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Complexity of Types of Trapezoidal Graphs

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Abstract: The number of spanning trees in graphs (networks) is a fundamental invariant that plays a crucial role in measuring the reliability and connectivity of a network. It is particularly significant in various applications, including network design, circuit analysis, and structural stability assessments. In this paper, we derive explicit and simplified formulas for computing the complexity of specific classes of graphs, particularly trapezoidal graphs, using advanced techniques from linear algebra and matrix analysis. By leveraging Kirchhoff's matrix tree theorem and eigenvalue-based formulations, we establish efficient methods for determining the number of spanning trees. Additionally, we explore computational approaches such as Chio's condensation and Dodgson's method to enhance the accuracy and efficiency of determinant calculations related to graph Laplacians. The results obtained provide a deeper insight into the structural properties of trapezoidal graphs and their spanning tree enumeration, offering potential applications in combinatorial optimization, network topology analysis, and applied mathematics.

Keywords: Edge contraction, spanning trees, trapezoidal graphs.

1. Introduction

We provide some fundamental definitions and lemmas in this work. We work with undirected graphs $G = (V, E)$ that are simple and finite, where the vertex set is V and the edge set is E. A tree that has the same set of vertices as graph G is said to be a spanning tree in graph G. The number of spanning trees in G, commonly known as the graph's complexity and represented by $\tau(G)$, is a quantity that has been extensively explored and found in many different applications. The three most common application domains are measuring the number of Eulerian circuits in a graph [1], identifying specific chemical isomers [2], and network dependability [3-5].

For $G = (V, E)$, the number of spanning trees may be found using the standard result of Kirchhoff [6]. The Kirchhoff matrix H is defined as the $n \times n$ characteristic matrix $H = D - A$, where D is the diagonal matrix of the degrees of G and A is its adjacency matrix. $H = [a_{ij}]$ is defined as follows: When $i = j$, then (i) $a_{ij} = -1$, v_i and v_j are nearby, and (ii) a_{ij} equals the degree of vertex a_{ij}; otherwise, (iii) $a_{ij} = 0$. Every co-factor of H is equivalent to $\tau(G)$. There are several ways to compute $\tau(G)$. let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_p$ indicate the eigenvalues of H matrix of a p point graph. Then, it can be demonstrated with ease that $\mu_p = 0$. Then it is easily shown that 0. Furthermore, Kelmans and Chelnokov [7] shown that, $\tau(G) = 1/p \prod_{k=1}^{p-1} \mu_k$. The formula for the number of spanning trees in a d-

regular graph G can be expressed as $\tau(G) = 1/p\prod_{k=1}^{p-1}(d-\mu_k)$ where $\lambda_0 = d, \lambda_1, \lambda_2, \ldots, \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. On the other hand, for a small number of unique graph families, there are straightforward formulas that greatly simplify the computation and determination of the number of related spanning trees, particularly for high numbers. One of the first such result is due to Cayley [8] who showed that complete graph on n vertices, K_n has n^{n-2} spanning trees that he showed $\tau(K_n) = n^{n-2}, n \geq 2$ and $\tau(K_{p,q}) = p^{q-1}q^{p-1}p, q \geq 1$ where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. It is well known, as in e.g., [9,10]. For Sierpiński graphs and data center networks, Zhang et al. [11] computed the spanning tree entropy. Linear-time techniques for calculating the quantity and mean size of connected sets in a planar 3-tree were introduced by Luo et al. [12]. Sotirov et al. [13] constructed a series of QMSTP relaxations of increasing complexity and quality by utilizing an expanded formulation for the minimal spanning tree problem. Mohamed et al. [14] obtained basic formulae for the number of spanning trees of specific types of cyclic snake graphs by applying matrix analysis and linear algebra techniques. Daoud [15] computed the number of spanning trees of the Cartesian and Composition products of certain families' graphs.

In this paper, we use matrix analysis and linear algebra techniques to obtain simple expressions of the complexity of specific graphs, including trapezoidal graph types.

