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Static Response of Rail Track supported by Linear and Non-Linear Stiffness

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Abstract:

Introduction: The railway system can be viewed as a collection of structural components, each with distinct mechanical properties. These components are defined by their strength, stiffness, natural and forced frequency response function, which is influenced by their mass and elastic characteristics. These parameters determine the static and dynamic response or behavior of the rail. To analyze the behavior of the track, it is essential to understand the mechanical properties of the primary rail track elements.

Objectives: This paper focuses on the Discretely Supported Rail Beam Model with Linear Spring Stiffness and Nonlinear Spring Stiffness, compared the results using ANSYS for Static Analysis.

Methods: The Rails are linear elements of infinite length, making them suitable to be modeled as beams. They possess flexural stiffness in both the vertical and lateral directions, as well as compressive stiffness in the longitudinal direction. Additionally, rails have shear stiffness, although this is often overlooked in modeling. The rail fastening system commonly used with prestressed concrete sleepers consists of a resilient spring fastener, which works in parallel with a much stiffer rail pad. The load/deflection behavior of the fastening system is non-linear; however, some linearization of this behavior can be justified. Rail pads are primarily subjected to compression, which is constant due to the fastening system and rail traffic. The inclusion of rail pads helps reduce the impact of this force by lowering the effective track mass acting on the sleepers and ballast. The track components, initially considered to be exhibiting the linear behaviors, and later for more accuracy, they were studied for their nonlinear behavior by many researchers. The Static Response is often determined for the Track Modulus, an important parameter with respect to design, and maintenance of the railway track structure. There are different mathematical models suggested for obtaining the Track Modulus. The traditional method is Beam on Continuous Elastic Foundation (BOEF) Model, most widely discussed and studied in different aspects. The Winkler's Hypothesis, considers Elastic Foundation as a system of Identical, Independent, Closely Spaced, Discrete and Linear Elastic Springs. In actual, the track components do not behave linearly.

Results: The Rail Beam Model with Linear Spring Stiffness and Nonlinear Spring Stiffness are compared against the results using ANSYS for Static Analysis.

Conclusions: Experimental and simulated results validate the Non-Linear Stiffness varies non-linearly having a quadratic behavior. The Band of Track between the Zero Deflection and Maximum Displacement with the Rail having Non-Linear Stiffness is TWO times against the Rail with Linear Stiffness.

Keywords: Track Modulus, Visco-Elastic Material, Beam on Continuous Elastic Foundation – BOEF, Discretely Supported Beam, Track Modulus, FEM.

1. Introduction

The structure of the Railway Track consists of (i) Super-Structure - the Rail, Fastening System and Sleepers; (ii) Sub-Structure - non-cohesive Granular Materials viz stone Ballast. The Railway Track is a Continuous Beam Structure supported on the Elastic Foundation having Sleepers, GRP, Liners and Elastic Clips tied to the Rail. The standard rail sections in use on Indian Railways are 60 kg, and 52 kg.

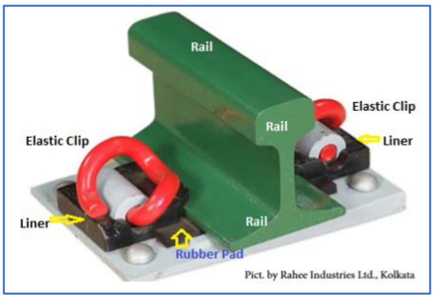


Figure 1: A schematic of Rail Track and Rail Fastening System using ERC

[Source: http://www.pandrolrahee.com/]

The traditional method is Beam on Continuous Elastic Foundation (BOEF) Model. The Winkler's Hypothesis, considers Elastic Foundation as a system of Identical, Independent, Closely Spaced, Discrete and Linear Elastic Springs. Another modelling approach is using the continuously supported rail beam and those that represent the track as a discretely supported rail beam. Continuously supported models of infinite length are based on the beam on elastic foundation theory.

