Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

A New Insight with Trigonometric Coefficients of Additive-Quadratic Functional Equations and its Stability Analysis

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract: This study introduces a novel framework for analyzing the Ulam-Hyers stability of mixed-type additive-quadratic functional equations with trigonometric constant coefficients in Banach spaces. Employing advanced analytical techniques and leveraging the unique properties of trigonometric functions, we derive sufficient conditions for the stability of these equations. The intricate relationship between additive and quadratic components is rigorously examined, emphasizing the pivotal role of trigonometric coefficients in influencing stability behavior. Our results provide fresh insights into the structural stability of functional equations and broaden the scope of existing stability theories. This work lays the groundwork for future research on mixed-type functional equations in both theoretical and applied mathematical contexts

Keywords: Additive, quadratic functional equations, generalized Hyers - Ulam - Rassias stability

1. Introduction

The study of functional equations and their stability has been a fundamental aspect of mathematical analysis for decades. The concept of stability in functional equations originated with Stanisław Ulam in 1940 [1], who posed the question of whether an approximate solution to a functional equation could be approximated by an exact solution. In 1941, Donald Hyers [2] provided the first affirmative answer to Ulam's question, establishing the stability of linear functional equations. This foundational result, now known as Ulam-Hyers stability, has since been generalized to a wide range of functional equations [3, 4, 5], including quadratic, cubic, and mixed-type equations. Mixed-type functional equations, which integrate distinct mathematical structures such as additive and quadratic components, have garnered significant attention due to their applications in fields like physics, economics, and engineering. The inclusion of trigonometric coefficients introduces additional complexity and depth to the analysis, as the inherent periodicity and symmetry of trigonometric functions play a crucial role in shaping stability properties. Despite their theoretical and practical significance, the stability of mixed-type functional equations with trigonometric coefficients remains a relatively underexplored area of research [6, 7, 8].

The study of Ulam-Hyers stability has seen significant advancements, emerging as a crucial area of research in functional analysis and its applications [9, 10, 11, 12, 13]. Functional equations with mixed structures, such as additive-quadratic forms, present unique challenges and opportunities for mathematical investigation. These equations naturally arise in various fields, modeling systems where linear and nonlinear behaviors interact. This paper focuses on a novel class of mixed-type additive-quadratic functional equations featuring trigonometric constant coefficients. The incorporation of trigonometric terms introduces distinctive properties, making the stability analysis both complex and fascinating. Trigonometric coefficients are not only mathematically significant but also hold practical relevance in modeling periodic and oscillatory phenomena, such as wave functions and signal processing. The primary objective of this study is to establish the Ulam-Hyers stability of these functional equations within the framework of Banach spaces. By employing advanced techniques in functional analysis and leveraging the inherent properties of trigonometric coefficients, this work offers new insights into the stability of such equations. Furthermore, this research enhances the broader understanding of how trigonometric factors influence the stability of mixed-type functional equations, paving the way for further theoretical developments and practical applications. Recently, Agilan et al. have explored stability results for various additive functional equations across different normed spaces, as evidenced in [14, 15, 16, 17, 18, 19, 20, 21,22].

Through these motivations, the paper aims to deepen the understanding of the stability properties of functional equations in Banach spaces and to demonstrate the effectiveness of combining direct and fixed point methods in such analyses. Authors have proved the generalized Ulam - Hyers stability of a mixed type general additive quadratic functional equation

$$\begin{aligned} &\mathcal{Q}_1\left(\mathcal{X}cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)+2\mathcal{Y}sec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)+\mathcal{Q}_1\left(\mathcal{Y}sec^{\mathcal{L}}\left(\frac{\pi}{4}\right)-\mathcal{X}cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right) \\ &\quad+\mathcal{Q}_1\left(\mathcal{X}cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)-\mathcal{Y}sec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right) \\ &=\left(\frac{cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)+3cosec^{2\mathcal{L}}\left(\frac{\pi}{4}\right)}{2}\right)\mathcal{Q}_1(\mathcal{X})+\left(\frac{3cosec^{2\mathcal{L}}\left(\frac{\pi}{4}\right)-cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)}{2}\right)\mathcal{Q}_1(-\mathcal{X}) \\ &\quad+\left(sec^{\mathcal{L}}\left(\frac{\pi}{4}\right)+3sec^{2\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{Q}_1(\mathcal{Y})+\left(3sec^{2\mathcal{L}}\left(\frac{\pi}{4}\right)-sec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{Q}_1(-\mathcal{Y}) \end{aligned} \tag{1}$$

in Banach spaces.