1.1 Chio's Condensation

Is technique to calculate an $n \times n$ determinant using $(n-1) \times (n-1)$ determinants; see to [16], [17]:

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix} \quad (1)$$

1.2 Dodgson's Condensation Method

By defining determinants of size $(n-1) \times (n-1)$ in terms of those of size $(n-2)\times(n-2)$, and so on, Dodgson's condensation method computes determinants of size $n \times n$ (see [18]). This method is based on Dodgson and Chio's method, but the difference between them is that this new method is resolved by calculating 4 unique determinants of $(n-1) \times (n-1)$ Order, (which can be derived from determinants of $n \times n$ order, if we remove first row and first column or first row and last column or last row and first column or last row and last column, elements that belongs to only one of unique determinants we should call them unique elements), and one determinant of $(n-2) \times (n-2)$ order which is formed from $n \times n$ order determinant with elements a_{ij} with $i, j = 1, n$, on condition that the determinant of $(n-2) \times (n-2) = 0$.

$$A = \frac{1}{\begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix}} \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} & a_{12} & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,2} & a_{n-1,n} \\ \vdots & & \vdots & \vdots & \vdots \\ a_{21} & \cdots & a_{2,n-1} & a_{22} & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} & a_{n,2} & a_{nn} \end{vmatrix}$$

Lemma 1.2.1 A basic graph G with n vertices has the following Laplacian matrix L_{nxn} definition $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$: where D is the graph's degree matrix and A is its adjacency matrix. Directed graphs can employ either the in degree or the out degree, depending on the application. The elements of $L_{i,j}$ are given by:

$$L_{i,j} = \begin{cases} \deg(v_i) & if\ i = j \\ -1 & if\ i \neq j\ and\ v_i\ is\ adjacent\ to\ v_j, \\ 0 & otherwise \end{cases}$$

where $\deg(v_i)$ is degree of the vertex i.

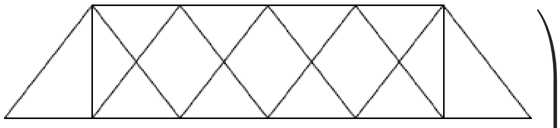
Definition 1.2.2 The intersection family of trapezoids, or **trapezoid graphs**, are those in which each trapezoid has two opposite sides that are parallel to each other.

These graphs, which are precisely between co-comparability graphs (where the complement has a transitive orientation) and permutation graphs (where the trapezoids are lines), have drawn a lot of attention [19]. For further studies about some other applications that could employ some other graphs, the reader may refer to the references [20,21,22,23,24,25,26,27,28,29].

2. Main Results

Theorem 2.1. Let $T(P_n^{n-2})$ be a trapezoidal graph of $2n-2$ vertices with triangulations

$$\tau\left(T(P_n^{n-2})\right) = \tau\left(\quad \right)$$



such that

$$\tau\left(T(P_n^{n-2})\right) = 100 \times 8^k,\ \ k \geq 1,$$

where k is the number of blocks.

Proof. By applying Eq. (1), we can perform the proof by the following steps:

- By mathematical induction, we prove the statement is true at $m = 1$, $\tau(G)$ is a 2×2-matrix with both rows the same:

$$\tau(G) = \frac{1}{520} \begin{bmatrix} 480 & 400 \\ 400 & 1200 \end{bmatrix} = 800.$$

- We prove that the statement is true at $m = k + 1$, that is straightforward induction using properties of determinants and Dodgson and Chio method and applying Eq. (2), we have:

$$\tau(G) = \begin{vmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 4 & -1 & \ddots & \cdots & \vdots & -1 & \ddots & \ddots & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & -1 \\ 0 & 0 & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & -1 & -1 \\ -1 & -1 & -1 & \ddots & \ddots & 0 & \ddots & 0 & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & \cdots & \ddots & \ddots & -1 & \ddots & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 & -1 & 4 \end{vmatrix}_{(2k+5)\times(2k+5)}$$

Consequently, we can have

$$\tau(G) = \frac{1}{(65 + (k-1) \times 15) \times 8^k} \begin{vmatrix} 60 \times 8^k & 50 \times 8^k \\ 50 \times 8^k & (150 + (k-1) \times 25) \times 8^k \end{vmatrix} = 100 \times 8^k,$$