Zimmermann (1888) presented a solution for the Winkler Hypothesis (Model) by assuming the rail - as 1 – D Analysis of a railway structure supported by elastic springs at discrete points of a beam laid on a continuous support (soil subgrade or foundation). In 1970, Meacham et. al. (1970) introduced a track model, assuming the track components to be replaced by a series of springs, with individual spring constants. Lundgren and Martin (1970), and Thompson and Tayabji (1976) considered discrete support for the rail in the static model of the track. Ahlbeck et. al (1978) simulated linear railroad track structures [based on linear theory] using one or two degrees of freedom with lumped-parameter model. The above developments have improved the Winkler model by taking into consideration the discrete properties of the support system of the rail. However, Kerr (1976) [2, 19] proved that the results obtained from the discrete support model are not significantly different from those obtained from the Winkler Model. Hetenyi [21] studied the different foundations and modelled a Hetenyi Foundation which is widely accepted and used by many researchers. Sandeep Shinde et. al. (2020) [1] presented a Compact Review – Mathematical Modelling of Rail Track stating the different Analytical and Numerical Mathematical Modelling of Rail Track that gave a good overview of the Rail Modelling.

Following is the Mathematical Modelling of Train – Track Dynamic interaction are discussed in Railway Track Dynamics - a survey by Tore Dahlberg [2], as follows:

a) Beam (Rail) on Continuous Elastic Foundation
b) Vehicle – Bridge Interaction (Moving Mass on Simply Supported Beam)
c) Beam (Rail) on Discrete Supports

d) Beam (Rail) on Discrete Supports including Ballast Model and

e) Beam (Rail) on Sleepers Embedded in Continuum, including 3-D FEM Models

2. BoEF as Continuously Supported Models:

The Rail as a Beam on Elastic Foundation (BOEF) can be modeled using either the classical beam theories using Euler-Bernoulli Beam Theory or the Rayleigh-Timoshenko Beam Theory. The Rayleigh-Timoshenko accounts for the rotary inertia of the beam's cross-section and the beam deformations caused by shear forces. Additionally, a longitudinal (axial) force in the rail can be incorporated into these models. The BOEF model, first introduced by Winkler (1867) for railway system modeling, uses the Euler-Bernoulli beam theory to describe an infinitely long rail beam resting on a uniform elastic foundation.

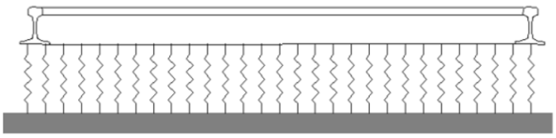


Figure 2: BoEF

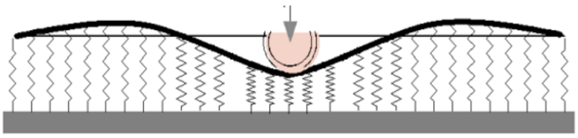


Figure 3: BoEF with Point Load

2.1. Continuously Supported Models

In this model, the rail is treated as a beam with bending stiffness (EI) supported by a continuous elastic foundation (figure 2), which represents all track components and is modeled using evenly distributed linear spring stiffness. The distributed force supporting the beam is directly proportional to the beam's deflection. This approach allows the calculation of rail deflection and bending moment under a moving load. Beam on elastic foundation loaded with a point force P from the wheel. The thick line indicates beam (rail) deflection due to the wheel load P, (figure 3)

This is the simplest track model and is still in use for easy and quick track deflection calculations for track design and analysis purposes. The vertical deflection y, for a given point load P as a function of the distance from the load x, is:

$$y(x) = -\frac{P\,.\,\beta}{2\,u}\, e^{-\beta x}\,[\cos(\beta x) + \sin(\beta x)] \qquad \textbf{– (1)}$$

The bending moment is

$$M = -\frac{P}{4\,\beta}\, e^{-\beta x}\,[\cos(\beta x) - \sin(\beta x)] \qquad \textbf{– (2)}$$

where, $\beta = \left(\frac{u}{4\,E\,I}\right)^{1/4}$ **– (3)**

P is the load

x is the distance along the track

E is the Young's modulus of elasticity of the rail steel (beam) and

Iz is the second moment of area of the beam.

u is the estimate of the track modulus

Using this model, it is also possible to calculate, for instance, the individual sleeper reaction by multiplying the sleeper's support stiffness by its deflection, as the load is proportional to the deflection. The load is typically distributed among three sleepers, with the sleeper directly beneath the load bearing 40-50% of the axle load. Originally developed for longitudinally sleeper tracks, the BOEF method has since been adapted for transversely sleeper tracks. It is widely regarded as the most reliable method for analysing rail foot stress and rail deflections.

