Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol X No. Y (2025)

A Study on Signed Fuzzy Sets and Relations

Sankar C [1*], Kalaivani C [2], Saravanan M [3], Sujatha R [4]

[*1] Department of Mathematics, St. Joseph's College of Engineering, Chennai, Tamilnadu, India,

email: csankar26@gmail.com

[2] Department of Mathematics, SSN College of Engineering, Chennai, Tamilnadu, India,

email: kalaivanic@ssn.edu.in

[3] PG and Research Department of Mathematics, Mannar Thirumalai Naciker College, Madurai, Tamil Nadu, India,

email: msaran81@gmail.com

[4] School of Science and Humanities(Mathematics), Shiv Nadar University, Chennai, Tamilnadu, India,

email: sujathar@snuchennai.edu.in

Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract:

Introduction: Signed fuzzy sets are defined and illustrated in this article. Definitions are given for the union and intersection of signed fuzzy sets, the inclusion of two signed fuzzy sets in a signed fuzzy set, and the collection of signed fuzzy sets. We also describe how a list of items and an arbitrary signed fuzzy set can intersect and union. Some of the aspects of Signed-FRs are described, along with their definition, composition of two Signed-FRs, and inverse. The notion of semiotic fuzzy relations that are reflexive, transitive, and symmetric is suggested. We talk about α-cut, strong $\alpha-$cut, and some of its characteristics.

Keywords: Signed Fuzzy Set (SFS), Signed Fuzzy Relation(Signed-FR), Signed Fuzzy Equivalence Relation.

1. Introduction

The range $[0,1]$ contains the fuzzy membership values. When the property is not present, a membership value of zero is allocated. If an extra attribute, such acceptance or resistance, is desired when converting a real-life problem, we might incorporate a positive or negative sign with the membership values, respectively. This symbol shows that we are taking into account the value of membership in addition to other characteristics that determine acceptance or rejection.

Imagine a scenario in which the government want to prolong a project and there are both proponents and opponents of the project. A positive sign indicates support, and a negative one indicates opposition. The values of membership have no bearing on these indicators. The Signed-FR extends the idea of a fuzzy set.

Membership values in a Signed-FR can be between -1 and 1. A member who objects to the necessary property is given a membership value of -1. Better data interpretation and mathematical modelling are made possible by this method, which addresses both the advantages and disadvantages. All positive degree values, indeterminacy, and membership values for non-membership are included.

For example, let's look at the answer to the question, "How are you?" We can represent the response in a crisp set if it is either "I am okay" or "I am not okay." The response "I am somewhat okay" can be represented as a fuzzy set. If the response is "I am okay, but I have some problems," a Signed-FR can be used to model it.

In a fuzzy set, the loss may have a membership value of zero when translating profit and loss problems. Negative values lose their relevance in this method, though. Three cases can be defined: normal, low, and high. The membership values can be set to 0, -1, and 1, respectively. Assigning membership values from [–1, 1] is possible if we take into account the range from low to high, for example, from loss to profit. Signed-FRs are the name given to certain kinds of fuzzy sets.

In 1965, Zadeh first introduced the concept of a fuzzy set as a way to generalize a crisp set. In order to tackle the imprecision inherent in numerous real-world issues, Zadeh created the notions of fuzzy sets, fuzzy orderings, and similarity relations in 1971. Partial orderings and sharp equivalence relations, which are essential ideas in many fields of pure and applied science, were generalized in these ways. Since then, a lot of researchers have looked into fuzzy relations. Dib and Youssef defined the fuzzy Cartesian product of two ordinary sets, X and Y, as the set of all L-fuzzy sets of $X \times Y$, where I is the unit closed interval and $L = I \times I$. Specifically, Lee used the concept of fuzzy relations established by Dib and Youssef to reach several results.

In 1994, Wen-Ran Zhang studied bipolar fuzzy sets and relations, which are a type of Signed-FR. The concept of both positive and negative membership values was introduced in this work. Gorzalczany established the concept of interval-valued fuzzy sets in 1987. According to Atanassov (1986), intuitionistic fuzzy sets take into account both membership and non-membership values. A 1991 study by Kaufmann and Gupta examined fuzzy sets and associated methods. The use of fuzzy sets in relational databases and knowledge representation was expanded in 2004 by Yang and Singh with the introduction of fuzzy bipolar relational models.

The article defines a Signed-FR and provides an example. Signed-FR union and intersection, Signed-FR complement, and Signed-FR inclusion between two Signed-FRs are defined. Additionally, we describe the union and intersection of arbitrary Signed-FRs and provide a list of their properties. Along with the terminology Signed-FR, composition of two Signed-FRs, and inverse of an Signed-FR, some of these qualities are explored. We present the notions of reflexive, transitive, and symmetric Signed-FRs. Additionally discussed are $\alpha-$cut, strong $\alpha-$cut, and other traits of $\alpha-$cuts.

2. Preliminary Concepts

We present the idea of signed fuzzy sets in this section. We start by establishing signed fuzzy sets and examining their traits and attributes. After that, we provide a number of basic definitions for signed fuzzy sets, each supported by examples to help make the point clear. These examples will give a better idea of the practical utility of signed fuzzy sets by showing how they can be used in different situations.

An element in a crisp set is either a member of the set or not. Membership has two possible values: 0 and 1.

Example 2.1.

Consumer feedback indicates if a consumer is happy (1) or unhappy (0). Diverse levels of satisfaction are not captured by this binary classification. Partial membership is permitted by fuzzy sets, in which the membership value of an element falls between 0 and 1.

Example 2.2.

Customer Feedback: Although a customer's feedback may indicate varying degrees of satisfaction (e.g., 0.7 or 0.3), it does not express conflicting opinions at the same time.

By permitting both positive and negative membership values, ranging from -1 to 1, signed fuzzy sets expand on fuzzy sets. The degrees of opposition and membership are represented by this.

Example 2.3.

Customer feedback is categorized as follows: Neutral = 0, Highly unsatisfied = -1, Somewhat satisfied = 0.5, Highly satisfied = 1, and Neutral = 0.

Due to their ability to overcome the limitations of both standard fuzzy sets and crisp sets, signed fuzzy sets are essential. Due to its capability for dual-dimension membership values, signed fuzzy sets provide a thorough representation of elements in complex settings. Making educated decisions in practical applications such as sentiment analysis, risk management, environmental impact assessments, and healthcare requires an understanding of both positive and negative aspects.

Definition 2.1.

Let g be any particular element of X, and let X be the universal set. $\Im_{\pm1}$ is a signed fuzzy set defined on X. Let $\Im_{\pm1}$ be a collection of ordered pairs, such that $\Im_{\pm1} = \left\{g, \mu_{\Im_{\pm1}}(g) / g \in X\right\}$. where the membership function is denoted by $\mu_{\Im_{\pm1}}(g) : X \to [-1,1]$.

Example 2.4.

In Celsius, let $X = \{5,10,15,20,25,30\}$ be the temperature. $\Im_{\pm1} = \left\{\left(g, \mu_{\Im_{\pm1}}(g)\right) / g \in X\right\}$, $\mu_{\Im_{\pm1}}(g) : X \to [-1,1]$, where $\mu_{\Im_{\pm1}}(5) = -0.9$, $\mu_{\Im_{\pm1}}(10) = -0.7$, and $\mu_{\Im_{\pm1}}(15) = -0.4$, $\mu_{\Im_{\pm1}}(20) = 0$. In this case, $\mu_{\Im_{\pm1}}(25) = 0.3$, $\mu_{\Im_{\pm1}}(30) = 0.6$, and $\mu_{\Im_{\pm1}}(35) = 0.8$.

Definition 2.2.

Given a nonempty set X, let $\Im_{\pm1}$ and $\Im_{\pm2}$ be signed fuzzy sets in X.

i. $\Im_{\pm1}(g) \subseteq \Im_{\pm2}(g)$, $\forall g \in X$ iff $\Im_{\pm1}(g) \le \Im_{\pm2}(g)$.

ii.the complement of $\Im_{\pm 1}$ is $\Im_{\pm 1}^{C}(g)=\begin{cases}1-\Im_{\pm 1}(g) & \text{if } \mu_{\Im_{\pm 1}(\mathrm{g})}>0\\ -1-\Im_{\pm 1}(g) & \text{if } \mu_{\Im_{\pm 1}(\mathrm{g})}<0.\end{cases}$

iii. $\Im_{\pm 1}(g)=\Im_{\pm 2}(g)$, if $\Im_{\pm 1}(g)\subseteq\Im_{\pm 2}(g)$ and $\Im_{\pm 1}(g)\supseteq\Im_{\pm 2}(g), g\in X$.

iv. $\left(\Im_{\pm 1}\cap\Im_{\pm 2}\right)(g)=minimum\left\{\Im_{\pm 1}(g),\Im_{\pm 2}(g)\right\}, g\in X$.

v. $\left(\Im_{\pm 1}\cup\Im_{\pm 2}\right)(g)=maximum\left\{\Im_{\pm 1}(g),\Im_{\pm 2}(g)\right\}, g\in X$.

Theorem 2.1.