where k is the number of blocks. So, we have

$$\tau(G) = \frac{1}{\begin{vmatrix} 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & -1 & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & \ddots & 0 & \ddots & \ddots & -1 \\ -1 & -1 & \ddots & \ddots & 0 & \ddots & -1 & \ddots & \ddots \\ -1 & 0 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \cdots & \ddots & 0 & -1 & \ddots & 0 & -1 & 4 \end{vmatrix}_{(2k+3)*(2k+3)}} \begin{vmatrix} \begin{vmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\ -1 & 4 & \ddots & \ddots & \ddots & \cdots & -1 & \ddots & \ddots & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \ddots & \ddots & -1 & \ddots & 0 & \ddots & \ddots & -1 \\ -1 & -1 & -1 & \ddots & \ddots & 0 & \ddots & -1 & \ddots & 0 \\ 0 & \ddots & 0 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 0 & -1 & \ddots & \cdots & -1 & 4 \end{vmatrix}_{(2k+4)*(2k+4)} & \begin{vmatrix} -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 4 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \cdots & \vdots \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & \ddots & 0 & \ddots & \ddots & \ddots & -1 \\ -1 & -1 & \ddots & \ddots & 0 & \ddots & -1 & \ddots & \ddots & 0 \\ -1 & 0 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & -1 & \ddots & -1 & 0 & \cdots & -1 & 4 & -1 \end{vmatrix}_{(2k+4)*(2k+4)} \\ \begin{vmatrix} -1 & 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & -1 & \ddots & 0 & \ddots & \ddots & -1 \\ -1 & -1 & -1 & \ddots & \ddots & 0 & \ddots & -1 & \ddots & 0 \\ 0 & \ddots & 0 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & 0 & -1 & \ddots & \ddots & \ddots & 4 \\ 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \ddots & \ddots & -1 \end{vmatrix}_{(2k+4)*(2k+4)} & \begin{vmatrix} 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & \ddots & 0 & \ddots & \ddots & -1 & -1 \\ -1 & -1 & \ddots & \ddots & 0 & \ddots & -1 & \ddots & \ddots & 0 \\ -1 & 0 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 & -1 & 4 \end{vmatrix}_{(2k+4)*(2k+4)} \end{vmatrix}$$

in which

$$\frac{1}{S} = \frac{1}{\begin{vmatrix} 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & -1 & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & \ddots & 0 & \ddots & \ddots & -1 \\ -1 & -1 & \ddots & \ddots & 0 & \ddots & -1 & \ddots & \ddots \\ -1 & 0 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \cdots & \ddots & 0 & -1 & \ddots & 0 & -1 & 4 \end{vmatrix}_{(n-3)\times(n-3)}}$$

$$A = \begin{vmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\ -1 & 4 & \ddots & \ddots & \ddots & \cdots & -1 & \ddots & \ddots & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \ddots & \ddots & -1 & \ddots & 0 & \ddots & \ddots & -1 \\ -1 & -1 & -1 & \ddots & \ddots & 0 & \ddots & -1 & \ddots & 0 \\ 0 & \ddots & 0 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 0 & -1 & \ddots & \cdots & -1 & 4 \end{vmatrix}_{(n-2)\times(n-2)},$$

$$B = \begin{vmatrix} -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 4 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \cdots & \vdots \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & \ddots & 0 & \ddots & \ddots & \ddots & -1 \\ -1 & -1 & \ddots & \ddots & 0 & \ddots & -1 & \ddots & \ddots & 0 \\ -1 & 0 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & -1 & \ddots & -1 & 0 & \cdots & -1 & 4 & -1 \end{vmatrix}_{(n-2)\times(n-2)}$$

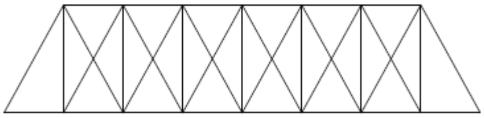
$$B^T = \begin{vmatrix} -1 & 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & -1 & \ddots & 0 & \ddots & \ddots & -1 \\ -1 & -1 & -1 & \ddots & \ddots & 0 & \ddots & -1 & \ddots & 0 \\ 0 & \ddots & 0 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & 0 & -1 & \ddots & \ddots & \ddots & 4 \\ 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \ddots & \ddots & -1 \end{vmatrix}_{(n-2)\times(n-2)}$$

and

$$C = \begin{vmatrix} 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & \ddots & 0 & \ddots & \ddots & -1 & -1 \\ -1 & -1 & \ddots & \ddots & 0 & \ddots & -1 & \ddots & \ddots & 0 \\ -1 & 0 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 & -1 & 4 \end{vmatrix}_{(n-2)\times(n-2)}$$

Theorem 2.2. Let $T(P_n^{n-2})$ be a trapezoidal graph of $(2n\ -2)$ vertices with triangulations

$$\tau\big(T(P_n^{n-2})\big) = \tau$$



such that

$$\tau\big(T(P_n^{n-2})\big) = 100 \times 12^{k-1}, \qquad k \geq 1$$

where k is the number of blocks.