However, this model is only suitable for static loading of tracks supported by soft materials, such as wooden sleepers, as it does not account for dynamic effects due to the absence of mass. Notably, the pinned-pinned frequency of approximately 800-1000 Hz, where the rail vibrates with nodes at the sleepers, is not captured by this model. To better simulate the function of rail pads, sleepers, and ballast, multiple continuous layers must replace the single layer shown in figure 2 and figure 3.

A continuous model like this was used to examine the behaviour of track support. The impact of changes in the support, particularly rail pads, was studied, revealing that softer rail pads more effectively isolate sleepers and significantly reduce sleeper strains. While the basic BOEF model, which represents the track with a single moving load, provides an initial understanding of track vibrations at steady state, it oversimplifies track behaviour under high-frequency impacts typical of modern railway systems. Several limitations arise from using the BOEF model in such scenarios.

2.2. Rail as Discretely Supported Beam Models

Modern railway tracks are typically composed of precisely positioned continuous welded rails and sleepers, which simplifies the mechanical modelling of the track as two parallel beams supported by sleepers at uniform intervals (Figure 4). Discretely Supported Models share similarities with continuously supported models, but with key differences:

They account for the discrete spacing of the sleepers.

They often incorporate multiple layers, representing elements like rail pads, sleepers, ballast, sub-ballast, and subgrade.

As a result, the rails exhibit vibration modes that correspond to this discrete support structure. One of the most significant vibration modes resembles bending between the discrete support points or "pins," where the rail vibrates with nodes at the sleeper locations (Figure 4). With certain simplifying assumptions, this pin-pin vibration resonance occurs at a specific frequency, fpp, can be calculated as follows

$$\boldsymbol{f_{pp} = \frac{\pi}{2l^2}\sqrt{\frac{EI}{m}}} \qquad \text{– (4)}$$

where

I : distance between two supports [m]

EI: bending stiffness of the rail (static) [Nm2]

m : mass of the rail per unit length [kg/m]

This assumption is based on an infinitely long beam perfectly supported at small, rigid points, which does not accurately reflect reality. In practice, the rail rests elastically on the sleeper with an elastic rail pad in between, and the sleeper itself is seated elastically within the ballast. These factors, along with other elements, can influence the pin-pin resonance. They introduce additional significant vertical resonances, with the three most dominant ones typically occurring between 30.0 Hz and 2000.0 Hz in most track structures

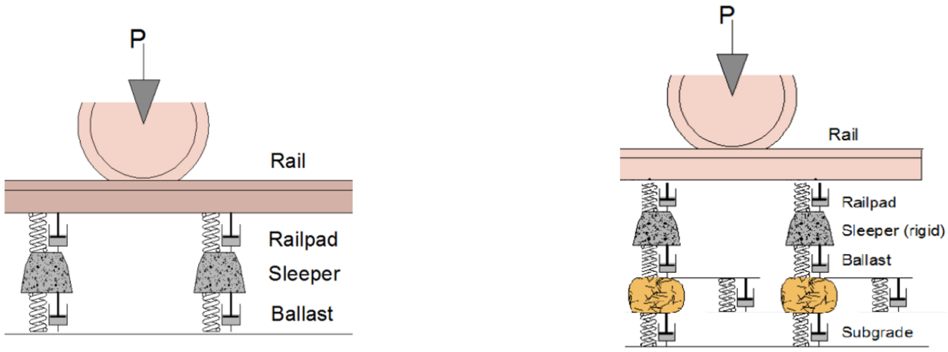


Figure 4: Discretely Supported Rail Beam

The supports could then be either discrete spring – damper systems or spring – mass – spring systems, modelling Rail pads, sleepers and ballast bed.

The ballast and subgrade masses influence each other, indicating that a deflection at one point (at one sleeper) will influence the deflection at the neighbouring sleepers.

