mathfrak{N}_{\mathcal{T}}^{b_n}(\mathfrak{p}, \mathfrak{q}, a_1^k) = 0$

for all $a_1, a_2, \dots, a_k > 0$.

Using this fact in (3), we obtain $\mathfrak{N}(\mathfrak{x}, \mathfrak{y}, a_1^k) = 0$ for all $a_1, a_2, \dots, a_k > 0$, that is, $\mathfrak{x} = \mathfrak{y}$. Therefore, the fixed point of $\mathfrak{T}$ is unique.

For the existence of a fixed point of $\mathfrak{T}$, we choose $\mathfrak{p}_0 \in \mathfrak{M}$ and define an iterative sequence $\{\mathfrak{p}_n\}$ by $\mathfrak{p}_n = \mathfrak{T}\mathfrak{p}_{n-1}$for all $n \in \mathcal{N}$. If $\mathfrak{p}_n = \mathfrak{p}_{n-1}$ for some $n \in \mathcal{N}$, then $\mathfrak{p}_n$ is the unique fixed point of $\mathfrak{T}$. Therefore, we may assume that $\mathfrak{p}_n \neq \mathfrak{p}_{n-1}$ for all $n \in \mathcal{N}$. For any $n \in \mathcal{N}$ and $a_1, a_2, \dots, a_k > 0$, we have

$$\mathfrak{N}(\mathfrak{p}_n, \mathfrak{p}_{n+1}, a_1^k) = \mathfrak{N}(\mathfrak{T}\mathfrak{p}_{n-1}, \mathfrak{T}\mathfrak{p}_n, a_1^k) \leq \mathfrak{N}\left(\mathfrak{x}, \mathfrak{y}, a_1, a_2, .., a_{\mathcal{T}-1}, \frac{a_{\mathcal{T}}}{\lambda}, a_{\mathcal{T}+1}, .., a_k\right)$$

$$= \mathfrak{N}_{\mathcal{T}}^{\lambda}(\mathfrak{p}_{n-1}, \mathfrak{p}_n, a_1^k)$$

By repeating this process, we obtain

$$\mathfrak{N}\left(\mathfrak{p}_n, \mathfrak{p}_{n+1}, a_1^{k}\right) \leq \mathfrak{N}_{\mathcal{T}}^{\lambda^n}\left(\mathfrak{p}_0, \mathfrak{p}_1, a_1^{k}\right) \tag{4}$$

for all $n \in \mathcal{N}$. For each $n \in \mathcal{N}$ and $a_1, a_2, \ldots, a_k > 0$ and $x > 0$, we have

$$\mathfrak{N}\left(\mathfrak{p}_n, \mathfrak{p}_{n+x}, a_1^{k}\right) \leq \begin{Bmatrix} \mathfrak{N}\left(\mathfrak{p}_n, \mathfrak{p}_{n+1}, a_1, a_2, .., a_{\mathcal{T}-1}, \frac{a_{\mathcal{T}}}{2}, a_{\mathcal{T}+1}, .., a_k\right) \\ \oplus \mathfrak{N}\left(\mathfrak{p}_{n+1}, \mathfrak{p}_{n+x}, a_1, a_2, .., a_{\mathcal{T}-1}, \frac{a_{\mathcal{T}}}{2}, a_{\mathcal{T}+1}, .., a_k\right) \end{Bmatrix}$$

$$\leq \begin{Bmatrix} \mathfrak{N}_{\mathcal{T}}^{2}\left(\mathfrak{p}_n, \mathfrak{p}_{n+1}, a_1^{k}\right) \oplus \mathfrak{N}\left(\mathfrak{p}_{n+1}, \mathfrak{p}_{n+2}, a_1, a_2, .., a_{\mathcal{T}-1}, \frac{a_{\mathcal{T}}}{2^2}, a_{\mathcal{T}+1}, .., a_k\right) \\ \oplus \mathfrak{N}\left(\mathfrak{p}_{n+2}, \mathfrak{p}_{n+x}, a_1, a_2, .., a_{\mathcal{T}-1}, \frac{a_{\mathcal{T}}}{2^2}, a_{\mathcal{T}+1}, .., a_k\right) \end{Bmatrix}$$