let us consider $\mathcal{X}$ and $\mathcal{Y}$ to be a normed space and a Banach space, respectively. Define a mapping $\mathcal{Q}:\mathcal{H}\rightarrow\mathcal{J}$ by

$$\begin{aligned} \mathcal{Q}(\mathcal{X},\mathcal{Y})=&\mathcal{Q}_1\left(\mathcal{X}cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)+2\mathcal{Y}sec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)+\mathcal{Q}_1\left(\mathcal{Y}sec^{\mathcal{L}}\left(\frac{\pi}{4}\right)-\mathcal{X}cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right) \\ &+\mathcal{Q}_1\left(\mathcal{X}cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)-\mathcal{Y}sec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right) \\ &-\left(\frac{cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)+3cosec^{2\mathcal{L}}\left(\frac{\pi}{4}\right)}{2}\right)\mathcal{Q}_1(\mathcal{X})-\left(\frac{3cosec^{2\mathcal{L}}\left(\frac{\pi}{4}\right)-cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)}{2}\right)\mathcal{Q}_1(-\mathcal{X}) \\ &-\left(sec^{\mathcal{L}}\left(\frac{\pi}{4}\right)+3sec^{2\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{Q}_1(\mathcal{Y})-\left(3sec^{2\mathcal{L}}\left(\frac{\pi}{4}\right)-sec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{Q}_1(-\mathcal{Y}) \end{aligned}$$

for all $\mathcal{X},\mathcal{Y}\in\mathcal{H}$.

2. Stability Results: Odd case

Theorem 2.1 Let $\mathcal{T}: \mathcal{X}^2 \to [0,\infty)$ be a function such that

$$\sum_{\mathcal{U}=0}^{\infty} \frac{\mathcal{T}\left(\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}} \mathcal{X}, \left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}} \mathcal{Y}\right)}{\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}}} \text{ converges in } \mathcal{R} \quad \text{and}$$

$$\lim_{\mathcal{U}\to\infty} \frac{\mathcal{T}\left(\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}} \mathcal{X}, \left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}} \mathcal{Y}\right)}{\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}}} = 0 \tag{1}$$

for all $\mathcal{X}, \mathcal{Y} \in \mathcal{H}$. Let $\mathcal{Q}_a: \mathcal{H} \to \mathcal{I}$ be an odd function satisfying the inequality

$$\|\mathcal{Q}_a(\mathcal{X}, \mathcal{Y})\| \leq \mathcal{T}(\mathcal{X}, \mathcal{Y}) \tag{2}$$

for all $\mathcal{X}, \mathcal{Y} \in \mathcal{H}$. Then there exists a unique additive mapping $A: \mathcal{H} \to \mathcal{I}$ such that

$$\|\mathcal{Q}_a(\mathcal{X}) - A(\mathcal{X})\| \leq \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)} \sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}} \mathcal{X}, 0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}} \tag{3}$$

for all $\mathcal{X} \in \mathcal{H}$. The mapping $A(\mathcal{X})$ is defined by

$$A(\mathcal{X}) = \lim_{\mathcal{U}\to\infty} \frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}} \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}}} \tag{4}$$

for all $\mathcal{X} \in \mathcal{H}$.

Proof. Replacing $(\mathcal{X}, \mathcal{Y})$ by $(\mathcal{X}, 0)$ in (2) and using oddness of $\mathcal{Q}_a$, we get

$$\left\|\mathcal{Q}_a(\mathcal{X}) - \frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)}\right\| \leq \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)} \mathcal{T}(\mathcal{X}, 0) \tag{5}$$

for all$\mathcal{X} \in \mathcal{H}$. Now replacing $\mathcal{X}$ by $\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\mathcal{X}\right)$ and dividing by $\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)$ in (5), we obtain

$$\left\|\frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)} - \frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2 \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\right\| \leq \frac{\mathcal{T}\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X}, 0\right)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2} \tag{6}$$

for all$\mathcal{X} \in \mathcal{H}$. It follows from (5) and (6) that

$$\left\|\mathcal{Q}_a(\mathcal{X}) - \frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2 \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\right\| \leq \left\|\mathcal{Q}_a(\mathcal{X}) - \frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)}\right\| \tag{7}$$