3. Elastic Beams on Non-Linear Foundations

The analysis of a linear elastic beam supported on a linear elastic foundation and subjected to lateral loads has been accomplished by many different techniques [2, 3]. The basic differential equation describing the elastic deflection curve of a beam on a linear elastic foundation, considering only flexural deformations of the beam, is [2]

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2y}{dx^2}\right) + P\,\frac{d^2y}{dx^2} + Ky = w \qquad -(5)$$

where y defines the deflection of the beam, E is the modulus of elasticity of the material of construction for the beam, Z is the moment of inertia of the cross-section at distance x, P is the axial load applied to the beam, w is the applied distributed load and k is the foundation modulus. The beam may have a variable cross-section over all or any portion of its length; thus, Z or dZ/dx may be a discontinuous function. The solution of eq. (5) is complicated by discontinuities in the applied loads, both P and w and the foundation modulus k. Discontinuities in the deflection curve or one of its derivatives may also be introduced into the problem by interior support constraints or special construction constraints. The problem is further complicated if the foundation supporting the beam behaves nonlinearly. Soil is not an linear material and to model it as an linear material could lead to considerable error in an analysis. Soil, in fact, has a stress-strain relationship that has been described with a hyperbolic curve [3]; a similar load-deflection relationship could be assumed to exist, as shown in Fig. 5, when modelling the reaction of a soil foundation. The nonlinear load-deflection relationship for a soil foundation can be approximated with a bilinear curve as illustrated in Fig. 6. Hence, the foundation modulus, k, of eq.

(1) becomes a nonlinear coefficient. It is the purpose of this paper to review a numerical method that can be used to solve the higher order differential equation, eq. (5), and to describe a technique for handling the nonlinear aspects of the problem which provides for rather quick convergence of the solution.

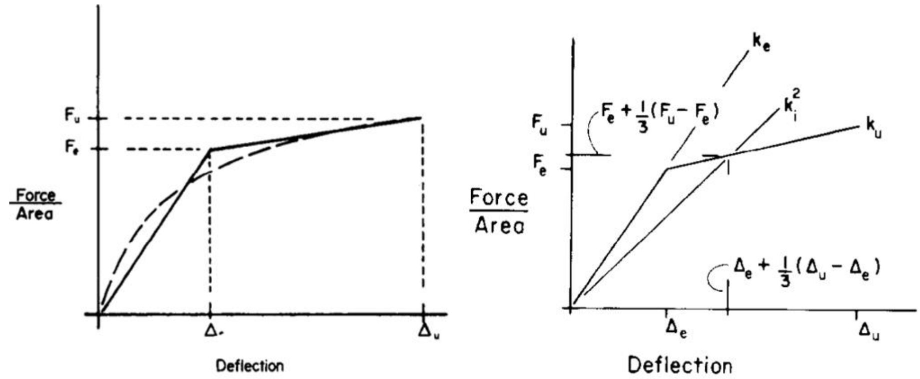


Figure 5: Approximation of Load – Deflection Curve;

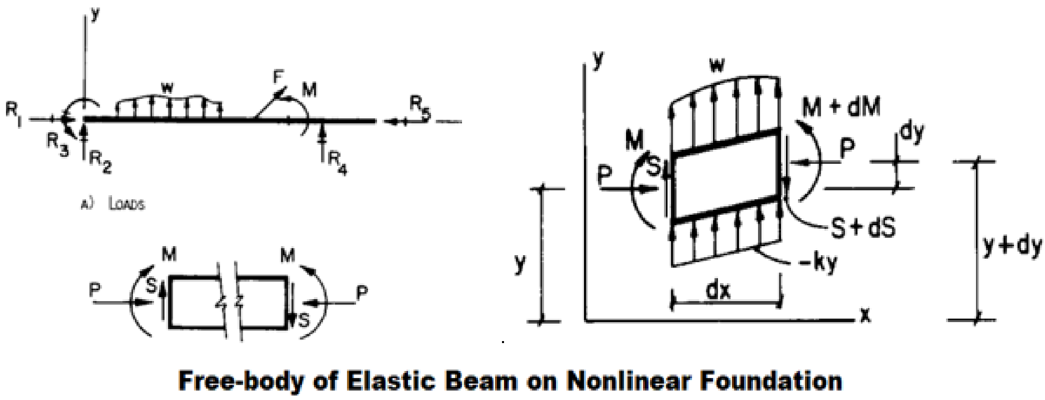


Figure 6: Free Body Diagram of Elastic Beam on Nonlinear Foundation

The midpoint difference method is formulated on the concept of replacing the higher order differential equation with a set of first order differential equations. In the case of eqn (I), it can be replaced with the following set of four first order differential equations:

$$\frac{d}{d_x}\begin{bmatrix} Y^1 \\ Y^2 \\ Y^3 \\ Y^4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & {}^{1}/_{EI} & 0 \\ 0 & -P & 0 & 1 \\ -K & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} Y^1 \\ Y^2 \\ Y^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$ where

$$Y^1 = Deflection \qquad (7a) \qquad\qquad Y^2 = \frac{dy}{dx} = Slope \qquad (7b)$$

$$Y^3 = EI\frac{d^2y}{dx^2} = Bending\ Moment \quad (7c) \qquad Y^4 = \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right) + P\frac{dy}{dx} = Shear \quad (7d)$$