$$\leq \begin{Bmatrix} \mathfrak{N}_{\mathcal{T}}^{2}\left(\mathfrak{p}_n, \mathfrak{p}_{n+1}, a_1^{k}\right) \oplus \mathfrak{N}_{\mathcal{T}}^{2^2}\left(\mathfrak{p}_{n+1}, \mathfrak{p}_{n+2}, a_1^{k}\right) \oplus \ldots \oplus \\ \mathfrak{N}_{\mathcal{T}}^{2x-1}\left(\mathfrak{p}_{n+x-2}, \mathfrak{p}_{n+x+1}, a_1^{k}\right) \oplus \mathfrak{N}^{2}\left(\mathfrak{p}_{n+x-1}, \mathfrak{p}_{n+x}, a_1^{k}\right) \end{Bmatrix}$$

By using (4), we obtain

$$\mathfrak{N}\left(\mathfrak{p}_n, \mathfrak{p}_{n+x}, a_1^{k}\right) \leq \mathfrak{N}_{\mathcal{T}}^{2\lambda^n}\left(\mathfrak{p}_0, \mathfrak{p}_1, a_1^{k}\right) \oplus \mathfrak{N}_{\mathcal{T}}^{2^2\lambda^{n+1}}\left(\mathfrak{p}_0, \mathfrak{p}_1, a_1^{k}\right) \oplus \ldots \oplus \mathfrak{N}_{\mathcal{T}}^{2^{x-1}\lambda^{n+x-1}}\left(\mathfrak{p}_0, \mathfrak{p}_1, a_1^{k}\right)$$

Since $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is $\mathcal{T}$ −natural, it follows from the above inequality that $\lim_{n\to+\infty} \mathfrak{N}\left(\mathfrak{p}_n, \mathfrak{p}_{n+x}, a_1^{k}\right) = 0$.

Therefore, $\{\mathfrak{p}_n\}$ is a $\mathbb{G}$ −Cauchy sequence. By the $\mathbb{G}$ −completeness of $(\mathfrak{M}, \mathfrak{N}, \oplus)$, there exists $\mathfrak{x} \in \mathfrak{M}$ such that

$$\lim_{n\to+\infty} \mathfrak{N}\left(\mathfrak{p}_n, \mathfrak{x}, a_1^{k}\right) = 0, \text{ for all } a_1, a_2, \ldots, a_k > 0.$$

We will show that $\mathfrak{x}$ is a fixed point of $\mathfrak{T}$. For each $a_1, a_2, \ldots, a_k > 0$, we have

$$\mathfrak{N}\left(\mathfrak{x}, \mathfrak{T}\mathfrak{x}, a_1^{k}\right) \leq \mathfrak{N}_{\mathcal{T}}^{2}\left(\mathfrak{x}, \mathfrak{p}_n, a_1^{k}\right) \oplus \mathfrak{N}_{\mathcal{T}}^{2}\left(\mathfrak{p}_n, \mathfrak{T}\mathfrak{x}, a_1^{k}\right)$$

$$= \mathfrak{N}_{\mathcal{T}}^{2}\left(\mathfrak{x}, \mathfrak{p}_n, a_1^{k}\right) \oplus \mathfrak{N}_{\mathcal{T}}^{2}\left(\mathfrak{T}\mathfrak{p}_{n-1}, \mathfrak{T}\mathfrak{x}, a_1^{k}\right) \leq \mathfrak{N}_{\mathcal{T}}^{2}\left(\mathfrak{x}, \mathfrak{p}_n, a_1^{k}\right) \oplus \mathfrak{N}_{\mathcal{T}}^{2\lambda}\left(\mathfrak{p}_{n-1}, \mathfrak{x}, a_1^{k}\right)$$

By using (5) in the above inequality, we obtain $\mathfrak{N}\left(\mathfrak{x}, \mathfrak{T}\mathfrak{x}, a_1^{k}\right) = 0$ for all $a_1, a_2, \ldots, a_k > 0$, that is, $\mathfrak{x} = \mathfrak{T}\mathfrak{x}$. Thus, $\mathfrak{x}$ is the unique fixed point of $\mathfrak{T}$.