$$+ \left\|\frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)} - \frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2 \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\right\|$$

$$\leq \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)}\left[\mathcal{T}(\mathcal{X}, 0) + \frac{\mathcal{T}\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X}, 0\right)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)}\right] \tag{8}$$

for all$\mathcal{X} \in \mathcal{H}$. In general for any positive integer N , we get

$$\left\|\mathcal{Q}_a(\mathcal{X}) - \frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N}\right\| \leq \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)} \sum_{\mathcal{H}=0}^{\mathcal{U}-1} \frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}} \mathcal{X}, 0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}}$$

$$\leq \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)} \sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}} \mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}} \tag{9}$$

for all$\mathcal{X} \in \mathcal{H}$. In order to prove the convergence of the sequence

$$\left\{\frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}}\right\},$$

replace $\mathcal{X}$ by $\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{M} \mathcal{X}$ and divide by $\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{M}$ in (1), for any $M, N > 0$, to deduce

$$\left\|\frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{M} \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{M}} - \frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N+M} \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{(N+M)}}\right\|$$

$$= \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{M}} \left\|\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{M} \mathcal{X}) - \frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \cdot \left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{M} \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}}\right\|$$

$$\leq \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)} \sum_{\mathcal{H}=0}^{\mathcal{U}-1} \frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}+M} \mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}+M}}$$

$$\leq \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)} \sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}+M} \mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}+M}}$$

$$\to 0 \quad as \quad M \to \infty$$

for all $\mathcal{X} \in \mathcal{H}$. Hence the sequence $\left\{\frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}}\right\}$ is a Cauchy sequence. Since $\mathcal{J}$ is complete, there exists a mapping $A: \mathcal{H} \to \mathcal{J}$ such that

$$A(\mathcal{X}) = \lim_{\mathcal{U}\to\infty} \frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}} \quad \forall \ \mathcal{X} \in \mathcal{H}.$$

Letting $N \to \infty$ in (1) we see that (3) holds for all$\mathcal{X} \in \mathcal{H}$. To prove A satisfies (1), replacing $(\mathcal{X}, \mathcal{Y})$ by $(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X}, \left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{Y})$ and dividing by $\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}$ in (2), we obtain

$$\frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}} \| \mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X}, \left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{Y}) \|$$

$$\leq \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}} \mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X}, \left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{Y})$$

for all $\mathcal{X}, \mathcal{Y} \in \mathcal{H}$. Letting $N \to \infty$ in the above inequality and using the definition of $A(\mathcal{X})$, we see that

$$\mathcal{Q}_a(\mathcal{X}, \mathcal{Y}) = 0.$$

Hence A satisfies (1) for all $\mathcal{X}, \mathcal{Y} \in \mathcal{H}$. To show A is unique, let $B(\mathcal{X})$ be another additive mapping satisfying (1) and (3), then

$$\|A(\mathcal{X}) - B(\mathcal{X})\| = \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}} \left\| A\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X}\right) - B\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X}\right) \right\|$$

$$\leq \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}} \left\{ \left\| A\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X}\right) - \mathcal{Q}_a\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X}\right) \right\| \right.$$

$$\left. + \left\| \mathcal{Q}_a\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X}\right) - B\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N} \mathcal{X}\right) \right\| \right\}$$

$$\leq \frac{2}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)} \sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{T}\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}+N} \mathcal{X}, 0\right)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{(\mathcal{H}+N)}}$$

$$\to 0 \quad as \quad N \to \infty$$

for all $\mathcal{X} \in \mathcal{H}$. Hence A is unique.