The nonlinear nature of the problem requires that an iterative approach be taken. Using the bilinear curve to model the load-displacement response of the foundation. all foundation springs are assumed to be active and to have a stiffness equal to the slope, & of the first segment of the load-displacement relationship in the first iteration of the analysis. Once the beam has been analyzed, the displacement of the midpoint of each real segment attached to a foundation spring is checked to determine (a) if the spring is in tension (positive deflection) or (b) if the displacement of the spring has exceeded the elastic displacement A, If the spring is in tension, the beam has tried to lift off of the foundation: hence, the spring stiffness is set to zero for the next iteration. If the spring displacement has exceeded the elastic displacement ∇_e, a new spring stiffness is assumed. Using the new stiffnesses for those foundation springs that had to be modified, a second analysis is made. Once again the midpoint displacements are checked to determine (a) if an active spring is in tension, (b) if an inactive spring, i.e. a spring of zero stiffness, is in compression and should be a active once again, (c) if the displacement of a spring has exceed A, and (d) if the assumed spring stiffness is compatible with the actual load-displacement response of the foundations for the computed displacement. Before the weighted averages scheme can be utilized two iterations of analysis must be carried out. For the first iteration all foundation springs are assumed to have a spring stiffness of $\boldsymbol{K_i^1} = \boldsymbol{K_{ie}} = \frac{\boldsymbol{F_{ie}}}{\propto_{ie}}$ (**8**)

For those springs where the displacement of the midpoint of the segment exceed; the elastic limit A,, a new spring stiffness is arbitrarily defined as $\boldsymbol{K_i^2} = \boldsymbol{K_{ie}} = \frac{2\,\boldsymbol{F_{ie}} + \boldsymbol{F_{iu}}}{2\,\Delta_{ie} + \Delta_{iu}}$ (**9**)

4. Rail Track – Euler-Bernoulli (E-B) with BOEF

The track model is consisting of a beam on a spring foundation representing the Beam on Elastic Foundation (BoEF), discretely supported by Springs having Stiffness represented as Linear (Constant) and Non-Linear Stiffness (Load-Deflection Curve) – as studied by Grassie, Sakdirat, Konthe, Thompson [5, 6, 7, 8]. For this study, this paper considers the Non-Linear Stiffness of Grooved Rubber Pad by considering Load-Deflection Curve obtained from the Experiment and validated by ANSYS Software. The UIC60 cross section is designed as the specifications given by the Indian Railways, figure 7.

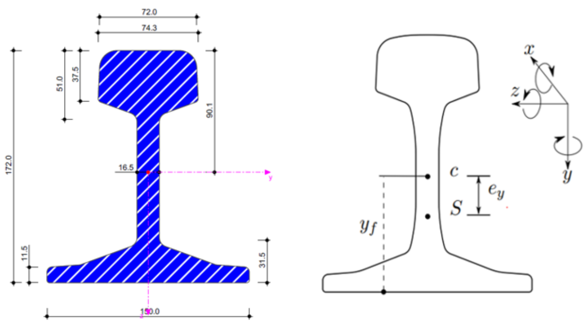


Figure 7: UIC60 Rail Cross Section Specifications in mm and Free Body Diagram

4.1. Non – Linear behavior of the Grooved Rail Pad - GRP:

The track model is consisting of a beam on a spring foundation representing the rail pad. Static Stiffness increases linearly with load but due to Non-Linear properties of Visco-Elastic Material (Elastomers), Dynamic Stiffness at higher frequencies or higher load steps is generally higher than the Static Stiffness. Non-Linear properties of Grooved Rubber Pad – GRP; can be determined by performing material modelling like Mooney – Rivlin, Neo-Hookean and Ogden Material Models etc.