Consider the signed fuzzy sets $\Im_{\pm 1},\Im_{\pm 2}$ and $\Im_{\pm 3}$ in X . Then

1. Idempotent Laws: $\Im_{\pm 1}\cup\Im_{\pm 1}=\Im_{\pm 1},\Im_{\pm 1}\cap\Im_{\pm 1}=\Im_{\pm 1}$.
2. Commutative Laws: $\Im_{\pm 1}\cup\Im_{\pm 2}=\Im_{\pm 2}\cup\Im_{\pm 1},\Im_{\pm 1}\cap\Im_{\pm 2}=\Im_{\pm 2}\cap\Im_{\pm 1}$.
3. Associative Laws: $\Im_{\pm 1}\cup(\Im_{\pm 2}\cup\Im_{\pm 3})=(\Im_{\pm 1}\cup\Im_{\pm 2})\cup\Im_{\pm 3}$, $\Im_{\pm 1}\cap(\Im_{\pm 2}\cap\Im_{\pm 3})=(\Im_{\pm 1}\cap\Im_{\pm 2})\cap\Im_{\pm 3}$.
4. Distributive Laws: $\Im_{\pm 1}\cap(\Im_{\pm 2}\cup\Im_{\pm 3})=(\Im_{\pm 1}\cap\Im_{\pm 2})\cup(\Im_{\pm 1}\cap\Im_{\pm 3})$, $\Im_{\pm 1}\cup(\Im_{\pm 2}\cap\Im_{\pm 3})=(\Im_{\pm 1}\cup\Im_{\pm 2})\cap(\Im_{\pm 1}\cup\Im_{\pm 3})$.
5. Absorption Laws: $\Im_{\pm 1}\cap(\Im_{\pm 1}\cup\Im_{\pm 2})=\Im_{\pm 1}$, $\Im_{\pm 1}\cup(\Im_{\pm 1}\cap\Im_{\pm 2})=\Im_{\pm 1}$.
6. Demorgan's Laws: $(\Im_{\pm 1}\cup\Im_{\pm 2})^{C}=\Im_{\pm 1}^{C}\cap\Im_{\pm 2}^{C}$, $(\Im_{\pm 1}\cap\Im_{\pm 2})^{C}=\Im_{\pm 1}^{C}\cup\Im_{\pm 2}^{C}$.
7. $(\Im_{\pm 1}^{C})^{C}=\Im_{\pm 1}$.
8. $\Im_{\pm 1}\cap\Im_{\pm 2}\subset\Im_{\pm 1}$ and $\Im_{\pm 1}\cap\Im_{\pm 2}\subset\Im_{\pm 2}$.
9. $\Im_{\pm 1}\subset\Im_{\pm 1}\cup\Im_{\pm 2}$ and $\Im_{\pm 2}\subset\Im_{\pm 1}\cup\Im_{\pm 2}$.
10. $\Im_{\pm 1}\subset\Im_{\pm 2}$ and $\Im_{\pm 2}\subset\Im_{\pm 3}$ then $\Im_{\pm 1}\subset\Im_{\pm 3}$.
11. $\Im_{\pm 1}\subset\Im_{\pm 2}$ then $\Im_{\pm 1}\cap\Im_{\pm 3}\subset\Im_{\pm 2}\cap\Im_{\pm 3}$ and $\Im_{\pm 1}\cup\Im_{\pm 3}\subset\Im_{\pm 2}\cup\Im_{\pm 3}$.

Definition 2.3.

If X is a nonempty set, then $\Im_{\pm i}\subset Signed-FS(X),\forall i\in I$.

1. The intersection of $\Im_{\pm i},\forall i\in I$ is denoted as $\bigcap_{i\in I}\Im_{\pm i}$, a signed fuzzy set in X defined for each $g\in X$, $\left(\bigcap_{i\in I}\Im_{\pm i}\right)(g)=min\left\{\Im_{\pm 1}(g),\Im_{\pm 2}(g),...\right\}$.
2. The union of $\Im_{\pm i}$, $\forall i\in I$ is denoted as $\bigcup_{i\in I}\Im_{\pm i}$, a signed fuzzy set in X defined for each $g\in X$, $\left(\bigcup_{i\in I}\Im_{\pm i}\right)(g)=max\left\{\Im_{\pm 1}(g),\Im_{\pm 2}(g),...\right\}$.

Definition 2.4.

Let $\Im_{\pm 1}$ be a signed fuzzy set in X and let $\Im_{\pm i}$, $\forall i\in I\subset$ Signed-FS X . Then

1. Distributive law in general $\mathfrak{I}_{\pm 1} \cup \left(\bigcap_{i \in I} \mathfrak{I}_{\pm i} \right) = \bigcap_{i \in I} \left(\mathfrak{I}_{\pm 1} \cup \mathfrak{I}_{\pm i} \right)$.
2. De Morgan's law in general $\left(\bigcup_{i \in I} \mathfrak{I}_{\pm i} \right)^{c} = \bigcap_{i \in I} \mathfrak{I}_{\pm 1}^{c}$.

3. Signed Fuzzy Relations

This section discusses the results of defining Signed-FRs using examples. Additionally, examples are provided to define the composition of signed fuzzy sets, and the outcomes are displayed.

A Signed Fuzzy Relations(Signed-FR) $\ddot{u}_{\pm}$ is a mapping from $X \times Y$ to the interval $[-1,1]$. The membership function of the relation $\ddot{u}_{\pm} \subseteq X \times Y$, $\mu_{\ddot{u}_{\pm}}$ varies over $[-1,1]$, expressing the strength of the mapping.

Definition 3.1.

Let two nonempty sets be X and Y. Then, $\ddot{u}_{\pm} = \left\{ (g,h), \mu_{\ddot{u}_{\pm}}(g,h) | \, g \in X, h \in Y \right\}$ is the definition of the Signed-FR $\ddot{u}_{\pm}$, where $\ddot{u}_{\pm} \in Signed - FR(X \times Y)$.

Definition 3.2.

Let $\ddot{u}_{\pm} \in Signed - FR(X \times Y)$. Then

1. The formula for $\ddot{u}_{\pm}^{-1}(g,h) = \ddot{u}_{\pm}(h,g)$ is the inverse of $\ddot{u}_{\pm}$.
2. The complement of $\ddot{u}_{\pm}(g,h)$ is $\ddot{u}_{\pm}^{c}(g,h) = \begin{cases} 1 - \mu_{\ddot{u}_{\pm}}(g,h) & \text{if } \mu_{\ddot{u}_{\pm}(\mathrm{g,h})} > 0 \\ -1 - \mu_{\ddot{u}_{\pm}}(g,h) & \text{if } \mu_{\ddot{u}_{\pm}(\mathrm{g,h})} < 0. \end{cases}$
3. $\mu_{\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 3}}(g,h) = \text{maximum } \left\{ \mu_{\ddot{u}_{\pm 1}}(g,h), \mu_{\ddot{u}_{\pm 3}}(g,h) \right\}$ is the union of $\ddot{u}_{\pm 1}$ and $\ddot{u}_{\pm 3}$.
4. $\mu_{\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 3}}(g,h) = \text{minimum } \left\{ \mu_{\ddot{u}_{\pm 1}}(g,h), \mu_{\ddot{u}_{\pm 3}}(g,h) \right\}$ is the intersection of $\ddot{u}_{\pm 1}$ and $\ddot{u}_{\pm 3}$.
5. the sum of $\ddot{u}_{\pm 1}$ and $\ddot{u}_{\pm 3}$, is, $\mu_{\ddot{u}_{\pm 1} + \ddot{u}_{\pm 3}}(g,h) = \left\{ \mu_{\ddot{u}_{\pm 1}}(g,h) + \mu_{\ddot{u}_{\pm 3}}(g,h) - \mu_{\ddot{u}_{\pm 1}}(g,h).\mu_{\ddot{u}_{\pm 3}}(g,h) \right\}$.
6. the multiplication of $\ddot{u}_{\pm 1}$ and $\ddot{u}_{\pm 3}$, is, $\mu_{\ddot{u}_{\pm 1}.\ddot{u}_{\pm 3}}(g,h) = \mu_{\ddot{u}_{\pm 1}}(g,h).\mu_{\ddot{u}_{\pm 2}}(g,h)$.

Theorem 3.1.

Let $\ddot{u}_{\pm 1}, \ddot{u}_{\pm 2}, \ddot{u}_{\pm 3} \in Signed - FR(X \times Y)$. Then

1. $\left(\ddot{u}_{\pm 1}^{c} \right)^{-1} = \left(\ddot{u}_{\pm 1}^{-1} \right)^{c}$.
2. $\left(\ddot{u}_{\pm 1}^{-1} \right)^{-1} = \ddot{u}_{\pm 1}$.
3. $\ddot{u}_{\pm 1} \subset \ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}$ and $\ddot{u}_{\pm 2} \subset \ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}$.
4. If $\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 2} \subset \ddot{u}_{\pm 1}$ and $\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 2} \subset \ddot{u}_{\pm 2}$.
5. If $\ddot{u}_{\pm 1} \subset \ddot{u}_{\pm 2}$ then $\ddot{u}_{\pm 1}^{-1} \subset \ddot{u}_{\pm 2}^{-1}$.

6. If $\ddot{u}_{\pm1} \subset \ddot{u}_{\pm2}$ and $\ddot{u}_{\pm2} \subset \ddot{u}_{\pm2}$, then $\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2} \subset \ddot{u}_{\pm2}$.

7. If $\ddot{u}_{\pm2} \subset \ddot{u}_{\pm1}$ and $\ddot{u}_{\pm2} \subset \ddot{u}_{\pm2}$, then $\ddot{u}_{\pm2} \subset \ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}$.

8. If $\ddot{u}_{\pm1} \subset \ddot{u}_{\pm2}$, then $\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2} = \ddot{u}_{\pm2}$ and $\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2} = \ddot{u}_{\pm1}$.

