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A Study on Perfect Rings Dominating Energy of Graphs

Prema M[1], Ruby Selestina M[2] and Purushothama S[3]

[1]Research Scholar, Department of Mathematics, Yuvaraja's College, University of Mysore, Mysuru

[2] Associate Professor, Department of Mathematics, Yuvaraja's College, University of Mysore, Mysuru

[3]Associate Professor, Department of Mathematics, MIT Mysore, Mysuru

Email: mpremamallaiah@gmail.com ruby.salestina@gmail.com psmandya@gmail.com

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Abstract:
A novel parameter called perfect rings domination has been developed in this study. The corresponding perfect rings dominating matrix is also generated. The energy of graphs is the sum of absolute value of their spectrum. This study investigates the spectrum and energy of certain family of graphs as well as ϑ-obrazom of graphs corresponding to this matrix. Moreover upper and lower bound are also established.

Keywords: PR Domination, PRD spectrum, PRD Energy

1. Introduction

Graphs with no isolated vertex are considered for this study. Graph energy is a significant topological index in the domain of chemical graph theory which can be used in chemistry. The absicth of graph energy was debuted by I. Gutman [9] in 1978 as summation of modulus values of spectrum of G, corresponds to adjacency matrix. A vast study of utilization on graph energy was pursued by I. Gutman and Balakrishnan [2, 10]. The consummation of energy and the equel with bounds can be beholded in vigous scrutinize of graph energy [12,13]. The necessary properties and vital chemical utilizations were evaluated in the molecular orbital theory of conjugated molecules [3-6]. For graph theoretic parlance one may refer Harary [11].

Let $S \subseteq V$. S is a dominating set if all $v \in V - S$ has a neighbor in S, minimum cardinality among such sets is called a minimum dominating set. A dominating set S is perfect if all v in G is dominated by strictly one element of S [13]. A dominating set S is rings domination if each v in V – S is adjacent to minimum two elements in V – S [1]. A dominating set A is a perfect rings dominating (PRD) set if (i) every $v \in G$ is dominated by strictly one element of A (ii) $\forall\, v \in V\backslash A, |N(v) \cap (V\backslash A)| \geq 2$. PRD set with minimum cardinality is the minimum PRD set of G and notate minimum PRD number by '$\mathfrak{p}$'.

This paper scrutinize the perfect rings dominating spectrum (say $\mathfrak{P}$-spectrum) and perfect rings dominating energy (say $\mathfrak{PE}$) of few classes of graphs as well as derive some bounds on $\mathfrak{PE}$.

1.1 Lemma [14]: Let $B = \begin{bmatrix} B_0 & B_1 \\ B_1 & B_0 \end{bmatrix}$ be a symmetric block matrix has order 2 with B_0 and B_1 are square matrices of same order. Then spectrum of B is the union of spectrum of $B_0 + B_1$ and $B_0 - B_1$.

2. PERFECT RINGS DOMINATING ENERGY

2.1 Definition: Consider $G = (p, q)$. Let $A \subseteq V(G)$ be its minimum perfect rings dominating (PRD) set. Then the perfect rings dominating matrix of G corresponding to A is a matrix $\mathfrak{P}_A$ has order p, defined as

$$\mathfrak{P}_A = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j,\ v_i \in A \\ 0 & \text{otherwise} \end{cases}$$

The characteristic polynomial of $\mathfrak{P}_A$ is $\varphi(\mathfrak{P}_A,\lambda) = \det(\mathfrak{P}_A - \lambda I)$. The $\mathfrak{P}$-spectrum of G is the eigenvalues of the matrix$\mathfrak{P}_A$. Let $\lambda_1, \lambda_2, \ldots \lambda_p$ be the spectrum of $\mathfrak{P}_A$. Then the perfect rings dominating energy $\mathfrak{PE}$ of G corresponding to A is defined as $\mathfrak{PE}_A(G) = \sum_{i=1}^{p} |\lambda_i|$.