Corollary 2.2 Let $\mathcal{T}$ and P be non negative real numbers. Let an odd function $\mathcal{Q}_{\mathrm{a}}: \mathcal{H} \to \mathcal{J}$ satisfy the inequality

$$\|\mathcal{Q}_{\mathrm{a}}(\mathcal{X}, \mathcal{Y})\| \leq \begin{cases} \mathcal{T}, & \\ \mathcal{T}\{||\mathcal{X}||^{\mathrm{P}} + ||\mathcal{Y}||^{\mathrm{P}}\}, & \mathrm{P} \neq 1; \\ \mathcal{T}\left\{||\mathcal{X}||^{\mathrm{P}}||\mathcal{Y}||^{\mathrm{P}} + \{||\mathcal{X}||^{2\mathrm{P}} + ||\mathcal{Y}||^{2\mathrm{P}}\}\right\}, & \mathrm{P} \neq \frac{1}{2}; \end{cases} \tag{10}$$

for all $\mathcal{X}, \mathcal{Y} \in \mathcal{H}$. Then there exists a unique additive function $\mathrm{A}: \mathcal{H} \to \mathcal{J}$ such that

$$\|\mathcal{Q}_1(\mathcal{X}) - \mathrm{A}(\mathcal{X})\| \leq \begin{cases} \dfrac{\mathcal{T}}{\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right) - 1}, \\ \dfrac{\mathcal{T}||\mathcal{X}||^{\mathrm{P}}}{\left|\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right) - \left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathrm{P}}\right|}, \\ \dfrac{\mathcal{T}||\mathcal{X}||^{2\mathrm{P}}}{\left|\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right) - \left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathrm{P}}\right|}, \end{cases} \tag{11}$$

for all $\mathcal{X} \in \mathcal{H}$

3.Stability Results: Even Case

Theorem 3.1 Let $\mathcal{T}: \mathrm{X}^2 \to [0, \infty)$ be a function such that

$$\sum_{\mathcal{U}=0}^{\infty} \frac{\mathcal{T}\left(\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}} \mathcal{X}, \left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}} \mathcal{Y}\right)}{\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathrm{N}}} \quad \text{converges in } \mathcal{R} \quad \text{and}$$

$$\lim_{\mathcal{U} \to \infty} \frac{\mathcal{T}\left(\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}} \mathcal{X}, \left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}} \mathcal{Y}\right)}{\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathrm{N}}} = 0 \tag{12}$$

for all $\mathcal{X}, \mathcal{Y} \in \mathcal{H}$. Let $\mathcal{Q}_q: \mathcal{H} \to \mathcal{J}$ be an even function satisfying the inequality

$$\|Q(\mathcal{X}, \mathcal{Y})\| \leq \mathcal{T}(\mathcal{X}, \mathcal{Y}) \tag{13}$$

for all $\mathcal{X}, \mathcal{Y} \in \mathcal{H}$. Then there exists a unique Quadratic mapping $Q: \mathcal{H} \to \mathcal{J}$ such that

$$\left\|\mathcal{Q}_q(\mathcal{X})-Q(\mathcal{X})\right\| \leq \frac{1}{3\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2} \sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}} \mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathcal{H}}} \tag{14}$$

for all $\mathcal{X} \in \mathcal{H}$. The mapping $A(\mathcal{X})$ is defined by

$$Q(\mathcal{X}) = \lim_{\mathcal{U}\to\infty} \frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{U}} \mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2N}} \tag{15}$$

for all $\mathcal{X} \in \mathcal{H}$.

Proof. Replacing $(\mathcal{X},\mathcal{Y})$ by $(\mathcal{X},0)$ in (13) and using evenness of $\mathcal{Q}_q$, we get

$$\left\|\mathcal{Q}_q(\mathcal{X}) - \frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\right\| \leq \frac{1}{3\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\mathcal{T}(\mathcal{X},0) \tag{16}$$

for all$\mathcal{X} \in \mathcal{H}$. Now replacing $\mathcal{X}$ by $\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X}$ and dividing by $\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2$ in (16), we obtain

$$\left\|\frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2} - \frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^4}\right\| \leq \frac{\mathcal{T}\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X},0\right)}{3\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^4} \tag{17}$$

for all$\mathcal{X} \in \mathcal{H}$. It follows from (16) and (17) that

$$\left\|\mathcal{Q}_q(\mathcal{X}) - \frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^4}\right\| \leq \left\|\mathcal{Q}_q(\mathcal{X}) - \frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\right\|$$

$$+\left\|\frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2} - \frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^4}\right\|$$

$$\leq \frac{1}{3\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\left[\mathcal{T}(\mathcal{X},0) + \frac{\mathcal{T}\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)\mathcal{X},0\right)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\right] \tag{18}$$

for all$\mathcal{X} \in \mathcal{H}$. In general for any positive integer N , we get