9. $(\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})^{-1} = \ddot{u}_{\pm1}^{\;-1} \cup \ddot{u}_{\pm2}^{\;-1}$, $(\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2})^{-1} = \ddot{u}_{\pm1}^{-1} \cap \ddot{u}_{\pm2}^{-1}$.

Proof.

1. Let $(g,h) \in X \times Y$. Then $\left(\ddot{u}_{\pm1}^{c}\right)^{-1}(g,h) = \left(\ddot{u}_{\pm1}^{c}\right)^{-}(h,g) \quad = -1 - \ddot{u}_{\pm1}^{-1}(h,g) \quad = -1 - (\ddot{u}_{\pm1}^{-1})^{-}(g,h)$ $= \left(\ddot{u}_{\pm1}^{-1}\right)^{c}(g,h)$.

2. Let $(g,h) \in X \times Y$. Then, $\left(\ddot{u}_{\pm1}\right)^{-1}(g,h) = \ddot{u}_{\pm1}(h,g)$. Therefore, $\left(\left(\ddot{u}_{\pm1}\right)^{-1}\right)^{-1}(g,h) = (\ddot{u}_{\pm1}(h,g))^{-1} = \ddot{u}_{\pm1}(g,h)$.

3. Let $(g,h) \in X \times Y$. Then $(\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})(g,h) = max\left\{\ddot{u}_{\pm1}(g,h), S_{\pm}(g,h)\right\} \geq \ddot{u}_{\pm1}(g,h)$ and $(\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})(g,h) = max\left\{\ddot{u}_{\pm1}(g,h), \ddot{u}_{\pm2}(g,h)\right\} \geq \ddot{u}_{\pm2}(g,h)$.

4. Let $(g,h) \in X \times Y$. If $(g,h) \in \ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}$, then (g,h) is in both $\ddot{u}_{\pm1}$ and $\ddot{u}_{\pm2}$. This implies $(g,h) \in \ddot{u}_{\pm1}$. Therefore $\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2} \subset \ddot{u}_{\pm1}$. Similarly $\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2} \subset \ddot{u}_{\pm2}$.

5. Let $(g,h) \in X \times Y$. If $\ddot{u}_{\pm1} \subset \ddot{u}_{\pm2}$, this implies that every pair (h,g) in $\ddot{u}_{\pm1}$ must also be in $\ddot{u}_{\pm2}$. Then, $\ddot{u}_{\pm1}^{-1}(g,h) = \ddot{u}_{\pm1}(h,g)$. Therefore $\ddot{u}_{\pm1}^{-1}(g,h) \subset \ddot{u}_{\pm2}^{-1}(g,h)$.

6. If $(g,h) \in \ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$, then (g,h) is either in $\ddot{u}_{\pm1}$, in $\ddot{u}_{\pm2}$ or in both $\ddot{u}_{\pm1}$ and $\ddot{u}_{\pm2}$. Since $\ddot{u}_{\pm1} \subset \ddot{u}_{\pm2}$, if $(g,h) \in \ddot{u}_{\pm1}$, then $(g,h) \in \ddot{u}_{\pm2}$. Similarly, since $\ddot{u}_{\pm2} \subset \ddot{u}_{\pm2}$, if $(g,h) \in \ddot{u}_{\pm2}$, then $(g,h) \in \ddot{u}_{\pm2}$. Therefore, $(g,h) \in \ddot{u}_{\pm1} \cup \ddot{u}_{\pm2} \subset \ddot{u}_{\pm2}$, which implies $\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2} \subset \ddot{u}_{\pm2}$.

7. If $g \in \ddot{u}_{\pm2}$, then $g \in \ddot{u}_{\pm1}$ because $\ddot{u}_{\pm2} \subset \ddot{u}_{\pm1}$. Similarly, if $g \in \ddot{u}_{\pm2}$, then $g \in \ddot{u}_{\pm2}$ since $\ddot{u}_{\pm2} \subset \ddot{u}_{\pm2}$. Since $g \in \ddot{u}_{\pm1}$ and $g \in \ddot{u}_{\pm2}$, it follows that $\ddot{u}_{\pm2} \subset \ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}$.

8. If $\ddot{u}_{\pm1} \subset \ddot{u}_{\pm2}$, then every element in $\ddot{u}_{\pm1}$ is also an element in $\ddot{u}_{\pm2}$. Since every element in $\ddot{u}_{\pm1}$ is already present in $\ddot{u}_{\pm2}$, the union $\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$ does not add any new elements to $\ddot{u}_{\pm2}$. Therefore $\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2} = \ddot{u}_{\pm2}$. Similarly, $\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2} = \ddot{u}_{\pm1}$. All elements in $\ddot{u}_{\pm1}$ are also in $\ddot{u}_{\pm2}$ because $\ddot{u}_{\pm1} \subset \ddot{u}_{\pm2}$. As a result, when we take the intersection $\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}$. we only include the elements in $\ddot{u}_{\pm1}$ that are identical to those in $\ddot{u}_{\pm2}$. Hence $\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2} = \ddot{u}_{\pm1}$.

9. If $(g,h) \in (\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})^{-1}$, then by definition~\ref{d1} $(h,g) \in \ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$. Therefore $(h,g) \in \ddot{u}_{\pm1}$ or $(h,g) \in \ddot{u}_{\pm2}$. If $(h,g) \in \ddot{u}_{\pm1}$ then $(g,h) \in \ddot{u}_{\pm1}^{-1}$. Similarly, if $(h,g) \in \ddot{u}_{\pm2}$ then $(g,h) \in \ddot{u}_{\pm2}^{-1}$, which implies $(g,h) \in \ddot{u}_{\pm1}^{-1} \cup \ddot{u}_{\pm2}^{-1}$. Therefore $(\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})^{-1} \subset \ddot{u}_{\pm1}^{-1} \cup \ddot{u}_{\pm2}^{-1}$. Conversely, if $(h,g) \in \ddot{u}_{\pm1}^{-1} \cup \ddot{u}_{\pm2}^{-1}$, then $(h,g) \in \ddot{u}_{\pm1}^{-1}$ or $(h,g) \in \ddot{u}_{\pm2}^{-1}$. If $(h,g) \in \ddot{u}_{\pm1}^{-1}$, then $(g,h) \in \ddot{u}_{\pm1}$. Similarly, if $(h,g) \in \ddot{u}_{\pm2}^{-1}$, then $(g,h) \in \ddot{u}_{\pm2}$. which implies $(g,h) \in \ddot{u}_{\pm1}^{-1} \cup \ddot{u}_{\pm2}^{-1}$. Therefore $\ddot{u}_{\pm1}^{-1} \cup \ddot{u}_{\pm2}^{-1} \subset (\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})^{-1}$. Finally, $(\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})^{-1} = \ddot{u}_{\pm1}^{\;-1} \cup \ddot{u}_{\pm2}^{\;-1}$. Similarly, $(\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2})^{-1} = \ddot{u}_{\pm1}^{-1} \cap \ddot{u}_{\pm2}^{-1}$.

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol X No. Y (2025)

Theorem 3.2.

Let $\ddot{u}_{\pm 1}, \ddot{u}_{\pm 2}, \ddot{u}_{\pm 3} \in Signed - FR(X \times Y)$ and let $(\ddot{u}_{\pm i})_{i \in I} \subset Signed - FR(X \times Y)$.

Then

1. $\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 1} = \ddot{u}_{\pm 1}$, $\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 1} = \ddot{u}_{\pm 1}$ Idempotent laws
2. $\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2} = \ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 1}$, $\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 2} = \ddot{u}_{\pm 2} \cap \ddot{u}_{\pm 1}$ Commutative laws
3. $\ddot{u}_{\pm 1} \cup (\ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 3}) = (\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}) \cup \ddot{u}_{\pm 3}$, $\ddot{u}_{\pm 1} \cap (\ddot{u}_{\pm 2} \cap \ddot{u}_{\pm 3}) = (\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 2}) \cap \ddot{u}_{\pm 3}$. Associative laws
4. $\ddot{u}_{\pm 1} \cup (\ddot{u}_{\pm 2} \cap \ddot{u}_{\pm 3}) = (\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}) \cap (\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 3})$, $\ddot{u}_{\pm 1} \cap (\ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 3}) = (\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 2}) \cup (\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 3})$
Distributive Laws
5. $\ddot{u}_{\pm 1} \cap \left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right) = \bigcup_{i \in I} (\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm i})$, $\ddot{u}_{\pm 1} \cup \left(\bigcap_{i \in I} R_{\pm i} \right) = \bigcap_{i \in I} (\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm i})$. Generalized distributive laws
6. $\ddot{u}_{\pm 1} \cup (\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 2}) = \ddot{u}_{\pm 1}$, $\ddot{u}_{\pm 1} \cap (\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}) = \ddot{u}_{\pm 1}$ Absorption Laws
7. $(\ddot{u} \cup \psi)^c = \ddot{u}^c \cap \psi^c$, $(\ddot{u} \cap \psi)^c = \ddot{u}^c \cup \psi^c$ De Morgan's laws
8. $\left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right)^c = \bigcap_{i \in I} \ddot{u}_{\pm i}^c$, $\left(\bigcap_{i \in I} \ddot{u}_{\pm i} \right)^c = \bigcup_{i \in I} \ddot{u}_{\pm i}^c$. Generalized De Morgan's laws
9. $(\ddot{u}_{\pm 1}^c)^c = \ddot{u}_{\pm 1}$. Involution

Proof.