Let $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_p$ be the spectrum of $\mathfrak{P}_A$ and they can be notated as $Spec_A(G) = \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_p \\ m_1 & m_2 & m_3 & \ldots & m_p \end{Bmatrix}$ where m_i is the algebraic multiplicity of eigenvaluesλ_i, for $1 \le i \le p$.

2.2 Remark: Though all the minimum PRD sets are of same cardinality, the perfect rings dominating energy $\mathfrak{PE}(G)$ need not be same for all PRD set.

2.3 Remark: If G has a unique minimum PRD set, then $\mathfrak{PE}_A(G)$ can be denoted as $\mathfrak{PE}(G)$.

3. PRD ENERGY OF GRAPHS

3.1 Theorem: For any complete graph K_p with $p \ge 3$,

$$\mathfrak{PE}_A(K_p) = p - 2 + \sqrt{p(p-2)+5}$$

Proof: Let $V(K_p) = \{v_1, v_2, \ldots, v_p\}$. Any arbitrary element of G will be a PRD-set. Consider $A = \{v_1\}$ as a PRD-set. Therefore the PRD-matrix has the structure

$$\mathfrak{P}_A(K_p) = \begin{pmatrix} 1 & J_{1\times p-1} \\ J_{p-1\times 1} & J_{p-1} - I_{p-1} \end{pmatrix}$$

Where J and I represents the matrix of 1's and identity matrix.

And the corresponding characteristic polynomial is $\varphi(K_p,\lambda) = \det(\mathfrak{P}_A(K_p) - \lambda I)$

$$\Rightarrow \varphi(K_p,\lambda) = (-1)^p(\lambda+1)^{p-2}(\lambda^2 - (p-1)\lambda - 1)$$

Then,

$$Spec_A(K_p) = \begin{Bmatrix} -1 & \frac{(p-2)+\sqrt{p^2-2p+5}}{2} & \frac{(p-2)-\sqrt{p^2-2p+5}}{2} \\ p-2 & 1 & 1 \end{Bmatrix}$$

Thus the PRD-energy is

$$\mathfrak{PE}_A(K_p) = (p-2)|-1| + \left|\frac{(p-2)+\sqrt{p^2-2p+5}}{2}\right| + \left|\frac{(p-2)-\sqrt{p^2-2p+5}}{2}\right|$$

$$= p - 2 + \sqrt{p^2-2p+5}$$

$$\mathfrak{PE}_A(K_p) = p - 2 + \sqrt{p(p-2)+5}$$

3.2 Theorem: For $K_{p,q}$ with $p, q \ge 2$,

$$\mathfrak{PE}_A(K_{p,q}) = \begin{cases} \sqrt{(p-1)^2+3} + \sqrt{(p+1)^2-4} & \text{if } p = q \\ 2(1+\sqrt{pq}) & \text{if } p < q \ \& \text{ if } p > q \end{cases}$$

Proof: Let $V(K_{p,q}) = \{v_1, \ldots, v_p, v_1', \ldots, v_q'\}$. Consider $A = \{v_1, v_1'\}$ as a PRD-set. Therefore the PRD-matrix has the structure

$$\mathfrak{P}_A(K_{p,q}) = \begin{pmatrix} 1 & J_{1\times p-1} \\ J_{p-1\times 1} & J_{p-1} - I_{p-1} \end{pmatrix}$$

i) Suppose $p = q$, then $\mathfrak{P}_A(K_{p,p})$ has the structure

$$\mathfrak{P}_A(K_{p,p}) = \begin{pmatrix} B_0 & B_1 \\ B_1 & B_0 \end{pmatrix}$$

Where $B_0 = \begin{pmatrix} 1 & O_{1\times p-1} \\ O_{p-1\times 1} & O_{p-1} \end{pmatrix}$ and $B_1 = (J_p)$, here J and O represents the matrix of 1's and 0's.
Therefore by Lemma 1.1, $Spec_A(K_{p,p}) = Spec(B_0 + B_1) \cup Spec(B_0 - B_1)$.
Consider, $B_0 + B_1$:

$$B_0 + B_1 = \begin{pmatrix} 2 & J_{1\times p-1} \\ J_{p-1\times 1} & J_{p-1} \end{pmatrix}$$

$$|(B_0 + B_1) - \lambda I| = (-1)^p \lambda^{p-2}\left(\lambda^2 - (p+1)\lambda + (p-1)\right)$$