$$\left\|\mathcal{Q}_q(\mathcal{X}) - \frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2N}}\right\| \leq \frac{1}{3\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\sum_{\mathcal{H}=0}^{\mathcal{U}-1} \frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathcal{H}}}$$

$$\leq \frac{1}{3\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathcal{H}}} \tag{19}$$

for all$\mathcal{X} \in \mathcal{H}$. In order to prove the convergence of the sequence

$$\left\{\frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2N}}\right\},$$

replace $\mathcal{X}$ by $\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2M}\mathcal{X}$ and divide by $\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2M}$ in (1), for any $M,N>0$, to deduce

$$\left\|\frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^M x)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2M}}-\frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N+M}\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2(N+M)}}\right\|$$

$$=\frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2M}}\left\|\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^M\mathcal{X})-\frac{\mathcal{Q}_a(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N\cdot\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^M\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2N}}\right\|$$

$$\leq\frac{1}{3\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\sum_{\mathcal{H}=0}^{\mathcal{U}-1}\frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}+M}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2(\mathcal{H}+M)}}$$

$$\leq\frac{1}{3\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\sum_{\mathcal{H}=0}^{\infty}\frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}+M}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2(\mathcal{H}+M)}}$$

$$\to 0 \quad as \quad M\to\infty$$

for all $\mathcal{X}\in\mathcal{H}$. Hence the sequence $\left\{\frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2N}}\right\}$ is a Cauchy sequence. Since $\mathcal{I}$ is complete, there exists a mapping $Q:\mathcal{H}\to\mathcal{I}$ such that

$$\mathcal{Q}_q(\mathcal{X})=\lim_{\mathcal{U}\to\infty}\frac{\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N\mathcal{X})}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2N}}\quad\forall\ \mathcal{X}\in\mathcal{H}.$$

Letting $N\to\infty$ in (1) we see that (14) holds for all$\mathcal{X}\in\mathcal{H}$. To prove Q satisfies (1), replacing $(\mathcal{X},\mathcal{Y})$ by $(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N\mathcal{X},\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N\mathcal{Y})$ and dividing by $\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2N}$ in (13), we obtain

$$\frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2N}}\parallel\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N\mathcal{X},\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N\mathcal{Y})\parallel$$

$$\leq\frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2N}}\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N\mathcal{X},\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^N\mathcal{Y})$$

for all $\mathcal{X},\mathcal{Y}\in\mathcal{H}$. Letting $N\to\infty$ in the above inequality and using the definition of $Q(\mathcal{X})$, we see that

$$\mathcal{Q}_q(\mathcal{X},\mathcal{Y})=0.$$

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

Hence Q satisfies (1) for all $\mathcal{X},\mathcal{Y} \in \mathcal{H}$. To show Q is unique, let $B(\mathcal{X})$ be another quadratic mapping satisfying (1) and (14), then

$$\|A(\mathcal{X})-B(\mathcal{X})\| = \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2N}}\left\|A(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}\mathcal{X}) - B(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}\mathcal{X})\right\|$$

$$\le \frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2N}}\left\{\left\|A(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}\mathcal{X}) - \mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}\mathcal{X})\right\|\right.$$

$$\left. + \left\|\mathcal{Q}_q(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}\mathcal{X}) - B(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{N}\mathcal{X})\right\|\right\}$$

$$\le \frac{2}{3\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2}}\sum_{\mathcal{H}=0}^{\infty}\frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}+N}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2(\mathcal{H}+N)}}$$

$$\to 0 \quad as \quad N \to \infty$$

for all $\mathcal{X} \in \mathcal{H}$. Hence Q is unique.

Corollary 3.2 Let $\mathcal{T}$ and P be non negative real numbers. Let an even function $\mathcal{Q}_{\mathrm{q}}: \mathcal{H} \to \mathcal{I}$ satisfy the inequality

$$\|Q(\mathcal{X},\mathcal{Y})\| \le \begin{cases} \mathcal{T}, & \\ \mathcal{T}\{||\mathcal{X}||^{\mathrm{P}} + ||\mathcal{Y}||^{\mathrm{P}}\}, & \mathrm{P} \neq 2; \\ \mathcal{T}\left\{||\mathcal{X}||^{\mathrm{P}}||\mathcal{Y}||^{\mathrm{P}} + \{||\mathcal{X}||^{2\mathrm{P}} + ||\mathcal{Y}||^{2\mathrm{P}}\}\right\}, & \mathrm{P} \neq 1; \end{cases} \qquad (20)$$