For $k = 1$, the above theorem reduces to the following result of Grabiec (1988).

Corollary 27

Let $(\mathfrak{M}, \mathfrak{N}, \oplus)$ be a $\mathbb{G}$ −complete revised fuzzy metric space such that

$$\lim_{t\to+\infty} \mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}) = 0 \text{ for all } \mathfrak{p}, \mathfrak{q} \in \mathfrak{M} \tag{6}$$

and $\mathfrak{T}: \mathfrak{M} \to \mathfrak{M}$be a mapping. Suppose that there exists $\lambda \in (0, 1)$ such that

$$\mathfrak{N}(\mathfrak{T}\mathfrak{p}, \mathfrak{T}\mathfrak{q}, \mathfrak{a}) \leq \mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}) \tag{7}$$

for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{M}$. Then, $\mathfrak{T}$ has a unique fixed point.

Remark 28

Let $(\mathfrak{M}, \mathfrak{N}, \oplus)$ be a revised fuzzy metric space and $\mathfrak{T}: \mathfrak{M} \to \mathfrak{M}$ be a mapping. The contractive condition (7) tells that the mapping $\mathfrak{T}$ contract the space with respect to the parameter t in the sense that the degree of the nearness of images of any two points under $\mathfrak{T}$ is not less than the degree of the nearness of corresponding points (obviously in case of stationery revised fuzzy metric spaces (see Gregori and Romaguera 2004) it is not applicable).

In Theorem 26, the mapping contracts the space with respect to only parameter $a_{\mathcal{T}}$ for some

$\mathcal{T} \in \{1, 2, \ldots, \mathcal{k}\}$ and it may not be contractive with respect to other parameters. Similarly, $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is assumed l-natural k-fuzzy metric space for at least one $\mathcal{T} \in \{1, 2, \ldots, \mathcal{k}\}$ only. The following example verifies the above remark.

Example 29

Let $\mathfrak{M} = [0, 1] \times [0, 1]$ and $\oplus$ be the product t-conorm and the revised fuzzy set $\mathfrak{N}$ on

$\mathfrak{M}^2 \times (0, \infty)^2$ be defined by

$$\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}_1, \mathfrak{a}_2) = 1 - \left[1 + \frac{|\mathfrak{q}_1 - \mathfrak{p}_1| + |\mathfrak{q}_2 - \mathfrak{p}_2|}{\mathfrak{a}_1}\right]^{-1}$$

for all $\mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_2)$, $\mathfrak{q} = (\mathfrak{q}_1, \mathfrak{q}_2) \in \mathfrak{M}$ and $\mathfrak{a}_1, \mathfrak{a}_2 > 0$. Then, $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is a $\mathbb{G}$ −complete revised fuzzy 2-metric space ($\mathcal{k} = 2$). Moreover, $\lim_{\mathfrak{a}_1 \to +\infty} \mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}_1, \mathfrak{a}_2) = 0$ for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{M}$, $\mathfrak{a}_2 > 0$,that is, $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is a 1 −natural revised fuzzy 2-metric space. Define a mapping $\mathfrak{T}: \mathfrak{M} \to \mathfrak{M}$ by

$$\mathfrak{N}(\mathfrak{T}\mathfrak{p}, \mathfrak{T}\mathfrak{q}, \lambda\mathfrak{a}_1, \mathfrak{a}_2) = 1 - \left[1 + \frac{|\mathfrak{q}_1 - \mathfrak{p}_1| + |\mathfrak{q}_2 - \mathfrak{p}_2|}{2\lambda\mathfrak{a}_1}\right]^{-1} \le 1 - \left[1 + \frac{|\mathfrak{q}_1 - \mathfrak{p}_1| + |\mathfrak{q}_2 - \mathfrak{p}_2|}{\mathfrak{a}_1}\right]^{-1} = \mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}_1, \mathfrak{a}_2)$$

for $\lambda \in [1/2, 1)$. By Theorem 26, $\mathfrak{T}$ has a unique fixed point. In this case, a point $(0,0) \in \mathfrak{M}$ is a fixed point of $\mathfrak{T}$.