1. Let us prove $\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 1} \subset \ddot{u}_{\pm 1}$. If $(g,h) \in \ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 1}$, then $(g,h) \in \ddot{u}_{\pm 1}$. Therefore, $\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 1} \subset \ddot{u}_{\pm 1}$. Conversely, if $(g,h) \in \ddot{u}_{\pm 1}$, then $(g,h) \in \ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 1}$. Thus $\ddot{u}_{\pm 1} \subset \ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 1}$. Combining these results, we have $\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 1} = \ddot{u}_{\pm 1}$. Similarly, we can show that $\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 1} = \ddot{u}_{\pm 1}$.
2. Let us show that $\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2} \subset \ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 1}$. If $(g,h) \in \ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}$, then $(g,h) \in \ddot{u}_{\pm 1}$ or $(g,h) \in \ddot{u}_{\pm 2}$, Therefore $(g,h) \in \ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 1}$. Hence, $\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2} \subset \ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 1}$. Similarly, we can show that $\ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 1} \subset \ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}$. Therefore, $\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2} = \ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 1}$. Likewise, we can demonstrate that $\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 2} = \ddot{u}_{\pm 2} \cap \ddot{u}_{\pm 1}$.
3. Let $(g,h) \in \ddot{u}_{\pm 1} \cup (\ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 3})$, Then $(g,h) \in \ddot{u}_{\pm 1}$ or $(g,h) \in \ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 3}$, which implies $(g,h) \in \ddot{u}_{\pm 1}$, $(g,h) \in \ddot{u}_{\pm 2}$ or $(g,h) \in \ddot{u}_{\pm 3}$. Therefore $(g,h) \in (\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}) \cup \ddot{u}_{\pm 3}$. Hence $\ddot{u}_{\pm 1} \cup (\ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 3}) = (\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}) \cup \ddot{u}_{\pm 3}$, Similarly $\ddot{u}_{\pm 1} \cap (\ddot{u}_{\pm 2} \cap \ddot{u}_{\pm 1}) = (\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 2}) \cap \ddot{u}_{\pm 1}$.
4. If $(g,h) \in \ddot{u}_{\pm 1} \cup (\ddot{u}_{\pm 2} \cap \ddot{u}_{\pm 3})$, then $(g,h) \in \ddot{u}_{\pm 1}$ or $(g,h) \in \ddot{u}_{\pm 2}$ and $(g,h) \in \ddot{u}_{\pm 3}$. This implies $(g,h) \in \ddot{u}_{\pm 1}$ or $(g,h) \in \ddot{u}_{\pm 2}$ and $(g,h) \in \ddot{u}_{\pm 1}$ or $(g,h) \in \ddot{u}_{\pm 3}$. Consequently, $(g,h) \in \ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}$ and $(g,h) \in \ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 3}$. Therefore $(g,h) \in (\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}) \cap (\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 3})$. Hence $\ddot{u}_{\pm 1} \cup (\ddot{u}_{\pm 2} \cap \ddot{u}_{\pm 3}) = (\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 2}) \cap (\ddot{u}_{\pm 1} \cup \ddot{u}_{\pm 3})$. Similarly $\ddot{u}_{\pm 1} \cap (\ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 3}) = (\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 2}) \cup (\ddot{u}_{\pm 1} \cap \ddot{u}_{\pm 3})$.

5. Let us prove $\ddot{u}_{\pm1} \cap \left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right) \subset \bigcup_{i \in I} (\ddot{u}_{\pm1} \cap \ddot{u}_{\pm i})$. Suppose $(g,h) \in \ddot{u}_{\pm1} \cap \left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right)$. Then $(g,h) \in \ddot{u}_{\pm1}$ and $(g,h) \in \left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right)$. Thus, there exists $i \in I$ such that $(g,h) \in \ddot{u}_{\pm i}$. Given that $(g,h) \in \ddot{u}_{\pm1}$, it follows that $(g,h) \in (\ddot{u}_{\pm1} \cap \ddot{u}_{\pm i})$. Therefore $(g,h) \in \bigcup_{i \in I} (\ddot{u}_{\pm1} \cap \ddot{u}_{\pm i})$. Hence $\ddot{u}_{\pm1} \cap \left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right) \subset \bigcup_{i \in I} (\ddot{u}_{\pm1} \cap \ddot{u}_{\pm i})$. Now, let's prove that $\bigcup_{i \in I} (\ddot{u}_{\pm1} \cap \ddot{u}_{\pm i}) \subset \ddot{u}_{\pm1} \cap \left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right)$. Suppose $(g,h) \in \bigcup_{i \in I} (\ddot{u}_{\pm1} \cap \ddot{u}_{\pm i})$. Then there exists $i \in I$ Such that $(g,h) \in \ddot{u}_{\pm1} \cap \ddot{u}_{\pm i}$. Thus $(g,h) \in \ddot{u}_{\pm1}$ and $(g,h) \in \ddot{u}_{\pm i}$. Hence $(g,h) \in \bigcup_{i \in I} \ddot{u}_{\pm i}$, and given that $(g,h) \in \ddot{u}_{\pm1}$, it follows that $(g,h) \in \ddot{u}_{\pm1} \cap \left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right)$. Therefore $\bigcup_{i \in I} (\ddot{u}_{\pm1} \cap \ddot{u}_{\pm i}) \subset \ddot{u}_{\pm1} \cap \left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right)$. Hence $\bigcup_{i \in I} (\ddot{u}_{\pm1} \cap \ddot{u}_{\pm i}) = \ddot{u}_{\pm1} \cap \left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right)$.

6. If $(g,h) \in (\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})^c$, then $(g,h) \notin \ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$. This implies $(g,h) \notin \ddot{u}_{\pm1}$ and $(g,h) \notin \ddot{u}_{\pm2}$. Therefore $(g,h) \in \ddot{u}_{\pm1}^c$ and $(g,h) \in \ddot{u}_{\pm2}^c$, which means $(g,h) \in \ddot{u}_{\pm1}^c \cap \ddot{u}_{\pm2}^c$. Hence $(\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})^c \subset \ddot{u}_{\pm1}^c \cap \ddot{u}_{\pm2}^c$. Similarly, if $(g,h) \in \ddot{u}_{\pm1}^c \cap \ddot{u}_{\pm2}^c$, then $(g,h) \in \ddot{u}_{\pm1}^c$ and $(g,h) \in \ddot{u}_{\pm2}^c$. This implies $(g,h) \notin \ddot{u}_{\pm1}$ and $(g,h) \notin \ddot{u}_{\pm2}$. Therefore $(g,h) \notin \ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$, which means $(g,h) \in (\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})^c$. Thus $\ddot{u}_{\pm1}^c \cap \ddot{u}_{\pm2}^c \subset (\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})^c$. Hence $\left(\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}\right)^c = \ddot{u}_{\pm1}^c \cap \ddot{u}_{\pm2}^c$. Similarly $\left(\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}\right)^c = \ddot{u}_{\pm1}^c \cup \ddot{u}_{\pm2}^c$.

7. Let us claim: $\ddot{u}_{\pm1} \cup \left(\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}\right) \subset \ddot{u}_{\pm1}$. If $(g,h) \in \ddot{u}_{\pm1} \cap (\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})$, then $(g,h) \in \ddot{u}_{\pm1}$ and $(g,h) \in (\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})$. This implies $(g,h) \in \ddot{u}_{\pm1}$. Therefore $\ddot{u}_{\pm1} \cup \left(\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}\right) \subset \ddot{u}_{\pm1}$. Now, suppose $\ddot{u}_{\pm1} \subset \ddot{u}_{\pm1} \cup \left(\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}\right)$. If $(g,h) \in \ddot{u}_{\pm1}$, then $(g,h) \in \ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$. Thus $(g,h) \in \ddot{u}_{\pm1}$ and $(g,h) \in \ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$, which means $(g,h) \in \ddot{u}_{\pm1} \cap (\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2})$, This implies $\ddot{u}_{\pm1} \subset \ddot{u}_{\pm1} \cup \left(\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}\right)$. Therefore $\ddot{u}_{\pm1} = \ddot{u}_{\pm1} \cup \left(\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}\right)$.

8. Let $(g,h) \in \left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right)^c$. This implies $(g,h) \in \bigcap_{i \in I} \ddot{u}_{\pm i}^c$. Hence $\left(\bigcup_{i \in I} \ddot{u}_{\pm i} \right)^c = \bigcap_{i \in I} \ddot{u}_{\pm i}^c$. Similarly, $\left(\bigcap_{i \in I} \ddot{u}_{\pm i} \right)^c = \bigcup_{i \in I} \ddot{u}_{\pm i}^c$.

9. Let $(g,h) \in (\ddot{u}_{\pm1}^c)^c$. This means $(g,h) \notin \ddot{u}_{\pm1}^c$, which implies $(g,h) \in \ddot{u}_{\pm1}$, Therefore $(\ddot{u}_{\pm1}^c)^c \subset \ddot{u}_{\pm1}$. Similarly $\ddot{u}_{\pm1} \subset (\ddot{u}_{\pm1}^c)^c$. Hence $\ddot{u}_{\pm1} = (\ddot{u}_{\pm1}^c)^c$.

The composition operation is the process of combining Signed-FRs across several product spaces. Here, we discuss two common operations: max-min composition and max-product composition. In essence, these operations let us merge two or more Signed-FR relations. Depending on the circumstance, there are numerous ways to define it.