for all $\mathcal{X},\mathcal{Y} \in \mathcal{H}$. Then there exists a unique quadratic function $\mathrm{Q}: \mathcal{H} \to \mathcal{I}$ such that

$$\|\mathcal{Q}_1(\mathcal{X}) - \mathrm{Q}(\mathcal{X})\| \le \begin{cases} \frac{\mathcal{T}}{3\left(\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2}-1\right)}, \\ \frac{\mathcal{T}||\mathcal{X}||^{\mathrm{P}}}{3\left|\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2}-\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathrm{P}}\right|}, \\ \frac{\mathcal{T}||\mathcal{X}||^{2\mathrm{P}}}{3\left|\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2}-\left(\mathrm{cosec}^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathrm{P}}\right|}, \end{cases} \qquad (21)$$

for all $\mathcal{X} \in \mathcal{H}$.

4 Stability Results: Mixed Case

Theorem 4.1 Let $\mathcal{T}: \mathrm{X}^2 \to [0,\infty)$ be a function satisfying (2.1) and (3.1) for all $\mathcal{X},\mathcal{Y} \in \mathcal{H}$. Let $\mathcal{Q}: \mathcal{H} \to \mathcal{I}$ be a function satisfying the inequality

$$\|\mathcal{Q}(\mathcal{X},\mathcal{Y})\| \le \mathcal{T}(\mathcal{X},\mathcal{Y}) \qquad (22)$$

for all $\mathcal{X},\mathcal{Y} \in \mathcal{H}$. Then there exists a unique additive mapping $\mathrm{A}: \mathcal{H} \to \mathcal{I}$ and a unique quadratic mapping $\mathrm{Q}: \mathcal{H} \to \mathcal{I}$ such that

$$\|\mathcal{Q}_1(\mathcal{X})-A(\mathcal{X})-Q(\mathcal{X})\| \leq \frac{1}{2}\left[\frac{1}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)}\sum_{\mathcal{H}=0}^{\infty}\left(\frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}}+\frac{\mathcal{T}(-\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}}\right)\right.$$

$$\left.+\frac{1}{3\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2}}\sum_{\mathcal{H}=0}^{\infty}\left(\frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathcal{H}}}+\frac{\mathcal{T}(-\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathcal{H}}}\right)\right] \quad (23)$$

for all $\mathcal{X}\in\mathcal{H}$. The mapping $A(\mathcal{X})$ and $Q(\mathcal{X})$ are defined in (4) and (15) respectively for all $\mathcal{X}\in\mathcal{H}$.

Proof. Let $\mathcal{Q}_o(\mathcal{X})=\frac{\mathcal{Q}_a(\mathcal{X})-\mathcal{Q}_a(-\mathcal{X})}{2}$ for all$\mathcal{X}\in\mathcal{H}$. Then $\mathcal{Q}_o(0)=0$ and $\mathcal{Q}_o(-\mathcal{X})=-\mathcal{Q}_o(\mathcal{X})$ for all$\mathcal{X}\in\mathcal{H}$. Hence

$$\|\mathcal{Q}_o(\mathcal{X},\mathcal{Y})\|\leq\frac{\mathcal{T}(\mathcal{X},\mathcal{Y})}{2}+\frac{\mathcal{T}(-\mathcal{X},-\mathcal{Y})}{2} \quad (24)$$

for all $\mathcal{X},\mathcal{X}\in\mathcal{H}$. By Theorem 2.1, we have

$$\|\mathcal{Q}_o(\mathcal{X})-A(\mathcal{X})\|\leq\frac{1}{2\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)}\sum_{\mathcal{H}=0}^{\infty}\left(\frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}}+\frac{\mathcal{T}(-\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}}\right) \quad (25)$$

for all $\mathcal{X}\in\mathcal{H}$. Also, let $\mathcal{Q}_e(\mathcal{X})=\frac{\mathcal{Q}_q(\mathcal{X})+\mathcal{Q}_q(-\mathcal{X})}{2}$ for all$\mathcal{X}\in\mathcal{H}$. Then $\mathcal{Q}_e(0)=0$ and $\mathcal{Q}_e(-\mathcal{X})=\mathcal{Q}_e(\mathcal{X})$ for all$\mathcal{X}\in\mathcal{H}$. Hence