The Load vs. Deflection data from test is given in tabular form as material characteristics to find out best curve fit values. The graph obtained from experimental study gives values of Load vs. Deflection to ANSYS material library as input and solved curve fit for polynomial 3rd degree equation. Values of all the constants have been obtained from the solutions. By considering the entire test parameters Load vs. Deflection plot obtained.

Figure 8: Material Modelling Using ANSYS with the Material Constants

Linear Elastic model fails to describe accurately the real material behavior, whose stress-strain relationship is independent of strain rate. Due to Non-Linear behavior the Stiffness of rail pad varies Non-Linearly and it has different value at every single point on the Load vs. Deflection curve. Practically it is difficult to find out the stiffness value for every instance of the loading step of curve. For the current study, the variable stiffness of GRP, polynomial of 3rd order approach to find out. The Mathematical form of strain-energy potential for the polynomial option is given as.

$$W = \sum_{i+j=1}^{N} Cij\ (I1-3)^{i}.(I2-3)^{j} + \sum_{k=1}^{N} \frac{1}{dk}(J-1)^{2k} \qquad \textbf{– (10)}$$

Where, W= strain energy potential.	
I1= First deviatric strain invariant	I2=second deviatric strain invariant
J= Determinant of the elastic deformation gradient F	N, Cij= material constants

In general number of constants depends on value of N.

N= 3, C10, C01, C20, C02, C30, C21, C12, C03, d2, d3.

The Load vs. Deflection nature obtained from the uniaxial loading test is considered as input to ANSYS software to find out above constants by curve fitting method. This modelling is carried out to validate the Load vs. Deflection curve obtained by ANSYS with experimental one.

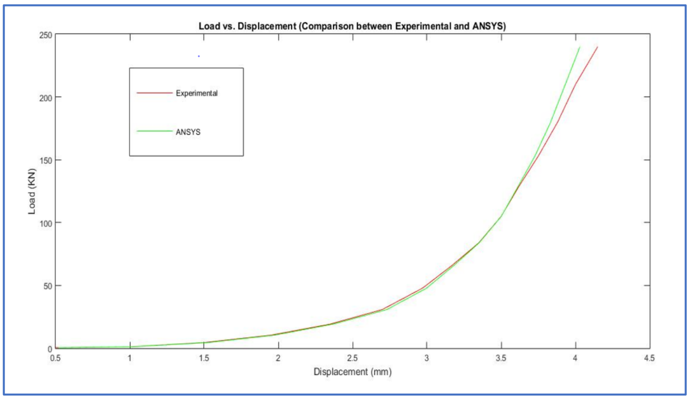


Figure 9: Load vs Displacement – Rubber Pad

The Stiffness of GRP is usually assumed to increase linearly with load to simplify analysis of track structure. Actual behaviour of GRP is determined by derivation of GRP Stiffness from the test results. The tangent stiffness is given by Kp= dp/dx. The Load vs Deflection equation is determined by performing curve fitting procedure in Matlab software. The Load vs Deflection and Stiffness vs Deflection equations are derived as

Power equation $\mathbf{(y = a.e^{bx}) \text{ is } y = 0.256580 * e^{1.694267 * x}}$ **– (11)**

$\mathbf{Kp = \frac{dy}{dx} = 0.4337 * e^{(1.694267 * x)}}$ **– (12)**

4.2. Determination of Rail Track – Euler-Bernoulli (E-B) with BOEF:

A beam model having cross-section, 60 kg Rail – UIC60, with Length of 36 m to account the variation over the long length. The rail pad under the rail considered as spring and stiffness of rail pad is given as input characteristics to the spring. The Distance between two springs is as kept 0.6 m. The model developed based on Beam on Elastic Foundation (BOEF) and simulated using ANSYS Software. The Influence of Track Stiffness on Track Behaviour under vertical Load demonstrated by Leposava [14]. The Single Rail Track – as BoEF, with the Influence of Non-Linear Behavior of Track Fastening System on Track Modulus presented by Sandeep Shinde et. al. [25]. In 2010, Al- Azzawi et. al. [26] worked on the deep beams on nonlinear elastic foundations using FEM.