Proof. By using Eq. (1), we can apply the proof by performing the following steps:

- By mathematical induction, we prove this statement at $m = 1$, i.e. $\tau(G)$ is a 2×2-matrix with both rows the same:

$$\tau(G) = \frac{1}{50}\begin{bmatrix} 60 & 50 \\ 50 & 125 \end{bmatrix} = 100.$$

- We prove the statement at $m = k + 1$, that is straightforward induction using properties of determinants and Dodgson and Chio method and applying Eq. (2), we have:

$$\tau(G) = \begin{vmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 4 & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & 5 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 5 & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\
0 & \ddots & \ddots & \ddots & -1 & 4 & 0 & \ddots & \ddots & -1 & -1 \\
-1 & -1 & -1 & \ddots & \ddots & 0 & 4 & -1 & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 5 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 5 & -1 \\
0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 & -1 & 4
\end{vmatrix}_{(2k+3)\times(2k+3)}$$

Consequently, we can have

$$\tau(G) = \frac{1}{(50 + (k-1) \times 15) \times 12^{k-1}}\begin{vmatrix} 60 \times 12^{k-1} & 50 \times 12^{k-1} \\ 50 \times 12^{k-1} & (125 + (k-1) \times 25) \times 12^{k-1} \end{vmatrix},$$

or

$$\tau(G) = 100 \times 12^{k-1},$$

where k is the number of blocks.

$$\tau(G)=\frac{1}{\begin{vmatrix} 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots \\ -1 & 5 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & 5 & -1 & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & 4 & 0 & \ddots & \ddots & -1 \\ -1 & -1 & \ddots & \ddots & 0 & 4 & -1 & \ddots & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 5 & \ddots & \vdots \\ 0 & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & -1 & -1 & 0 & \cdots & 0 & -1 & 5 \end{vmatrix}_{(2k+1)*(2k+1)}} \begin{vmatrix} \begin{vmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\ -1 & 4 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \vdots \\ 0 & \ddots & 5 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & 5 & -1 & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & \ddots & \ddots & -1 & 4 & 0 & \ddots & \ddots & -1 \\ -1 & -1 & -1 & \ddots & \ddots & 0 & 4 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 5 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & -1 & -1 & 0 & \cdots & -1 & 5 \end{vmatrix}_{(2k+2)*(2k+2)} & \begin{vmatrix} -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 4 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & \vdots \\ -1 & 5 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 5 & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & 4 & 0 & \ddots & \ddots & -1 & -1 \\ -1 & -1 & \ddots & \ddots & 0 & 4 & -1 & \ddots & \ddots & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 5 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & -1 & -1 & -1 & 0 & \cdots & -1 & 5 & -1 \end{vmatrix}_{(2k+2)*(2k+2)} \\ \begin{vmatrix} -1 & 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots \\ 0 & \ddots & 5 & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & 5 & -1 & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & \ddots & \ddots & -1 & 4 & 0 & \ddots & \ddots & -1 \\ -1 & -1 & -1 & \ddots & \ddots & 0 & 4 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 5 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 5 \\ 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 & -1 \end{vmatrix}_{(2k+2)*(2k+2)} & \begin{vmatrix} 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 \\ -1 & 5 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 5 & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & 4 & 0 & \ddots & \ddots & -1 & -1 \\ -1 & -1 & \ddots & \ddots & 0 & 4 & -1 & \ddots & \ddots & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 5 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 5 & -1 \\ 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 & -1 & 4 \end{vmatrix}_{(2k+2)*(2k+2)} \end{vmatrix}$$

for which

$$\frac{1}{S}=\frac{1}{\begin{vmatrix} 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots \\ -1 & 5 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & 5 & -1 & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & 4 & 0 & \ddots & \ddots & -1 \\ -1 & -1 & \ddots & \ddots & 0 & 4 & -1 & \ddots & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 5 & \ddots & \vdots \\ 0 & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & -1 & -1 & 0 & \cdots & 0 & -1 & 5 \end{vmatrix}_{(2k+1)\times(2k+1)}}$$