Therefore,

$$Spec_A(B_0 + B_1) = \begin{Bmatrix} 0 & \frac{(p+1) + \sqrt{(p-1)^2 + 3}}{2} & \frac{(p+1) - \sqrt{(p-1)^2 + 3}}{2} \\ p-2 & 1 & 1 \end{Bmatrix}$$

Consider, $B_0 - B_1$:

$$B_0 - B_1 = \begin{pmatrix} 0 & -J_{1\times p-1} \\ -J_{p-1\times 1} & -J_{p-1} \end{pmatrix}$$

$$|(B_0 - B_1) - \lambda I| = (-1)^p \lambda^{p-2}\left(\lambda^2 + (p-1)\lambda - (p-1)\right)$$

Therefore,

$$Spec_A(B_0 - B_1) = \begin{Bmatrix} 0 & \frac{(1-p) + \sqrt{(p+1)^2 - 4}}{2} & \frac{(1-p) - \sqrt{(p+1)^2 - 4}}{2} \\ p-2 & 1 & 1 \end{Bmatrix}$$

Hence

$$Spec_A(K_{p,p}) = \begin{Bmatrix} 0 & \frac{(p+1) + \sqrt{(p-1)^2 + 3}}{2} & \frac{(p+1) - \sqrt{(p-1)^2 + 3}}{2} & \frac{(1-p) + \sqrt{(p+1)^2 - 4}}{2} & \frac{(1-p) - \sqrt{(p+1)^2 - 4}}{2} \\ 2p-4 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

Now,

$$\mathfrak{PE}_A(K_{p,p}) = \left|\frac{(p+1) + \sqrt{(p-1)^2 + 3}}{2}\right| + \left|\frac{(p+1) - \sqrt{(p-1)^2 + 3}}{2}\right| + \left|\frac{(1-p) + \sqrt{(p+1)^2 - 4}}{2}\right| + \left|\frac{(1-p) - \sqrt{(p+1)^2 - 4}}{2}\right|$$

Thus,

$$\mathfrak{PE}_A(K_{p,p}) = \frac{2\sqrt{(p-1)^2 + 3}}{2} + \frac{2\sqrt{(p+1)^2 - 4}}{2}$$

$$\mathfrak{PE}_A(K_{p,p}) = \sqrt{(p-1)^2 + 3} + \sqrt{(p+1)^2 - 4}$$

ii) Suppose $p > q$, then the PRD matrix is of the form

$$\mathfrak{P}_A(K_{p,q}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & O_{1\times p-1} \\ O_{p-1\times 1} & O_{p-1} \end{pmatrix}$; $B = (J_{p\times q})$

$C = (J_{q\times p})$; $D = \begin{pmatrix} 1 & O_{1\times q-1} \\ O_{q-1\times 1} & O_{q-1} \end{pmatrix}$, here J and O represents the matrix of 1's and 0's.

And the corresponding characteristic polynomial is,

$$\varphi(K_{p,q},\lambda) = \det(\mathfrak{P}_A(K_{p,q}) - \lambda I)$$

$$\varphi(K_{p,q},\lambda) = (-1)^p \lambda^{p+q-4}\ \left[\lambda^4 - 2\lambda^3 - (pq-1)\lambda^2 + (2pq-(p+q))\lambda - [pq-(p+q)+1]\right]$$

$$\approx \lambda^{p+q-4}(\lambda - 1)\left[\lambda^3 - \lambda^2 - pq\lambda - (pq-(p+q))\right]$$

$$\varphi(K_{p,q},\lambda) \approx \lambda^{p+q-4}(\lambda-1)^2(\lambda^2 - pq)$$

iii) Suppose $p > q$, then the PRD matrix is of the form

$$\mathfrak{P}_A(K_{p,q}) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

where $P = \begin{pmatrix} 1 & O_{1\times p-1} \\ O_{p-1\times 1} & O_{p-1} \end{pmatrix}$; $Q = (J_{p\times q})$