In Theorem 26, corresponding to condition (2), we assume that the space $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is

$\mathcal{T}$ −natural. Notice that, for the existence of a fixed point, the $\mathcal{T}$ −naturalness cannot be replaced by the $\mathfrak{m}$ −naturalness with $\mathfrak{m} \neq \mathcal{T}$. The following example verifies this fact.

Example 30

Let $\mathfrak{M} = [0, 1] \times [0, 1]$ and $\oplus$ be the product t-conorm and the revised fuzzy set $\mathfrak{N}$ on

$\mathfrak{M}^2 \times (0, \infty)^2$ be defined by

$$\mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}_1, \mathfrak{a}_2) = 1 - \left[1 + \frac{|\mathfrak{q}_1 - \mathfrak{p}_1| + |\mathfrak{q}_2 - \mathfrak{p}_2|}{\mathfrak{a}_2}\right]^{-1}$$

for all $\mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_2)$, $\mathfrak{q} = (\mathfrak{q}_1, \mathfrak{q}_2) \in \mathfrak{M}$ and $\mathfrak{a}_1, \mathfrak{a}_2 > 0$. Then, $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is a $\mathbb{G}$ −complete revised fuzzy 2-metric space ($\mathcal{k} = 2$). Moreover,

$\lim_{\mathfrak{a}_1 \to +\infty} \mathfrak{N}(\mathfrak{p}, \mathfrak{q}, \mathfrak{a}_1, \mathfrak{a}_2) = 0$ for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{M}$, $\mathfrak{a}_1 > 0$,

That is, $(\mathfrak{M}, \mathfrak{N}, \oplus)$ is a 2 −natural revised fuzzy 2-metric space. Define a mapping $\mathfrak{T}: \mathfrak{M} \to \mathfrak{M}$ by

$\mathfrak{T}(\mathfrak{p}_1, \mathfrak{p}_2) = (\mathfrak{p}_1, \mathfrak{p}_2)$ for all $(\mathfrak{p}_1, \mathfrak{p}_2) \in \mathfrak{M}$.

Notice that, for any arbitrary $\lambda \in (0,1)$

$$\mathfrak{N}(\mathfrak{T}\mathfrak{p},\mathfrak{T}\mathfrak{q},\lambda\mathfrak{a}_1,\mathfrak{a}_2) \leq \mathfrak{N}(\mathfrak{p},\mathfrak{q},\mathfrak{a}_1,\mathfrak{a}_2)$$

But the fixed point of $\mathfrak{T}$ is not unique. Indeed, every point $(\mathfrak{p}_1,\mathfrak{p}_2) \in \mathfrak{M}$ is a fixed point of $\mathfrak{T}$.

Finally, we will prove a fixed-point result for a revised fuzzy $\mathscr{k}$ −contraction mapping. We begin with the definition of a revised fuzzy $\mathscr{k}$ −contraction mapping as follows:

Definition 31

Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a revised fuzzy $\mathscr{k}$ −metric space. A mapping $\mathfrak{T}: \mathfrak{M} \to \mathfrak{M}$ is called a revised fuzzy $\mathscr{k}$ −contraction mapping if

$$\mathfrak{N}\left(\mathfrak{T}\mathfrak{p},\mathfrak{T}\mathfrak{q},a_1^{\mathscr{k}}\right) \leq \lambda\left\{\mathfrak{N}\left(\mathfrak{p},\mathfrak{q},a_1^{\mathscr{k}}\right)\right\} \tag{8}$$

for all $(\mathfrak{p}_1,\mathfrak{p}_2) \in \mathfrak{M}anda_1,a_2,\ldots,a_{\mathscr{k}} > 0$, where $\lambda \in [0,1)$ is a constant.