Definition 3.3.

Let $\ddot{u}_{\pm 1} \in Signed - FR(X \times Y)$, and let $\ddot{u}_{\pm 2} \in Signed - FR(Y \times Z)$. Then, the composition of $\ddot{u}_{\pm 1}$ and $\ddot{u}_{\pm 2}$ is $\left(\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 1}\right)(g,i) = \bigvee_{h \in Y}\left[\ddot{u}_{\pm 1}(g,h) \wedge \ddot{u}_{\pm 2}(h,i)\right]$

Theorem 3.3.

1. If $\ddot{u}_{\pm 1} \in Signed - FR(X \times Y), \ddot{u}_{\pm 2} \in Signed - FR(Y \times Z)$ and $\ddot{u}_{\pm 3} \in Signed - FR(Z \times W)$, then $\ddot{u}_{\pm 1} \circ (\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 3}) = (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}) \circ \ddot{u}_{\pm 3}$.

2. If $\ddot{u}_{\pm 2}, \ddot{u}_{\pm 3} \in Signed - FR(X \times Y)$ and $\ddot{u}_{\pm 1} \in Signed - FR(Y \times Z)$, then $\ddot{u}_{\pm 1} \circ (\ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 3}) = (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}) \cup (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 3})$.

3. If $\ddot{u}_{\pm 1} \subset \ddot{u}_{\pm 2}$, then $\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 1} \subset \ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 2}$, where $\ddot{u}_{\pm 1}, \ddot{u}_{\pm 2} \in Signed - FR(X \times Y)$ and $\ddot{u}_{\pm 3} \in Signed - FR(Y \times Z)$.

4. If $\ddot{u}_{\pm 1} \in Signed - FR(X \times Y)$ and $\psi \in Signed - FR(Y \times Z)$, then $(\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 1})^{-1} = \ddot{u}_{\pm 1}^{-1} \circ \ddot{u}_{\pm 2}^{-1}$.

Proof.

1. Let $\ddot{u}_{\pm 1} \in Signed - FR(X \times Y)$, $\ddot{u}_{\pm 2} \in Signed - FR(Y \times Z)$ and $\ddot{u}_{\pm 3} \in Signed - FR(Z \times W)$. Suppose $(g,i) \in (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}) \circ \ddot{u}_{\pm 3}$. This means there exists a $h \in Y$ such that $(g,h) \in \ddot{u}_{\pm 1}$ and $(h,i) \in (\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 3})$. Similarly, if $i \in Z$ such that $(h,i) \in \ddot{u}_{\pm 2}$ and $(i,j) \in \ddot{u}_{\pm 3}$, then $(g,i) \in (\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 1})$. Similarly $(g,j) \in \ddot{u}_{\pm 3} \circ (\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 1})$. Therefore $(\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}) \circ \ddot{u}_{\pm 3} \subset \ddot{u}_{\pm 1} \circ (\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 3})$. Conversely, Let $(g,j) \in \ddot{u}_{\pm 1} \circ (\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 3})$. This implies that there exists a $i \in Z$ such that $(g,i) \in (\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 3})$ and $(i,j) \in \ddot{u}_{\pm 1}$. Similarly, if $h \in Y$ such that $(g,h) \in \ddot{u}_{\pm 3}$ and $(h,i) \in \ddot{u}_{\pm 2}$, then $(h,j) \in \ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}$. Consequently, $(g,j) \in (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}) \circ \ddot{u}_{\pm 3}$. Therefore $\ddot{u}_{\pm 1} \circ (\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 3}) \subset (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}) \circ \ddot{u}_{\pm 3}$. Thus, $\ddot{u}_{\pm 1} \circ (\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 3}) = (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}) \circ \ddot{u}_{\pm 3}$.

2. Let $(g,h) \in \ddot{u}_{\pm 1} \circ (\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 3})$, then there exists a $h \in \ddot{u}_{\pm 3}$ such that $(g,h) \in \ddot{u}_{\pm 1}$ and $(h,i) \in (\ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 3})$. Since $(h,i) \in \ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 3}$ it is a member of either $\ddot{u}_{\pm 2}$ or $\ddot{u}_{\pm 3}$. If $(h,i) \in \ddot{u}_{\pm 3}$, then $(g,i) \in \ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 3}$. If $(h,i) \in \ddot{u}_{\pm 3}$, then $(g,i) \in \ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 3}$. which implies $(g,i) \in (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}) \cup (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 3})$. Conversely, Suppose $(g,i) \in (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}) \cup (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 3})$. This implies that $(g,i) \in \ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}$ or $(g,i) \in \ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 3}$, which in turn means there exists a $h \in \ddot{u}_{\pm 3}$ such that $(g,h) \in \ddot{u}_{\pm 1}$ and $(h,i) \in \ddot{u}_{\pm 2}$ or $\ddot{u}_{\pm 3}$. Therefore, $(g,i) \in \ddot{u}_{\pm 1} \circ (\ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 3})$. Hence $\ddot{u}_{\pm 1} \circ (\ddot{u}_{\pm 2} \cup \ddot{u}_{\pm 3}) = (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2}) \cup (\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 3})$.

3. Let $\ddot{u}_{\pm 2}, \ddot{u}_{\pm 3} \in Signed - FR(X \times Y)$ and $\ddot{u}_{\pm 1} \in Signed - FR(Y \times Z)$, with the assumption that $\ddot{u}_{\pm 2} \subseteq \ddot{u}_{\pm 3}$ and $(g,i) \in X \times Z$, then $(\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2})(g,i) = \bigvee_{h \in Y}\left[\ddot{u}_{\pm 1}(g,h) \wedge \ddot{u}_{\pm 2}(h,i)\right]$ $\leq \bigvee_{h \in Y}\left[\ddot{u}_{\pm 3}(g,h) \wedge \ddot{u}_{\pm 2}(h,i)\right] = \ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 3}$. Therefore, $\ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 2} \subseteq \ddot{u}_{\pm 1} \circ \ddot{u}_{\pm 3}$.

4. Let $\ddot{u}_{\pm 1} \in Signed - FR(X \times Y)$ and $\ddot{u}_{\pm 2} \in Signed - FR(Y \times Z)$, with $(g,i) \in X \times Z$. Then $[(\ddot{u}_{\pm 2} \circ \ddot{u}_{\pm 1})^{-1}](i,g) = (\ddot{u}_{\pm 1}^{-1} \circ \ddot{u}_{\pm 2}^{-1})(g,i)$.

4. Signed Fuzzy Equivalence Relation

This section includes instances and definitions of transitive, symmetric, antisymmetric, reflexive, and antireflexive qualities. The definition of equivalency relations is also provided, along with the findings. Additionally, examples are used to clarify the definitions of $\alpha -$cut and strong $\alpha -$cut of signed fuzzy sets, along with the results.

Definition 4.1

A Signed-FR $\ddot{u}_{\pm}$ in $X \times X$ is reflexive if and only if $\mu_{\pm}(g,g)=1, \forall g \in X$.

Definition 4.2

If $\ddot{u}_{\pm}$ in $X \times X$ is a Signed-FR, then $\ddot{u}_{\pm}$ is antireflexive if and only if $\mu_{\ddot{u}_{\pm}}(g,g)=0, \forall g \in X$.

Theorem 4.1.

Let $\ddot{u}_{\pm1} \in Signed-FR(X \times X)$

1. $\ddot{u}_{\pm1}$ is reflexive if and only if $\ddot{u}_{\pm1}^{-1}$ is reflexive.
2. If $\ddot{u}_{\pm1}$ is reflexive, then $\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}$ is reflexive for each $\ddot{u}_{\pm2}$ in $Signed-FR(X\times X)$.
3. If $\ddot{u}_{\pm1}$ is reflexive, then $\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}$ is reflexive for each $\ddot{u}_{\pm2}$ in $Signed-FR(X \times X)$.

Proof.