$R = (J_{q\times p})$; $S = \begin{pmatrix} 1 & O_{1\times q-1} \\ O_{q-1\times 1} & O_{q-1} \end{pmatrix}$

here J and O represents the matrix of 1's and 0's.
Therefore the corresponding characteristic polynomial is,

$$\varphi(K_{p,q},\lambda) = \det(\mathfrak{P}_A(K_{p,q}) - \lambda I)$$

$$\varphi(K_{p,q},\lambda) = (-1)^p \lambda^{p+q-4}\ \left[\lambda^4 - 2\lambda^3 - (pq-1)\lambda^2 + (2pq-(p+q))\lambda - [pq-(p+q)+1]\right]$$

$$\Longrightarrow \varphi(K_{p,q},\lambda) \approx \lambda^{p+q-4}(\lambda-1)^2(\lambda^2 - pq)$$

For the both cases ii), iii) the characteristic polynomial remains same. Hence,

$$\mathrm{Spec}_A(K_{p,q}) = \begin{Bmatrix} 0 & 1 & \sqrt{pq} & -\sqrt{pq} \\ p+q-4 & 2 & 1 & 1 \end{Bmatrix}$$

$$\mathfrak{PE}_A(K_{p,p}) = 2(1+\sqrt{pq})$$

3.3 Theorem: For a crown graph ${S_p}^0$ with $p \geq 2$,

$$\mathfrak{PE}_A({S_p}^0) = 2(p-2) + \sqrt{p^2 - 2p + 5} + \sqrt{p^2 + 2p - 3}$$

Proof: Let $V({S_p}^0) = \{v_1, \ldots, v_p, v_1', \ldots, v_p'\}$, let $A = \{v_1, v_1'\}$ be a PRD set. Then $\mathfrak{P}_A({S_p}^0)$ has the form $\begin{pmatrix} B_0 & B_1 \\ B_1 & B_0 \end{pmatrix}$, where $B_0 = \begin{pmatrix} 1 & O_{1\times p-1} \\ O_{p-1\times 1} & O_{p-1} \end{pmatrix}$ and $B_1 = (J_p - I_p)$, here J, O and I represents the matrix of 1's, 0's and identity matrix.
Therefore by Lemma 1.1, $\mathrm{Spec}_A({S_p}^0) = \mathrm{Spec}(B_0 + B_1) \cup \mathrm{Spec}(B_0 - B_1)$.
Consider, $B_0 + B_1$:

$$B_0 + B_1 = \begin{pmatrix} 1 & J_{1\times p-1} \\ J_{p-1\times 1} & J_{p-1} - I_{p-1} \end{pmatrix}$$

$$|(B_0 + B_1) - \lambda I| = (-1)^p(\lambda+1)^{p-2}(\lambda^2 - (p-1)\lambda - 1)$$

Therefore,

$$\mathrm{Spec}_A(B_0 + B_1) = \begin{Bmatrix} -1 & \dfrac{(p-1)+\sqrt{p^2-2p+5}}{2} & \dfrac{(p-1)-\sqrt{p^2-2p+5}}{2} \\ p-2 & 1 & 1 \end{Bmatrix}$$

Consider, $B_0 - B_1$:

$$B_0 - B_1 = \begin{pmatrix} 1 & -J_{1\times p-1} \\ -J_{p-1\times 1} & -(J_{p-1} - I_{p-1}) \end{pmatrix}$$

$$|(B_0 - B_1) - \lambda I| = (-1)^p(\lambda-1)^{p-2}(\lambda^2 + (p-3)\lambda - (2p-3))$$