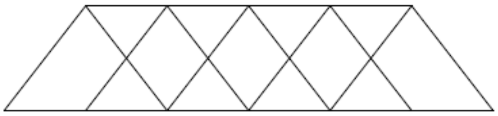
$$A=\begin{vmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\ -1 & 4 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \vdots \\ 0 & \ddots & 5 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & 5 & -1 & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & \ddots & \ddots & -1 & 4 & 0 & \ddots & \ddots & -1 \\ -1 & -1 & -1 & \ddots & \ddots & 0 & 4 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 5 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & -1 & -1 & 0 & \cdots & -1 & 5 \end{vmatrix}_{(2k+2)\times(2k+2)}$$

$$B=\begin{vmatrix} -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 4 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & \vdots \\ -1 & 5 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 5 & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & 4 & 0 & \ddots & \ddots & -1 & -1 \\ -1 & -1 & \ddots & \ddots & 0 & 4 & -1 & \ddots & \ddots & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 5 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & -1 & -1 & -1 & 0 & \cdots & -1 & 5 & -1 \end{vmatrix},$$

$$B^T = \begin{vmatrix} -1 & 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots \\ 0 & \ddots & 5 & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & 5 & -1 & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & \ddots & \ddots & -1 & 4 & 0 & \ddots & \ddots & -1 \\ -1 & -1 & -1 & \ddots & \ddots & 0 & 4 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 5 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 5 \\ 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 & -1 \end{vmatrix}_{(2k+2)*(2k+2)}$$

$$C = \begin{vmatrix} 4 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 \\ -1 & 5 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 5 & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \ddots & \ddots & -1 & 4 & 0 & \ddots & \ddots & -1 & -1 \\ -1 & -1 & \ddots & \ddots & 0 & 4 & -1 & \ddots & \ddots & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 5 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 5 & -1 \\ 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 & -1 & 4 \end{vmatrix}_{(2k+2)*(2k+2)}$$

Theorem 2.3. Let $T(P_n^{n-2})$ be a trapezoidal graph of $2n-2$ vertices with triangulations

$\tau\big(T(P_n^{n-2})\big) = \tau\Bigg($  $\Bigg)$

such that

$$\tau(T(P_n^{n-2})) = 9 \times 2^{3k+2}, \quad k \geq 1,$$

where k is the number of blocks.

Proof. By applying Eq. (1), we can perform the following steps:

- By mathematical induction, we prove this statement at $m = 1$, i.e. $\tau(G)$ is a 2×2 matrix with both rows the same:

$$\tau(T(P_n^{n-2})) = \frac{1}{216}\begin{bmatrix} 192 & 144 \\ 144 & 432 \end{bmatrix} = 288.$$

- We try to prove the statement at $m = k+1$; that is straightforward induction using properties of determinants and Dodgson and Chio method and applying Eq. (2), we have:

$$\tau(T(P_n^{n-2})) = \left|\begin{matrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 3 & \ddots & \ddots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & \ddots & 4 & \ddots & \ddots & \cdots & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 4 & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \vdots & \vdots & \ddots & -1 & 3 & 0 & \ddots & \ddots & -1 & 0 \\ -1 & \ddots & -1 & \ddots & \ddots & 0 & 3 & -1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 4 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 4 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 3 \end{matrix}\right|_{(2k+5)\times(2k+5)}$$

Consequently, we can have

$$\tau(T(P_n^{n-2})) = \frac{1}{(8^k \times (27 + 6\,(k-1)))}\left|\begin{matrix} 3 \times 8^{k+1} & 18 \times 8^k \\ 18 \times 8^k & 8^k \times (54 + 9\,(k-1)) \end{matrix}\right| = 9 \times 2^{3k+2},$$

where k is the number of blocks

$$-\tau(G) = \frac{1}{S}\left|\begin{matrix} A & B \\ B^T & C \end{matrix}\right|.$$