1. Assume $\ddot{u}_{\pm1}$ is reflexive. This means that $\mu_{\ddot{u}_{\pm1}}(g,g)=1$ for all $g \in X$. we need to show that $\ddot{u}_{\pm1}^{-1}$ is reflexive. For $\ddot{u}_{\pm1}^{-1}$ to be reflexive. we require $\mu_{\ddot{u}_{\pm1}^{-1}}(g,g)=1$ for all $g \in X$. By the definition of the inverse Signed-FR, $\mu_{\ddot{u}_{\pm1}^{-1}}(g,g)=\mu_{\ddot{u}_{\pm1}}(g,g)$. since $\mu_{\ddot{u}_{\pm1}}(g,g)=1$ for all $g \in X$, it follows that $\mu_{\ddot{u}_{\pm1}^{-1}}(g,g)=1$ for all $g \in X$. Therefore $\ddot{u}_{\pm1}^{-1}$ is reflexive. Conversely, Assume $\ddot{u}_{\pm1}^{-1}$ is reflexive. This means $\mu_{\ddot{u}_{\pm1}^{-1}}(g,g)=1$ for all $g \in X$. we need to show that $\ddot{u}_{\pm1}$ is reflexive. For $\ddot{u}_{\pm1}$ to be reflexive, we require $\mu_{\ddot{u}_{\pm1}}(g,g)=1$ for all $g \in X$. By the definition of the inverse fuzzy relation $\mu_{\ddot{u}_{\pm1}}(g,g)=\mu_{\ddot{u}_{\pm1}^{-1}}(g,g)$. Since $\mu_{\ddot{u}_{\pm1}^{-1}}(g,g)=1$ for all $g \in X$, it follows that $\mu_{\ddot{u}_{\pm1}}(g,g)=1$ for all $g \in X$. Therefore, $\ddot{u}_{\pm1}$ is reflexive.
2. Given, $\ddot{u}_{\pm1}$ is a reflexive fuzzy relation on X . To show that $\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$ is reflexive for any Signed-FR $\ddot{u}_{\pm2}$ on X . Since $\ddot{u}_{\pm1}$ is reflexive, we have $\mu_{\ddot{u}_{\pm1}}(g,g)=1$ for all $g \in X$. we need to show that $\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$ is reflexive, we require $\mu_{\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}}(g,g)=1$ for all $g \in X$. By the definition of union of Signed-FRs, the membership grade of (g,g) in $\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$ is $\mu_{\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}}(g,g)=max(\mu_{\ddot{u}_{\pm1}}(g,g),\mu_{\ddot{u}_{\pm2}}(g,g))$. Since $\mu_{\ddot{u}_{\pm1}}(g,g)=1$ for all $g \in X$, we have $\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$ is $max(\mu_{\ddot{u}_{\pm1}}(g,g),\mu_{\ddot{u}_{\pm2}}(g,g))=max(\mu_{1,\mu_{\ddot{u}_{\pm2}}}(g,g))=1$ for all $g \in X$. Thus, $\mu_{\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}}(g,g)=1$ for all $g \in X$. Therefore $\ddot{u}_{\pm1} \cup \ddot{u}_{\pm2}$ is reflexive.
3. Let us assume $\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}$ is reflexive, then ψ is reflexive. Assume $\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}$ is reflexive. This means $\mu_{\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}}(g,g)=1$ for all $g \in X$. $\mu_{\ddot{u}_{\pm1} \cap \ddot{u}_{\pm2}}(g,g)=1$ for all $g \in X$. By the definition of the

intersection of Signed-FRs, we have $\mu_{\text{ü}_{\pm1}\cup\text{ü}_{\pm2}}(g,g) = min(\mu_{\text{ü}_{\pm1}}(g,g), \mu_{\text{ü}_{\pm2}})$. Since $\text{ü}_{\pm1}$ is reflexive, we know $\mu_{\text{ü}_{\pm1}\cup\text{ü}_{\pm2}}(g,g) = 1$ for all $g \in X$. Thus, $min(\mu_{\text{ü}_{\pm1}}, \mu_{\text{ü}_{\pm2}}) = 1$ for all $g \in X$. The only way for this equality to hold is if $\mu_{\text{ü}_{\pm2}}(g,g) = 1$ for all $g \in X$. Therefore $\text{ü}_{\pm2}$ is reflexive. Conversely, If $\text{ü}_{\pm2}$ is reflexive, then $\text{ü}_{\pm1} \cap \text{ü}_{\pm2}$ is reflexive. Assume $\text{ü}_{\pm2}$ is reflexive. This means $\mu_{\text{ü}_{\pm2}}(g,g) = 1$ for all $g \in X$. we need to show that $\text{ü}_{\pm1} \cap \text{ü}_{\pm2}$ is reflexive. For $\text{ü}_{\pm1} \cap \text{ü}_{\pm2}$ to be reflexive, we require $\mu_{\text{ü}_{\pm1}\cap\text{ü}_{\pm2}}(g,g) = 1$ for all $g \in X$. By the definition of the intersection of fuzzy relations, we have $\mu_{\text{ü}_{\pm1}\cap\text{ü}_{\pm2}}(g,g) = min(\mu_{\text{ü}_{\pm1}}(g,g), \mu_{\psi}(g,g))$. Since both $\text{ü}_{\pm1}$ and $\text{ü}_{\pm2}$ are reflexive, we have $\mu_{\text{ü}_{\pm1}}(g,g) = 1$ and $\mu_{\text{ü}_{\pm2}}(g,g) = 1$ for all $g \in X$. Thus $min(1,1) = 1$ for all $g \in X$. Therefore $\mu_{\text{ü}_{\pm1}\cap\text{ü}_{\pm2}}(g,g) = 1$ for all $g \in X$. Hence $\text{ü}_{\pm1} \cap \text{ü}_{\pm2}$ is reflexive.

Theorem 4.2.

Let $\text{ü}_{\pm1} \in Signed - FR(X)$.

1. $\text{ü}_{\pm1}$ is antireflexive if and only if $\text{ü}_{\pm1}^{-1}$ is antireflexive.
2. If $\text{ü}_{\pm1}$ is antireflexive, then $\text{ü}_{\pm1} \cup \text{ü}_{\pm2}$ is antireflexive if and only if $\psi \in Signed - FR(X)$ is antireflexive.
3. If $\text{ü}_{\pm1}$ is antireflexive, then $\text{ü}_{\pm1} \cap \text{ü}_{\pm2}$ is antireflexive for each ψ in $Signed - FR(X)$.

Proof.

1. Assume $\text{ü}_{\pm1}$ is antireflexive. This means that $\mu_{\text{ü}_{\pm1}}(g,g) = 0$ for all $g \in X$. we need to show that $\text{ü}_{\pm1}^{-1}$ is antireflexive. For $\text{ü}_{\pm1}^{-1}$ to be aanti-reflexive, we require $\mu_{\text{ü}_{\pm1}}^{-1}(g,g) = 0$ for all $g \in X$. By the definition of the inverse fuzzy relation, $\mu_{\text{ü}_{\pm1}^{-1}}(g,g) = \mu_{\text{ü}_{\pm1}}(g,g)$. Since $\mu_{\text{ü}_{\pm1}^{-1}}(g,g) = 0$ for all $g \in X$, it follows that $\mu_{\text{ü}_{\pm1}^{-1}}(g,g) = 0$ for all $g \in X$. Therefore $\text{ü}_{\pm1}^{-1}$ is anti-reflexive. Conversely, Assume $\text{ü}_{\pm1}^{-1}$ is antireflexive. This means $\mu_{\text{ü}_{\pm1}^{-1}}(g,g) = 0$ for all $g \in X$. We need to show that $\text{ü}_{\pm1}$ is antireflexive. For $\text{ü}_{\pm1}$ to be antireflexive, we require $\mu_{\text{ü}_{\pm1}^{-1}}(g,g) = 0$ for all $g \in X$. By the definition of inverse Signed-FR $\mu_{\text{ü}_{\pm1}}(g,g) = \mu_{\text{ü}_{\pm1}^{-1}}(g,g)$. Since $\mu_{\text{ü}_{\pm1}^{-1}}(g,g) = 0$ for all $g \in X$, it follows that $\mu_{\text{ü}_{\pm1}}(g,g) = 0$ for all $g \in X$. Therefore $\text{ü}_{\pm1}$ is antireflexive.
2. Let us assume $\text{ü}_{\pm1} \cup \text{ü}_{\pm2}$ is antireflexive. This means that $\mu_{\text{ü}_{\pm1}\cup\text{ü}_{\pm2}}(g,g) = 0$ for all $g \in X$. By the definition of the union of Signed-FRs, we have $\mu_{\text{ü}_{\pm1}\cup\text{ü}_{\pm2}}(g,g) = max(\mu_{\text{ü}_{\pm1}}(g,g), \mu_{\text{ü}_{\pm2}})(g,g)$. Since $\text{ü}_{\pm1}$ is antireflexive, we know $\mu_{\text{ü}_{\pm1}}(g,g) = 0$ for all $g \in X$. Thus $max(0, \mu_{\text{ü}_{\pm2}}(g,g)) = 0$ for all $g \in X$. The only way for for this equality to hold is if $\mu_{\psi}(g,g) = 0$ for all $g \in X$. Therefore $\text{ü}_{\pm2}$ is antireflexive. Conversely, assume $\text{ü}_{\pm2}$ is antireflexive. This means $\mu_{\text{ü}_{\pm2}}(g,g) = 0$ for all $g \in X$. We need to show that $\text{ü}_{\pm1} \cup \text{ü}_{\pm2}$ is antireflexive. For $\text{ü}_{\pm1} \cup \text{ü}_{\pm2}$ to be antireflexive, we require $\mu_{\text{ü}_{\pm1}\cup\text{ü}_{\pm2}}(g,g) = 0$ for all $g \in X$. By the definition of union of Signed-FRs, we have

$\mu_{ü_{\pm1} \cup ü_{\pm2}} = max(\mu_{ü_{\pm1}}(g,g), \mu_{ü_{\pm2}}(g,g))$. Since both $ü_{\pm1}$ and $ü_{\pm2}$ are reflexive, we have $\mu_{ü_{\pm1}}(g,g)=0$ and $\mu_{ü_{\pm2}}(g,g)=0$ for all $g \in X$. Thus $max(0,0)=0$ for all $g \in X$. Therefore $\mu_{ü_{\pm1} \cup ü_{\pm2}}(g,g)=0$ for all $g \in X$. Hence $ü_{\pm1} \cup ü_{\pm2}$ is antireflexive.