Theorem 32

Let $(\mathfrak{M},\mathfrak{N},\oplus)$ be a $\mathbb{G}$ −complete revised fuzzy $\mathscr{k}$ −metric space and $\mathfrak{T}: \mathfrak{M} \to \mathfrak{M}$ is called a revised fuzzy $\mathscr{k}$ −contraction mapping. Then, $\mathfrak{T}$ has a unique fixed point.

Proof Let $\mathfrak{p}_0 \in \mathfrak{M}$ and define a sequence $\{\mathfrak{p}_n\}$ by $\mathfrak{p}_n = \mathfrak{T}\mathfrak{p}_{n-1}$ for all $n \in \mathcal{N}$. We will show that this sequence is a $\mathbb{G}$ −Cauchy sequence. For any $n \in \mathcal{N}$, we have

$$\mathfrak{N}\left(\mathfrak{p}_n,\mathfrak{p}_{n+1},a_1^{\mathscr{k}}\right) = \mathfrak{N}\left(\mathfrak{T}\mathfrak{p}_{n-1},\mathfrak{T}\mathfrak{p}_n,a_1^{\mathscr{k}}\right) \leq \lambda\left\{\mathfrak{N}\left(\mathfrak{p}_{n-1},\mathfrak{p}_n,a_1^{\mathscr{k}}\right)\right\}$$

By repeating in this manner, we obtain

$$\mathfrak{N}\left(\mathfrak{p}_n,\mathfrak{p}_{n+1},a_1^{\mathscr{k}}\right) \leq \lambda^n\left\{\mathfrak{N}\left(\mathfrak{p}_{n-1},\mathfrak{p}_n,a_1^{\mathscr{k}}\right)\right\} \tag{9}$$

for all $n \in \mathcal{N}$. Since $\lambda \in [0,1)$, we conclude from (9) that

$$\lim_{n\to+\infty}\left\{\mathfrak{N}\left(\mathfrak{p}_n,\mathfrak{p}_{n+1},a_1^{\mathscr{k}}\right)\right\} \geq 1,$$

that is,

$$\lim_{n\to+\infty}\mathfrak{N}\left(\mathfrak{p}_n,\mathfrak{p}_{n+1},a_1^{\mathscr{k}}\right) = 0, \tag{10}$$

for all $a_1,a_2,\ldots,a_{\mathscr{k}} > 0$. For each $n \in \mathcal{N}$, $x > 0$ and $a_1,a_2,\ldots,a_{\mathscr{k}} > 0$, we have

$$\mathfrak{N}\left(\mathfrak{p}_n,\mathfrak{p}_{n+1},a_1^{\mathscr{k}}\right) \leq \mathfrak{N}_{\mathcal{T}}^2\left(\mathfrak{p}_n,\mathfrak{p}_{n+1},a_1^{\mathscr{k}}\right)\oplus\mathfrak{N}_{\mathcal{T}}^2\left(\mathfrak{p}_n,\mathfrak{p}_{n+x},a_1^{\mathscr{k}}\right)$$

$$\leq \begin{Bmatrix} \mathfrak{N}_{\mathcal{T}}^2\left(\mathfrak{p}_n,\mathfrak{p}_{n+1},a_1^{\mathscr{k}}\right)\oplus\mathfrak{N}_{\mathcal{T}}^{2^2}\left(\mathfrak{p}_{n+1},\mathfrak{p}_{n+2},a_1^{\mathscr{k}}\right) \oplus \ldots \oplus \\ \mathfrak{N}_{\mathcal{T}}^{2^{x-1}}\left(\mathfrak{p}_{n+x-2},\mathfrak{p}_{n+x-1},a_1^{\mathscr{k}}\right)\oplus\mathfrak{N}_{\mathcal{T}}^{2^{x-1}}\left(\mathfrak{p}_{n+x-1},\mathfrak{p}_{n+x},a_1^{\mathscr{k}}\right) \end{Bmatrix} \tag{11}$$