$$\|\mathcal{Q}_e(\mathcal{X},\mathcal{Y})\|\leq\frac{\mathcal{T}(\mathcal{X},\mathcal{Y})}{2}+\frac{\mathcal{T}(-\mathcal{X},-\mathcal{Y})}{2} \quad (26)$$

for all $\mathcal{X},\mathcal{Y}\in\mathcal{H}$. By Theorem 3.1, we have

$$\|\mathcal{Q}_e(\mathcal{X})-Q(\mathcal{X})\|\leq\frac{1}{6\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2}}\sum_{\mathcal{H}=0}^{\infty}\left(\frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathcal{H}}}+\frac{\mathcal{T}(-\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathcal{H}}}\right) \quad (27)$$

for all $\mathcal{X}\in\mathcal{H}$. Define

$$Q(\mathcal{X})=\mathcal{Q}_e(\mathcal{X})+\mathcal{Q}_o(\mathcal{X}) \quad (28)$$

for all $\mathcal{X}\in\mathcal{H}$. From (25),(27) and (28), we arrive

$$\|\mathcal{Q}_1(\mathcal{X})-A(\mathcal{X})-Q(\mathcal{X})\|=\|\mathcal{Q}_e(\mathcal{X})+\mathcal{Q}_o(\mathcal{X})-A(\mathcal{X})-Q(\mathcal{X})\|$$

$$\leq\|\mathcal{Q}_o(\mathcal{X})-A(\mathcal{X})\|+\|\mathcal{Q}_e(\mathcal{X})-Q(\mathcal{X})\|$$

$$\leq\frac{1}{2\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)}\sum_{\mathcal{H}=0}^{\infty}\left(\frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}}+\frac{\mathcal{T}(-\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}}\right)$$

$$+\frac{1}{6\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2}\sum_{\mathcal{H}=0}^{\infty}\left(\frac{\mathcal{T}(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathcal{H}}}+\frac{\mathcal{T}(-\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{\mathcal{H}}\mathcal{X},0)}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2\mathcal{H}}}\right)$$

for all $\mathcal{X} \in \mathcal{H}$

Corollary 4.2 Let $\mathcal{T}$ and P be non negative real numbers. Let a function $\mathcal{Q}: \mathcal{H} \rightarrow \mathcal{I}$ satisfy the inequality

$$\|\mathcal{Q}(\mathcal{X},\mathcal{Y})\| \le \begin{cases}\mathcal{T}, & \\ \mathcal{T}\{||\mathcal{X}||^{\mathrm{P}}+||\mathcal{Y}||^{\mathrm{P}}\}, & \mathrm{P} \neq 1,2; \\ \mathcal{T}\left\{||\mathcal{X}||^{\mathrm{P}}||\mathcal{Y}||^{\mathrm{P}}+\{||\mathcal{X}||^{2\mathrm{P}}+||\mathcal{Y}||^{2\mathrm{P}}\}\right\}, & \mathrm{P} \neq \frac{1}{2},1;\end{cases} \tag{29}$$

for all $\mathcal{X},\mathcal{Y} \in \mathcal{H}$. Then there exists a unique additive function $\mathrm{A}: \mathcal{H} \rightarrow \mathcal{I}$ and a unique quadratic function $\mathrm{Q}: \mathcal{H} \rightarrow \mathcal{I}$ such that

$$\|\mathcal{Q}_1(\mathcal{X}) - A(\mathcal{X}) - \mathcal{Q}(\mathcal{X})\|$$

$$\le \begin{cases}\left(\frac{\mathcal{T}}{\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)-1}\right)+\left(\frac{\mathcal{T}}{\left(\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2-1\right)}\right), \\ \left[\frac{1}{|\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)-\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^P|}+\frac{1}{3|\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2-\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^P|}\right]||\mathcal{X}||^P, \\ \left[\frac{1}{|\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)-\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2P}|}+\frac{1}{3|\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^2-\left(cosec^{\mathcal{L}}\left(\frac{\pi}{4}\right)\right)^{2P}|}\right]||\mathcal{X}||^{2P}\end{cases} \tag{30}$$

for all $\mathcal{X} \in \mathcal{H}$

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