3. Since $ü_{\pm1}$ is antireflexive, we have $\mu_{ü_{\pm1}}(g,g)=0$ for all $g \in X$. We need to show that $ü_{\pm1} \cap ü_{\pm2}$ is antireflexive. For $ü_{\pm1} \cap ü_{\pm2}$ to be antireflexive. we require $\mu_{ü_{\pm1} \cap ü_{\pm2}}(g,g)=0$ for all $g \in X$. By the definition of the intersection of Signed-FRs, the membership grade of (g,g) in $ü_{\pm1} \cap ü_{\pm2}$ is $\mu_{ü_{\pm1} \cap ü_{\pm2}}(g,g) = min(\mu_{ü_{\pm1}}(g,g), ü_{\pm2}(g,g))$ for all $g \in X$. Since $\mu_{ü_{\pm1}}(g,g)=0$ for all $g \in X$, we have $min(0, \mu_{ü_{\pm2}}(g,g))=0$ for all $g \in X$. Thus $ü_{\pm1} \cap ü_{\pm2}$ is $\mu_{ü_{\pm1} \cap ü_{\pm2}}(g,g)=0$ for all $g \in X$. Therefore $ü_{\pm1} \cap ü_{\pm2}$ is antireflexive.

Definition 4.3.

If $ü_{\pm}$ in $X \times X$ is a Signed-FR, then $ü_{\pm}$ is symmetric if and only if $\mu_{ü_{\pm}}(g,h) = \mu_{ü_{\pm}}(h,g), \forall g,h \in X$.

Definition 4.4.

If $ü_{\pm}$ in $X \times X$ is a Signed-FR, then $ü_{\pm}$ is antisymmetric if and only if $\mu_{ü_{\pm}}(g,h) > 0$ then $\mu_{ü_{\pm}}(h,g) = 0, g,h \in X, g \neq h$.

Theorem 4.3.

Let $ü_{\pm}$ in $Signed-FR(X)$. Then $ü_{\pm}$ is symmetric if and only if $ü_{\pm} = ü_{\pm}^{-1}$.

Proof.

Assume $ü_{\pm}$ is symmetric. This means that $\mu_{ü_{\pm}}(g,h) = \mu_{ü_{\pm}}(h,g)$ for all $g,h \in X$. By inverse Signed-FR, we have $\mu_{ü_{\pm}}^{-1}(h,g) = \mu_{ü_{\pm}}(g,h)$ for all $g,h \in X$. Since $ü_{\pm}$ is symmetric, it follows that $\mu_{ü_{\pm}}(g,h) = \mu_{ü_{\pm}}(h,g)$ for all $g,h \in X$. Therefore $\mu_{ü_{\pm}}^{-1}(g,h) = \mu_{ü_{\pm}}(g,h)$ for all $g,h \in X$. Hence $ü_{\pm} = ü_{\pm}^{-1}$.

Conversely, assume $ü_{\pm} = ü_{\pm}^{-1}$. This means that $\mu_{ü_{\pm}}(g,h) = \mu_{ü_{\pm}^{-1}}(g,h)$ for all $g,h \in X$. By inverse Signed-FR, we have $\mu_{ü_{\pm}^{-1}}(g,h) = \mu_{ü_{\pm}}(g,h)$ for all $g,h \in X$. Since $ü_{\pm} = ü_{\pm}^{-1}$, it follows that $\mu_{ü_{\pm}}(g,h) = \mu_{ü_{\pm}}(h,g)$ for all $g,h \in X$. Therefore $ü_{\pm}$ is symmetric.

Theorem 4.4.

Let $ü_{\pm1}$, $ü_{\pm2}$ in Signed-FR(X). Then $ü_{\pm1}$ and $ü_{\pm2}$ are symmetric, then $ü_{\pm1} \cup ü_{\pm2}$ and $ü_{\pm1} \cap ü_{\pm2}$ are symmetric.

Proof.

Given $\text{ü}_{\pm 1}$ and $\text{ü}_{\pm 2}$ are symmetric Signed-FRs on X. To show that $\text{ü}_{\pm 1} \cup \text{ü}_{\pm 2}$ is symmetric, we need to show that for all $g, h \in X$, $\mu_{\text{ü}_{\pm 1} \cup \text{ü}_{\pm 2}}(g,h) = \mu_{\text{ü}_{\pm 1} \cup \text{ü}_{\pm 2}}(h,g)$. Since $\text{ü}_{\pm 1}$ and $\text{ü}_{\pm 2}$ are symmetric, we know $\mu_{\text{ü}_{\pm 1}}(g,h) = \mu_{\text{ü}_{\pm 1}}(h,g)$ and $\mu_{\text{ü}_{\pm 2}}(g,h) = \mu_{\text{ü}_{\pm 2}}(h,g)$ for all $g, h \in X$. Therefore $\text{ü}_{\pm 1} \cup \text{ü}_{\pm 2}$ is symmetric. Similarly, we can show that $\text{ü}_{\pm 1} \cap \text{ü}_{\pm 2}$ is symmetric.

Definition 4.5.

If $\text{ü}_{\pm}$ in $X \times X$ is a Signed-FR, then $\text{ü}_{\pm}$ is transitive in the sense of max-min if and only if $\mu_{\text{ü}_{\pm}}(g,i) \geq \bigvee_{h \in X} \left[\mu_{\text{ü}_{\pm}}(g,h) \wedge \mu_{\text{ü}_{\pm}}(h,i) \right], g, h, i \in X$.

Theorem 4.5.

Let $\text{ü}_{\pm}$ in $Signed - FR(X)$. If $\text{ü}_{\pm}$ is transitive, then $\text{ü}_{\pm}^{-1}$ is also transitive.

Proof.

Given $\text{ü}_{\pm}$ is transitive Signed-FR X. This means that $\mu_{\text{ü}_{\pm}}(g,i) \geq \bigvee_{h \in X} \left[\mu_{\text{ü}_{\pm}}(g,h) \wedge \mu_{\text{ü}_{\pm}}(h,i) \right]$, $g, h, i \in X$. To show that $\text{ü}_{\pm}^{-1}$ is transitive. We need to show $\mu_{\text{ü}_{\pm}^{-1}}(g,i) \geq \bigvee_{h \in X} \left[\mu_{\text{ü}_{\pm}^{-1}}(g,h) \wedge \mu_{\text{ü}_{\pm}^{-1}}(h,i) \right]$, $g, h, i \in X$. By inverse Signed-FR, we have $\mu_{\text{ü}_{\pm}^{-1}}(g,h) = \mu_{\text{ü}_{\pm}}(h,g)$ for all $g, h \in X$. We get $\mu_{\text{ü}_{\pm}}(g,i) \geq \bigvee_{h \in X} \left[\mu_{\text{ü}_{\pm}}(g,h) \wedge \mu_{\text{ü}_{\pm}}(h,i) \right]$, $g, h, i \in X$. Therefore $\text{ü}_{\pm}^{-1}$ is transitive.

Theorem 4.6.

Let $\text{ü}_{\pm}$ in $Signed - FR(X)$. If $\text{ü}_{\pm}$ is transitive, then $\text{ü}_{\pm}^{2}$ is transitive.

Theorem 4.7.

Let $\text{ü}_{\pm}$ and $\text{ü}_{\pm 2}$ in $Signed - FR(X)$. If $\text{ü}_{\pm}$ and $\text{ü}_{\pm 2}$ are transitive, then $\text{ü}_{\pm} \cap \text{ü}_{\pm 2}$ is transitive.

Definition 4.6.

A fuzzy relation $\text{ü}_{\pm} \subseteq X \times X$ is referred to as a "fuzzy equivalence relation" or "similarity relation" if it meets the specified requirements.

1. $\mu_{\pm}(g,g) = \pm 1, \forall g \in X$. (Reflexive Relation)
2. $\mu_{\text{ü}_{\pm}}(g,h) = \mu_{\text{ü}_{\pm}}(h,g), \forall g, h \in X$. (Symmetric Relation)
3. $\mu_{\text{ü}_{\pm}}(g,i) \geq \bigvee_{h \in X} \left[\mu_{\text{ü}_{\pm}}(g,h) \wedge \mu_{\text{ü}_{\pm}}(h,i) \right], g, h, i \in X$. (Transitive Relation)

Consequently, the matrix is a signed fuzzy equivalency relation. In this case, the symmetry of the matrix across the main diagonal indicates a symmetric relation. Moreover, because each element has a degree of membership of 1, it demonstrates reflexivity. The transitive attribute is likewise satisfied by the composition of relations.

Definition 4.7.

A Signed-FR $\Im_{\pm 1}$ defined on X. For any $\alpha \in [-1,1]$ the α-cut $\Im_{\pm 1}^{\alpha}$ is given by $\Im_{\pm 1}^{\alpha} = \left\{ g \in X \,/\, \Im_{\pm 1}(g) \geq \alpha \right\}$.

Definition 4.8.

A Signed-FR $\Im_{\pm 1}$ defined on X. For any $\alpha \in [-1,1]$ the strong $\alpha - cut \Im_{\pm 1}^{\alpha +}$ is given by $\Im_{\pm 1}^{\alpha +} = \left\{ g \in X \,/\, \Im_{\pm 1}(g) > \alpha \right\}$.

Definition 4.9.

A Signed-FR is convex if its α cuts are convex for all $\alpha \in [-1,1]$. That is, $g_1, g_2 \in \Im_{\pm 1}^{\alpha}$, then $\lambda g_1 + (1-\lambda) g_2 \in \Im_{\pm 1}^{\alpha}$ for $\lambda \in [-1,1]$.

Theorem 4.8.

A Signed-FR $\Im_{\pm 1}$ on $\ddot{u}_{\pm}$ is convex if and only if $\Im_{\pm 1}\left(\lambda g_1 + (1-\lambda) g_2 \geq min\left\{ \Im_{\pm 1}(g_1), \Im_{\pm 1}(g_2) \right\} \right)$, for all $g_1, g_2 \in \ddot{u}_{\pm}$ and $\lambda \in [-1,1]$.