From (10), we have,

$\lim_{n\to+\infty}\mathfrak{N}_x^{\mathcal{T}}\left(\mathfrak{p}_n,\mathfrak{p}_{n+1},a_1^{\mathscr{k}}\right) = 0$, for all $a_1,a_2,\ldots,a_{\mathscr{k}} > 0$ and $x > 0$,which together with inequality (11) yields,

$$\lim_{n\to+\infty}\mathfrak{N}\left(\mathfrak{p}_n,\mathfrak{p}_{n+x},a_1^{\mathscr{k}}\right) \leq 0 \oplus 0 \oplus \ldots \oplus 0 = 0, \text{ for all } a_1,a_2,\ldots,a_{\mathscr{k}} > 0 \text{ and } x > 0.$$

Therefore, the sequence $\{\mathfrak{p}_n\}$ is a $\mathbb{G}$ −Cauchy sequence in $\mathfrak{M}$. By the $\mathbb{G}$ −completeness of $\mathfrak{M}$, there exists $\mathfrak{x} \in \mathfrak{M}$ such that the sequence $\{\mathfrak{p}_n\}$ converges to $\mathfrak{x}$, that is,

$$\lim_{n\to+\infty} \mathfrak{N}\left(\mathfrak{p}_n, \mathfrak{x}, a_1^{k}\right) = 0, \tag{12}$$

for all $a_1, a_2, \dots, a_k > 0$.

Now, we will show that $\mathfrak{x}$ is a fixed point of $\mathfrak{T}$. For each $n \in \mathcal{N}$, we have

$$\mathfrak{N}\left(\mathfrak{p}_{n+1}, \mathfrak{T}\mathfrak{x}, a_1^{k}\right) = \mathfrak{N}\left(\mathfrak{T}\mathfrak{p}_n, \mathfrak{T}\mathfrak{x}, a_1^{k}\right) \le \lambda\left\{\mathfrak{N}\left(\mathfrak{p}_n, \mathfrak{x}, a_1^{k}\right)\right\}$$

By using (12), we have, $\lim_{n\to+\infty}\left\{\mathfrak{N}\left(\mathfrak{p}_{n+1}, \mathfrak{T}\mathfrak{x}, a_1^{k}\right)\right\} = 0$,that is,

$$\lim_{n\to+\infty} \mathfrak{N}\left(\mathfrak{p}_{n+1}, \mathfrak{T}\mathfrak{x}, a_1^{k}\right) = 0 \tag{13}$$

for all $a_1, a_2, \dots, a_k > 0$. For any $n \in \mathcal{N}$, we have

$\mathfrak{N}\left(\mathfrak{x}, \mathfrak{T}\mathfrak{x}, a_1^{k}\right) \le \mathfrak{N}_2^{T}\left(\mathfrak{x}, \mathfrak{x}_{n+1}, a_1^{k}\right) \oplus \mathfrak{N}_2^{k}\left(\mathfrak{p}_{n+1}, \mathfrak{T}\mathfrak{x}, a_1^{k}\right)$,

which together with (12) and (13) yields

$\mathfrak{N}\left(\mathfrak{x}, \mathfrak{T}\mathfrak{x}, a_1^{k}\right) = 0$ for all $w_1, w_2, \dots, w_k > 0$. That is, $\mathfrak{T}\mathfrak{x} = \mathfrak{x}$. Thus, $\mathfrak{x}$ is a fixed point of $\mathfrak{T}$.

$\mathfrak{N}\left(\mathfrak{x}, \mathfrak{y}, w_1^{k}\right) > 0$, that is, $\mathfrak{N}\left(\mathfrak{x}, \mathfrak{y}, w_1^{k}\right) < 1$.

Now, we have

$$\mathfrak{N}\left(\mathfrak{x}, \mathfrak{y}, w_1^{k}\right) = \mathfrak{N}\left(\mathfrak{T}\mathfrak{x}, \mathfrak{T}\mathfrak{y}, w_1^{k}\right) \le \lambda\left\{\mathfrak{N}\left(\mathfrak{x}, \mathfrak{y}, w_1^{k}\right)\right\}$$

Since $\lambda < 1$, the above inequality yields a contradiction.

Therefore, we must have, $\mathfrak{x} = \mathfrak{y}$. Thus, the fixed point of T is unique.

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