As a result, we get

$$\tau(G) = \frac{1}{\left|\begin{matrix} 3 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots \\ -1 & 4 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & 4 & -1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 & 3 & 0 & \ddots & \ddots & -1 \\ 0 & -1 & \ddots & \ddots & 0 & 3 & -1 & \ddots & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 4 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \cdots & -1 & 0 & -1 & 0 & 0 & -1 & 4 \end{matrix}\right|_{(2k+3)*(2k+3)}} \left\|\begin{matrix} \left|\begin{matrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\ -1 & 3 & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 4 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & 4 & -1 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & -1 & 3 & 0 & \ddots & \ddots & -1 \\ -1 & \ddots & -1 & \ddots & \ddots & 0 & 3 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 4 \end{matrix}\right|_{(2k+4)*(2k+4)} & \left|\begin{matrix} -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 3 & \ddots & \ddots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -1 & 4 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 4 & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & -1 & 3 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & -1 & \ddots & \ddots & 0 & 3 & -1 & \ddots & \ddots & \vdots \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 4 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & -1 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \end{matrix}\right|_{(2k+4)*(2k+4)} \\ \left|\begin{matrix} -1 & 3 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots \\ 0 & \ddots & 4 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & 4 & -1 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & -1 & 3 & 0 & \ddots & \ddots & -1 \\ -1 & \ddots & -1 & \ddots & \ddots & 0 & 3 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 4 \\ 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 \end{matrix}\right|_{(2k+4)*(2k+4)} & \left|\begin{matrix} 3 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\ -1 & 4 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 4 & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & -1 & 3 & 0 & \ddots & \ddots & -1 & 0 \\ 0 & -1 & \ddots & \ddots & 0 & 3 & -1 & \ddots & \ddots & \vdots \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 4 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 4 & -1 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 3 \end{matrix}\right|_{(2k+4)*(2k+4)} \end{matrix}\right|$$

for which

$$\frac{1}{S} = \frac{1}{\left|\begin{matrix} 3 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots \\ -1 & 4 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & 4 & -1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 & 3 & 0 & \ddots & \ddots & -1 \\ 0 & -1 & \ddots & \ddots & 0 & 3 & -1 & \ddots & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 4 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \cdots & -1 & 0 & -1 & 0 & 0 & -1 & 4 \end{matrix}\right|_{(2m+3)\times(2m+3)}},$$

$$\mathrm{A}=\begin{vmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\ -1 & 3 & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 4 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & 4 & -1 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & -1 & 3 & 0 & \ddots & \ddots & -1 \\ -1 & \ddots & -1 & \ddots & \ddots & 0 & 3 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 4 \end{vmatrix}_{(2m+4)\times(2m+4)},$$

$$\mathrm{B}=\begin{vmatrix} -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 3 & \ddots & \ddots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -1 & 4 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 4 & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & -1 & 3 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & -1 & \ddots & \ddots & 0 & 3 & -1 & \ddots & \ddots & \vdots \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 4 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & -1 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \end{vmatrix}_{(2m+4)*(2m+4)}$$

$$B^T=\begin{vmatrix} -1 & 3 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots \\ 0 & \ddots & 4 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & 4 & -1 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & -1 & 3 & 0 & \ddots & \ddots & -1 \\ -1 & \ddots & -1 & \ddots & \ddots & 0 & 3 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 4 \\ 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 \end{vmatrix}_{(2m+4)*(2m+4)},$$

and

$$\mathrm{C}=\begin{vmatrix} 3 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\ -1 & 4 & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 4 & -1 & \ddots & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & -1 & 3 & 0 & \ddots & \ddots & -1 & 0 \\ 0 & -1 & \ddots & \ddots & 0 & 3 & -1 & \ddots & \ddots & \vdots \\ -1 & \ddots & \ddots & \ddots & \ddots & -1 & 4 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 & \ddots & \ddots & \ddots & 4 & -1 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 3 \end{vmatrix}_{(2m+4)*(2m+4)}$$

3. Conclusion

An essential invariant and significant indicator of a network's dependability is the number of spanning trees in graphs or networks. In this work, we used matrix analysis and linear algebra to construct elementary formulae for the complexity of several kinds of special graphs, including trapezoidal graph forms. The derived expressions provide a computationally efficient approach to evaluating the structural properties of these graphs. Furthermore, our results contribute to a deeper understanding of network resilience and optimization, which can be beneficial in various applications, such as communication networks, circuit design, and combinatorial optimization.

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