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Fredholm Integral Equation Solving by Different Methods

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Abstract:

For many branches of applied mathematics, integral equations have proven essential tools. This article examines various numerical techniques for resolving Fredholm integral problems. The purpose is to classify chosen techniques and evaluate their efficacy. Integral equations are one of the most significant subfields of mathematical analysis in a number of branches of mathematical physics and mechanics. This research will discuss the Fredholm integral equation, its solutions, and its uses.

Keywords: Fredholm integral Equations (FIE), First kind, second kind.Direct Computation Method (DCM).

Introduction: Numerous scientific and engineering fields naturally involve integral equations [1–5]. A computer approach to solving integral equations is essential in scientific research. Integral equations are widely used in many different fields, including:, electricity and magnetism, kinetic theory of gases, geophysics and biology, radiation, optimization, mathematical economics, population genetics, queuing theory, medicine, optimal control systems ,acoustics, fluid mechanics, steady state heat conduction, fracture mechanics, radiative heat transfer problems, quantum mechanics, communication theory and many more.

An integral equation of extreme significance is the Fredholm integral equation. Equations that arise from the transformation of points in a certain vector space of integrable functions to points in the same space using particular integral operators are known as integral equations. There are a number of approximation techniques for solving integral equations..Solving integral equations computationally is an essential role in scientific study.

Integral equations that are closely connected to differential equations represent nearly all of Fredholm Integral Equations [6].Thus, boundary value problems for differential equations are the source of FIE, which are subsequently resolved using a variety of simplified techniques. Fredholm was highly motivated to find this kind of equation. After being found by Fredholm, the equations were given the name Fredholm Integral Equations. It served as the foundation for addressing significant challenges preventing mathematics from progressing [5]. We conclude by pointing out that FIE can also be found in both linear and non-linear forms, such as the homogeneous and non-homogeneous varieties [1, 3].

FIE:

A linear IE of the form

$g(\propto)\, u(\propto) = f(\propto) + \lambda \int_a^b K(\propto,\tau)\, u(\tau) d\tau$

a,b are both constants.$g(\propto), f(\propto) and\ k(\propto,\tau)$ and are known functions while $u(\propto)$ are unknown function.The function $k(\propto,\tau)$is known as the kernel of the IE.

FIE of the 1st kind:

A linear IE of the form $g(\propto) = 0$ in equation

$f(\propto) + \lambda \int_a^b K(\propto,\tau)\, u(\tau)\, d\tau$= 0

FIE of the 2nd kind:

A linear IE of the form $g(\propto) = 1$ in equation

$$u(\propto) = f(\propto) + \lambda \int_a^b K(\propto,\tau)\, u(\tau) d\tau$$

Methods to solve FIE of the 2nd kind :

We will discuss some analytical and numerical techniques in this study to solve second-kind FIE.

1.Direct Computation Method (DCM)

2.Variation Iteration Method (VIM)

1.1 Direct Computation Method (DCM): In this part of the study, the FIEs will be solved using the DCM. For the provided IE, this approach provides an accurate answer in closed form. It is important to keep in consideration that this method can be applied to the form's degenerate and separable kernels of the form,

$$K(\gamma,\tau) = \sum_{i=1}^{n} g_k(\gamma)\, h_k(\tau) \qquad (1)$$

DCM can apply in following form

$$u(\gamma) = f(\gamma) + \lambda\, g(\gamma) \int_a^b h(\tau)\, u(\tau) d\tau \qquad (2)$$

The R.H.S of equation (2) is depends on one variable t.

Let $\int_a^b h(\tau)\, u(\tau) d\tau$= α (3)

$$u(\gamma) = f(\gamma) + \lambda\, g(\gamma)\, \alpha \qquad (4)$$

Substitute equation (4) into (3)

1.2 Variational Iteration mehod (VIM):

If a solution does exist, the VIM provides consecutive approximations of it that may converge quickly to the exact answer. It is possible to use the resulting approximation for numerical purposes. It is necessary to first convert the integral equation to its corresponding integro differential equation in order to solve the Fredholm integral problem. This method is effective if the kernel can be separated of the form,

$$K(\gamma,\tau) = g(\gamma)\, h(\tau) \quad ..(5)$$

$$u(\gamma) = f(\gamma) + g(\gamma) \int_a^b h(\tau)\, u(\tau) d\,\tau$$

Differentiate both sides of (5) with respect to γ we get,

$$u'(\gamma) = f'(\gamma) + g'(\gamma) \int_a^b h(\tau)\, u(\tau) d\,\tau$$

$u_{n+1}(\gamma) = u_n(\gamma) + \int_0^\gamma \lambda\,(\tau)\,\{ u'_n(\tau) - f'(\tau) - g'(\tau) \int_a^b h\,(\mathrm{r})\, u_n\,(r)\,\mathrm{dr}\}\,\mathrm{d}\,\tau$
(6)