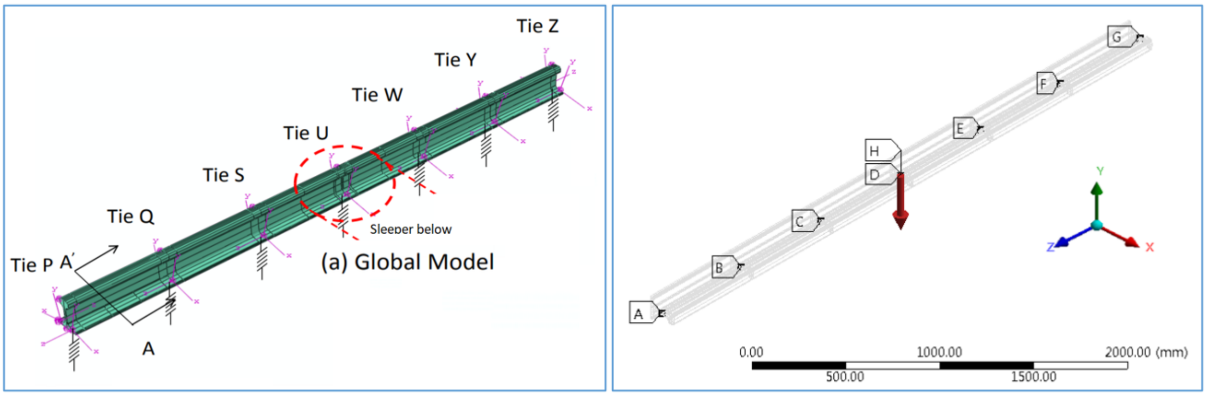
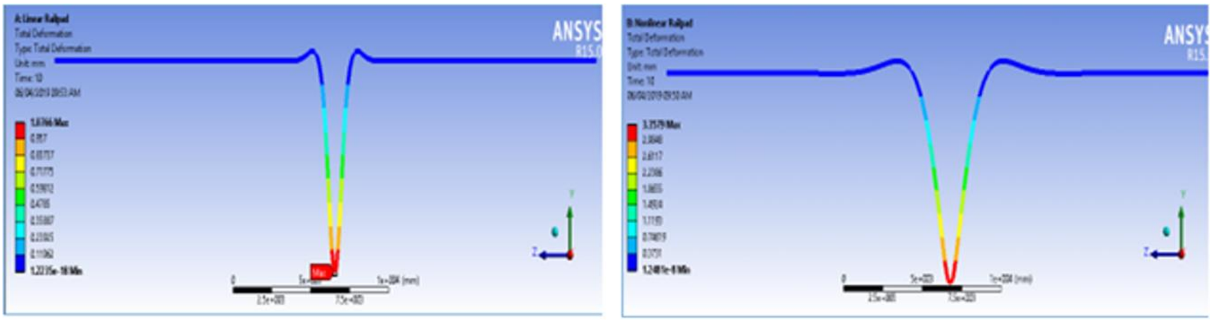


Figure 10: Modelling of Rail Track – 1– D Modelling – Boundary Conditions

Considering point load at center; simulated the model for various load steps from (0 – 250 KN) and obtained plot of Length vs. Deflection from it. The Linear Model consists of the Linear Spring Stiffness. The Non-Linear Model consists of the Non-Linear Spring Stiffness [based on the Load – Deflection Curve and the Non-Linear Spring Stiffness [based on the Load – Deflection Curve] of the GRP.

4.3. Results of Rail Track – Beam on Elastic Foundation (BOEF):

Following fig. describes the Rail Deflection based on the Linear and Non-Linear Spring effects.



Linear behavior of springs **Non-Linear behavior of springs**

Rail Length vs. Deflection

Figure 11: Deflection of Rail Track – Linear and Non-Linear Springs

The Maximum Displacement Results are compared between the Theoretical and Numerical Results found using Ansys Software vary by 3.65% when the Linear Stiffness specified for the Rail Pad. whereas, the Maximum Displacement Results are compared between the Theoretical and Numerical Results found using Ansys Software vary by 3.88% when the Non-Linear Stiffness specified for the Rail Pad.