Proof.

Let $\Im_{\pm 1}$ represents a Signed-FR on $\ddot{u}_{\pm}$. Assume that $\Im_{\pm 1}$ is convex, meaning its α cuts are also convex, and $\Im_{\pm 1}^{\alpha}$ is convex. If $g_1, g_2 \in \Im_{\pm 1}^{\alpha}$, then $\lambda g_1 + (1-\lambda) g_2 \in \Im_{\pm 1}^{\alpha}$ for $\lambda \in [-1,1]$. Now consider $\Im_{\pm 1}(g_1) \leq \Im_{\pm 1}(g_2)$. Let $\alpha = \Im_{\pm 1}(g_1)$. $\Im_{\pm 1}(\lambda g_1 + (1-\lambda) g_2) \geq \alpha = \Im_{\pm 1}(g_1) = min[\Im_{\pm 1}(g_1), \Im_{\pm 1}(g_2)]$. Hence we have $\Im_{\pm 1}(\lambda g_1 + (1-\lambda) g_2) \geq min[\Im_{\pm 1}(g_1), \Im_{\pm 1}(g_2)]$, for all $g_1, g_2 \in \ddot{u}_{\pm}$ and $\lambda \in [-1,1]$.

Conversely, suppose a Signed-FR satisfies $\Im_{\pm 1}(\lambda g_1 + (1-\lambda) g_2) \geq min[\Im_{\pm 1}(g_1), \Im_{\pm 1}(g_2)]$, forall $g_1, g_2 \in \ddot{u}_{\pm}$ and $\lambda \in [-1,1]$. To show that $\Im_{\pm 1}$ is convex, meaning $\Im_{\pm 1}^{\alpha}$ is convex. Let us take $g_1, g_2 \in \Im_{\pm 1}^{\alpha}$, implying $\lambda g_1 + (1-\lambda) g_2 \in \Im_{\pm 1}^{\alpha}$. Since $\Im_{\pm 1}(g_1) \geq \alpha$ and $\Im_{\pm 1}(g_2) \geq \alpha$. we have $\Im_{\pm 1}(\lambda g_1 + (1-\lambda) g_2) \geq min(\Im_{\pm 1}(g_1), \Im_{\pm 1}(g_2)) \geq min(\alpha, \alpha) = \alpha$. Thus, if $g_1, g_2 \in \Im_{\pm 1}^{\alpha}$ then $\lambda g_1 + (1-\lambda) g_2 \in \Im_{\pm 1}^{\alpha}$. Therefore $\Im_{\pm 1}^{\alpha}$ is convex. which implies that $\Im_{\pm 1}$ is convex.

Theorem 4.9.

Let $\Im_{\pm 1}$, $\Im_{\pm 2}$ be a signed fuzzy set X. Then the following properties holds for all $\alpha, \beta \in \left[-1,1\right]$

1. $\Im_{\pm 1}^{\alpha +} \subseteq \Im_{\pm 1}^{\alpha}$.
2. $\alpha \leq \beta$, then $\Im_{\pm 1}^{\beta} \subseteq \Im_{\pm 1}^{\alpha}$ and $\Im_{\pm 1}^{\beta +} \subseteq \Im_{\pm 1}^{\alpha +}$.
3. $\left(\Im_{\pm 1} \cap \Im_{\pm 2} \right)^{\alpha} = \Im_{\pm 1}^{\alpha} \cap \Im_{\pm 2}^{\alpha}$ and $\left(\Im_{\pm 1} \cup \Im_{\pm 2} \right)^{\alpha} = \Im_{\pm 1}^{\alpha} \cup \Im_{\pm 2}^{\alpha}$.
4. $\left(\Im_{\pm 1} \cap \Im_{\pm 2} \right)^{\alpha +} = \Im_{\pm 1}^{\alpha +} \cap \Im_{\pm 2}^{\alpha +}$ and $\left(\Im_{\pm 1} \cup \Im_{\pm 2} \right)^{\alpha +} = \Im_{\pm 1}^{\alpha +} \cup \Im_{\pm 2}^{\alpha +}$.

Proof.

1. If $g \in \Im_{\pm1}^{\alpha+}$, which implies $\Im_{\pm1}(g) > \alpha$, then $g \in \Im_{\pm1}^{\alpha+}$. Hence $\Im_{\pm1}^{\alpha+} \subseteq \Im_{\pm1}^{\alpha}$.
2. If $\alpha \le \beta$, then $\Im_{\pm1}^{\beta} \subseteq \Im_{\pm1}^{\alpha}$. Let $\alpha \le \beta \le \Im_{\pm1}(g)$, which implies $\alpha \le \Im_{\pm1}(g)$. Therefore $g \in \Im_{\pm1}^{\alpha}$. Hence $\Im_{\pm1}^{\beta} \subseteq \Im_{\pm1}^{\alpha}$. Similarly $\Im_{\pm1}^{\beta+} \subseteq \Im_{\pm1}^{\alpha+}$.
3. If $g \in \left(\Im_{\pm1} \cap \Im_{\pm2}\right)^{\alpha}$, then $\left(\Im_{\pm1} \cap \Im_{\pm2}\right)(g) \ge \alpha$. Therefore, $min\left[\Im_{\pm1}(g), \Im_{\pm2}(g)\right] \ge \alpha$. This implies $\Im_{\pm1}(g) \ge \alpha$ and $\Im_{\pm2}(g) \ge \alpha$. Hence, $g \in \Im_{\pm1}^{\alpha}$ and $g \in \Im_{\pm2}^{\alpha}$. Therefore, $g \in \Im_{\pm1}^{\alpha} \cap \Im_{\pm2}^{\alpha}$. Hence, $\left(\Im_{\pm1} \cap \Im_{\pm2}\right)^{\alpha} \subseteq \Im_{\pm1}^{\alpha} \cap \Im_{\pm2}^{\alpha}$. Similarly, Let $g \in \Im_{\pm1}^{\alpha} \cap \Im_{\pm2}^{\alpha}$, which implies $g \in \Im_{\pm1}^{\alpha}$ and $g \in \Im_{\pm2}^{\alpha}$. Then, $\Im_{\pm1}^{\alpha}(g) \ge \alpha$ and $\Im_{\pm1}^{\alpha}(g) \ge \beta$. Therefore, $min\left[\Im_{\pm1}(g), \Im_{\pm2}(g)\right] \ge \alpha$, so $\left(\zeta \cap \vartheta\right)(g) \ge \alpha$. Thus $g \in \left(\Im_{\pm1} \cap \Im_{\pm2}\right)^{\alpha}$. Hence, $\Im_{\pm1}^{\alpha} \cap \Im_{\pm2}^{\alpha} \subseteq \left(\Im_{\pm1} \cap \Im_{\pm2}\right)^{\alpha}$. Therefore, $\left(\Im_{\pm1} \cap \Im_{\pm2}\right)^{\alpha} = \Im_{\pm1}^{\alpha} \cap \Im_{\pm2}^{\alpha}$. Similarly $\left(\Im_{\pm1} \cup \Im_{\pm2}\right)^{\alpha} = \Im_{\pm1}^{\alpha} \cup \Im_{\pm2}^{\alpha}$.
4. Similar to (3).

6. Conclusion

This chapter provides an example and definition of a signed fuzzy set. Examples of signed fuzzy sets' complement, union, intersection, and composition are provided, and the outcomes are examined. Examples of Signed-FRs are shown, and the outcomes are examined. There is a discussion of the definitions of transitive, symmetric, reflexive, and irreflexive relations. Results are also presented for alpha-cuts and strong alpha-cuts of Signed-FRs.

This idea can be expanded into neutrosophic sets, plithogenic sets, and interval-valued Signed-FRs. Numerous real-time applications can make use of it.

7. Acknowledgment

The authors express gratitude to the editor and anonymous referees for reviewing this manuscript and providing helpful suggestions and comments

8. References

[1] Zadeh, Lotfi A, *Fuzzy sets*, (Information and control, 1965), pp. 338–353.

[2] Zadeh, Lotfi A, *Similarity relations and fuzzy orderings*, (Information sciences, 1971), pp. 177–200.

[3] Atanassov, Krassimir, *Intuitionistic fuzzy sets*, (Fuzzy Sets and Systems, 1986), pp. 87–96.

[4] Gorza lczany, Marian B, *A method of inference in approximate reasoning based on interval-valued fuzzy sets*, (Fuzzy Sets and Systems, 1987), pp. 1–17.

[5] Dib, KA and Youssef, Nabil, L, *Fuzzy Cartesian product, fuzzy relations and fuzzy functions*, (Fuzzy Sets and Systems, 1991), pp. 299–315.

[6] Kaufman, Arnold and Gupta, Madan M, *Introduction to fuzzy arithmetic, 1991*.

[7] Zhang, W-R, *Bipolar fuzzy sets*, (IEEE international conference on fuzzy systems proceedings, 1998), pp. 835–840.

[8] Zhang, Wen-Ran, *Bipolar fuzzy sets and relations: a computational framework for cognitive modelling and multiagent decision analysis*, (Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intelligence, 1994), pp. 305–309.

[9] Zhang, Wen-Ran and Zhang, Lulu, *YinYang bipolar logic and bipolar fuzzy logic*, (Information Sciences, 2004), pp. 265–287.

[10] Lee, Jeong-Gon and Hur, Kul, *Bipolar fuzzy relations*, (Mathematics, 2019).