Where λ is Lagrange multiplier. In VIM we follow two steps, first we determine the λ

1.3 Adomian Decomposition Method (ADM) for FIE of 2nd kind :

The ADM was developed by George Adomian decomposing the unknown U(x) of every equation into the sum of an infinite number of components described by the decomposition series is the basic concept of the ADM. components defined by

$$u(\propto) = \sum_{i=0}^{n} u_n(\propto)$$

$$\sum_{i=0}^{n} u_n(\propto) = f(\propto) + \lambda \int_a^b K(\propto,\tau)\, u_n(\tau) d\tau \quad (7)$$

Compairing $u_0(\tau) = f(\propto)$

$$u_{n+1}(\tau) = \lambda \int_a^b K(\propto,\tau)\, u_n(\tau) \mathrm{d}\tau \quad (8)$$

Which is equivalent to$u_0(\tau) = f(\propto)$

$$u_{n+1}(\propto) = \lambda \int_a^b K(\propto,\tau)\, u_n(\tau) \mathrm{d}\tau$$

$$u_1(\propto) = \lambda \int_a^b K(\propto,\tau)\, u_0(\tau) \mathrm{d}\tau \quad (9)$$

$$u_2(\propto) = \lambda \int_a^b K(\propto,\tau)\, u_1(\tau) \mathrm{d}\tau \quad (10)$$

And so on.

It is obvious that the decomposition technique transformed the IE into an advanced computation of the individual components. Formally, it has been demonstrated that the derived series rapidly converges to the precise solution if there is one. in order to confirm that the resulting series converges quickly,numerous researchers researched deep into the convergence idea of the decomposition series. However, in the case of actual issues, where a closed form solution is unattainable, numerical objectives usually involve a simplified number of terms. We get higher accuracy.

1.4 M-ADM For FIE Of 2nd Kind:

Domain decomposition methods provide solutions through an infinite series of components.

The FIE (3),

$$u(\propto) = f(\propto) + \lambda \int_a^b K(\propto, \tau)\, u(\tau) d\, \tau$$

Using recurrence relation $u_0(\tau) = f(\propto)$

$$u_{n+1}(\tau) = \lambda \int_a^b K(\propto, \tau)\, u_n(\tau) \mathrm{d}\tau \qquad (11)$$

consists of one or more terms in a polynomial. In the occurrence that the function f(x) is formed from two or more polynomials combined, hyperbolic functions or trigonometric functions and the evaluation of component $u_j, j \geq 0$.

The M-ADM depends mainly on splitting the function $f(\propto)$ into two parts, therefore it cannot be used if the $f(\propto)$consists of only one term. The M-ADM presents a small variation to the recurrence relation to determine the component of $u(\propto)$ in an faster and easier way.In many examples the function $f(\propto)$ can be set as the sum of two partial functions namely,$f_1(\propto)$and $f_2(\propto)$. The M-ADM admits the use of the modified recurrence relation.

$$u_0(\tau) = f_1(\propto) \qquad (12)$$

$$u_1(\tau) = f_2(\propto) + \lambda \int_a^b K(\propto, \tau)\, u_0(\tau) \mathrm{d}\tau \qquad (13)$$

$$u_{n+1}(\tau) = \lambda \int_a^b K(\propto, \tau)\, u_n(\tau) \mathrm{d}\tau \quad , n \geq 1 \qquad (14)$$

Some example on this method

Example 1: $u(\gamma) = \gamma e^{\gamma} - \gamma + \gamma \int_0^1 u(\tau)\, \mathrm{d}\tau$

By direct method equation (i) as put

$$\int_0^1 u(\tau)\, \mathrm{d}\tau = \alpha \qquad (ii)$$

Therefore, $u(\gamma) = \gamma e^{\gamma} - \gamma + \gamma\alpha$ (iii)

Put these value of $u(\gamma)$ in equation (ii)

$$\alpha = \int_0^1 (\tau e^{\tau} - \tau + \tau\alpha)\, d\, \tau$$

$$\alpha = \frac{\alpha}{2} + \frac{1}{2}$$

$$\alpha = 1$$

substitute α= 1 in equation (4) we get exact solution,

$$u(\gamma) = \gamma e^{\gamma} - \gamma + \gamma$$

$$u(\gamma) = \gamma e^{\gamma}$$

By variation iteration method,

$$u(\gamma) = \gamma e^{\gamma} - \gamma + \gamma \int_0^1 u(\tau)\, d\tau \qquad \text{(i)}$$

Differentiate from both sides of equation (i) with respect to γ

$$u' = \gamma e^{\gamma} + \gamma - 1 + \int_0^1 u(\tau)\, d\tau \qquad \text{(iv)}$$

$$u_0(\gamma) = 0$$

$$u_{n+1}(\gamma) = u_n(\gamma) - \int_0^{\gamma} \{u'_n(\tau) - \tau e^{\tau} - e^{\tau} + 1 - \int_0^1 u_n(\psi)\, d\psi\}\, d\tau \qquad \text{(v)}$$