Track Modulus based on Linear behavior of the GRP:

$E = 2 \times 10^5\, N/mm^2 \;= 2 \times 10^{11}\, N/m^2$ $\qquad I = 3.047 \times 10^7\, mm^4 = 3.047 \times 10^{-5} m^4$

$k = 60000 \frac{N}{mm} = 60 \times 10^6\, N/m$ $\qquad$ **The Deflection of Rail is given as** $\boldsymbol{y = \frac{\beta . p}{2K}}$

Length of BOEF is $\boldsymbol{\beta = \sqrt[4]{\frac{K}{4EI}} = \sqrt[4]{\frac{60\times10^6}{4\times2\times10^{11}\times3.047\times10^{-5}}} = 1.2525\, m}$

Displacement of Rail				
y1= 0.1043 mm	y2= 0.2087 mm	y3= 0.3131 mm	y4= 0.4175 mm	y5= 0.5218 mm
y6= 0.6262 mm	y7= 0.7306 mm	y8= 0.835 mm	y9= 0.9393 mm	y10= 1.043 mm

Theoretical and ANSYS Results for Maximum Displacement in Rail are tabulated as fig.12:

	Theoretical	ANSYS	% Difference
Linear	1.013	1.05	-3.65%
Non-Linear	3.225	3.35	-3.88%

Figure 12: Theoretical and ANSYS Results for Maximum Displacement in Rail

The displacement of Rail is plotted against the Track Length in the figure 11 with the Linear and Non-Linear behavior of the Rail Pad.

Figure 13: Displacement Variation along the Length of Rail

(comparison of Rail supported by Linear and Non-Linear Stiffness)

BLUE Colured Curve represents Linear Stiffness [Points 1 to 7] and ORANGE Colured Curve represents Non-Linear Stiffness [Points 1' to 7']			
Point	**Description of Curve**	**Linear Stiffness**	**Non-Linear Stiffness**
Point 1 - 3 and (Symmetrical after Max Displacement) Point 7 - 5	The **π/β - ratio of π against the Length of BOEF** for the **Linear Stiffness Behavior** of ERC and GRP	**1.0 meter** approximately	**4.0 meters** approximately
Point 2 and Point 6	The **Maximum Lift of the Track** - BOEF for the **Linear Stiffness Behavior** of ERC and GRP	**0.08 mm** approximately	**0.2 mm** approximately
Point 4	The **Maximum Dispacement** of the Rail with **Linear Stiffness Behavior** of ERC and GRP	**1.08 mm** approximately	**3.4 mm** approximately
Point 3 - 7	The **Band of Track between the Zero Deflection** - BOEF for the **Linear Stiffness Behavior** of ERC and GRP	**2.0 meters of Length** approximately	**4.2 meters of Length** approximately
Point 1 - 5	The **Length of BOEF** that is affected due to Point Load, (for the **Linear Stiffness Behavior** of ERC and GRP)	**4.0 meters** approximately	**12.5 meters** approximately
Point 4 - 5	The **π/β - ratio of π against the Length of BOEF** for the **Linear Stiffness Behavior** of ERC and GRP	**2.0 meters** approximately	**6.2 meters** approximately
Point 4 - 7	The **Band of Track between the Zero Deflection and Maximum Displacement** is given as **$\pi/4\beta$ - ratio of π against the Length of BOEF** for **Linear Stiffness Behavior** of ERC and GRP	**1.0 meter** approximately	**2.0 meters** approximately

Figure 14: Close Look at the Displacement Variation along the Length of Rail

5. Conclusion:

A close examination of Fig.13 and Fig 14 reveals following observations:

1. Experimental and simulated results validate the Non-Linear Stiffness varies Non-Linearly having a quadratic behaviour.
2. The Maximum Displacement of the Rail with Non-Linear Stiffness (3.4 mm) is approximately THREE times against the Rail with Linear Stiffness behaviour (1.08 mm).
3. The Maximum upward Lift of the Rail with Non-Linear Stiffness (2.0 mm) is approximately THREE times against the Rail with Linear Stiffness (0.08 mm).
4. The Distance [Band or Width] between Zero Deflection points of the Rail with Non-Linear Stiffness (4.28 m) is approximately TWO times against the Rail with Linear Stiffness (2.0 mm).
5. The Band of Track between the Zero Deflection and Maximum Displacement is given as; $\pi/4\beta$ - ratio of π against the Length of BOEF of the Rail with Non-Linear Stiffness (2.0 m) is approximately TWO times against the Rail with Linear Stiffness (1.0 m).
6. The Length of BOEF of the Rail, which is affected due to Point Load, of the Rail with Non-Linear Stiffness (12.5 m) is approximately THREE times against the Rail with Linear Stiffness (4.0 mm).

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