Put $\lambda = -1$

$$u_1(\gamma) = u_0(\gamma) - \int_0^{\gamma} \{u'_0(\tau) - \tau e^{\tau} - e^{\tau} + 1 - \int_0^1 u_0(\psi)\, d\psi\}\, d\tau$$

$$== \gamma e^{\gamma} - \gamma \qquad \text{(vi)}$$

$$u_2(\gamma) = u_1(\gamma) - \int_0^{\gamma} \{u'_1(\tau) - \tau e^{\tau} - e^{\tau} + 1 - \int_0^1 u_1(\psi)\, d\psi\}\, d\tau$$

$$== \gamma e^{\gamma} - \frac{1}{2}\gamma$$

(vii)

$$u_3(\gamma) = u_2(\gamma) - \int_0^{\gamma} \{u'_2(\tau) - \tau e^{\tau} - e^{\tau} + 1 - \int_0^1 u_2(\psi)\, d\psi\}\, d\tau \qquad \text{(viii)}$$

We get, $u_n(\gamma) = \gamma e^{\gamma} - \frac{1}{2^{n-1}}\gamma$

$$u(\gamma) = \lim_{n\to\infty} u_n(\gamma)$$

$$u(\gamma) = \gamma e^{\gamma}$$

Example 2: FIE by ADM method

$u(\propto) = -\pi \propto + \sin \propto + \propto \int_0^{\pi} \tau u(\tau)\, d\tau$

Consider,

$u(\propto) = -\pi \propto\ + sin \propto\ + \propto\ \int_0^{\pi} \tau \mathrm{u}(\tau) d\tau$

$u_0(\propto)\ = -\pi \propto\ + sin \propto$

$$u_{n+1}(\propto) = \propto \int_0^{\pi} \tau\, u_n(\propto)$$

$= \propto \int_0^{\pi} \tau\, u_0(\tau) \mathrm{d}\tau$

$$= \propto \int_0^{\pi} \tau(\text{-}\pi\, \tau + sin\tau)\ d\tau$$

$$= -\propto \frac{\pi^4}{3} + \propto \pi$$

$$u_2(\propto) = \propto \int_0^{\pi} \tau\, u_1(\tau) \mathrm{d}\tau$$

$$= \ \propto \int_0^{\pi} \tau^2 \left(-\frac{\pi^4}{3} + \pi\right) d\tau$$

$= \propto \frac{\pi^4}{3} - \propto \frac{\pi^7}{3}$

From , $u(\propto) = \sum_{i=0}^{n} u_n\, (\propto)$

$u(\propto) = -\pi \propto\ + sin \propto - \propto \frac{\pi^4}{3} + \propto \pi + \propto \frac{\pi^4}{3} - \propto \frac{\pi^7}{3}$ …

Cancelling the noise terms we get,

$$u(\propto) = \ sin \propto$$

Example 3: The FIE by using M-ADM

$$u(\propto) = 3 \propto\ + e^{4\propto} - \frac{1}{16}(17 + 3e^{4\propto}) + \int_0^1 \tau \mathrm{u}(\tau) d\tau$$

Consider,

$$u(\propto) = 3 \propto\ + e^{4\propto} - \frac{1}{16}(17 + 3e^{4\propto}) + \int_0^1 \tau \mathrm{u}(\tau) d\tau$$

$f_1(\propto) = \ 3 \propto\ + e^{4\propto}$, $f_2(\propto) = -\frac{1}{16}(17 + 3e^{4\propto})$

$$u_0(\propto) = f_1(\propto) = \ 3 \propto\ + e^{4\propto}$$

$u_0(\tau) = 3\, \tau + e^{4\tau}$

$$u_1(\tau) = f_2(\propto) + \lambda \int_a^b K(\propto, \tau)\, u_0(\tau) \mathrm{d}\tau$$

$$u_1(\tau) = -\frac{1}{16}(17 + 3e^{4\tau}) + \int_0^1 \tau \mathrm{u}(\tau) d\tau = 0$$

$$u_{n+1}(\tau) = \lambda \int_a^b K(\propto, \tau)\, u_n(\tau) \mathrm{d}\tau \quad , n \geq 1$$

$= 0 \ , \quad n \geq 1$

Each component $u_j, j \geq 0$ is zero.

, $$u(\propto) = \sum_{i=0}^{n} u_n\,(\propto)$$

$$u(\propto) = 3 \propto \, + e^{4\propto}$$

Conclusion: In our study we discuss the various methods to find the solutions of the FIEs of various kind. We obtain M-ADM is more easy than other method.

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