Therefore,

$$\mathrm{Spec}_A(B_0 - B_1) = \begin{Bmatrix} 1 & \frac{(3-p)+\sqrt{p^2+2p-3}}{2} & \frac{(3-p)-\sqrt{p^2+2p-3}}{2} \\ p-2 & 1 & 1 \end{Bmatrix}$$

Hence

$$\mathrm{Spec}_A(S_p{}^0) = \begin{Bmatrix} -1 & 1 & \frac{(p-1)+\sqrt{p^2-2p+5}}{2} & \frac{(p-1)-\sqrt{p^2-2p+5}}{2} & \frac{(3-p)+\sqrt{p^2+2p-3}}{2} & \frac{(3-p)-\sqrt{p^2+2p-3}}{2} \\ p-2 & p-2 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

Now,

$$\mathfrak{PE}_A(S_p{}^0) = (p-2)|-1| + (p-2)|1| + \left|\frac{(p-1)+\sqrt{p^2-2p+5}}{2}\right| + \left|\frac{(p-1)-\sqrt{p^2-2p+5}}{2}\right| + \left|\frac{(3-p)+\sqrt{p^2+2p-3}}{2}\right| + \left|\frac{(3-p)-\sqrt{p^2+2p-3}}{2}\right|$$

Thus,

$$\mathfrak{PE}_A(S_p{}^0) = 2(p-2) + \sqrt{p^2-2p+5} + \sqrt{p^2+2p-3}$$

3.4 Theorem: For a barbell graph $B_{p,p}$ with $p > 3$, $\mathfrak{PE}_A(B_{p,p}) = 3p - 4 + \sqrt{(p-2)^2 + 8}$

Proof: Let the vertex set $V(B_{p,p}) = \{v_1, \ldots, v_p, v_1{}', \ldots, v_p{}'\}$. Consider the PRD set $A = \{v_1, v_1{}'\}$.
Then $\mathfrak{P}_A(B_{p,p})$ has the form $\begin{pmatrix} B_0 & B_1 \\ B_1 & B_0 \end{pmatrix}$,
where $B_0 = \begin{pmatrix} 1 & J_{1\times p-1} \\ J_{p-1\times 1} & J_{p-1} - I_{p-1} \end{pmatrix}$ and $B_1 = \begin{pmatrix} 1 & O_{1\times p-1} \\ O_{p-1\times 1} & O_{p-1} \end{pmatrix}$, here J and O represents the matrix of 1's and 0's.
Therefore by Lemma 1.1, $\mathrm{Spec}_A(B_{p,p}) = \mathrm{Spec}(B_0 + B_1) \cup \mathrm{Spec}(B_0 - B_1)$.
Consider, $B_0 + B_1$:

$$B_0 + B_1 = \begin{pmatrix} 2 & J_{1\times p-1} \\ J_{p-1\times 1} & J_{p-1} - I_{p-1} \end{pmatrix}$$

$$|(B_0 + B_1) - \lambda I| = (-1)^p(\lambda + 1)^{p-2}(\lambda^2 - p\lambda + (p-3))$$

Therefore,

$$\mathrm{Spec}_A(B_0 + B_1) = \begin{Bmatrix} -1 & \frac{p+\sqrt{(p-2)^2+8}}{2} & \frac{p-\sqrt{(p-2)^2+8}}{2} \\ p-2 & 1 & 1 \end{Bmatrix}$$

Consider, $B_0 - B_1$:

$$B_0 - B_1 = \begin{pmatrix} 0 & J_{1\times p-1} \\ J_{p-1\times 1} & J_{p-1} - I_{p-1} \end{pmatrix}$$

$$|(B_0 - B_1) - \lambda I| = (-1)^p(\lambda - (p-1))(\lambda + 1)^{p-1}$$

Therefore,

$$\mathrm{Spec}_A(B_0 - B_1) = \begin{Bmatrix} p-1 & -1 \\ 1 & p-1 \end{Bmatrix}$$

Hence

$$\mathrm{Spec}_A(B_{p,p}) = \begin{Bmatrix} -1 & p-1 & \frac{p+\sqrt{(p-2)^2+8}}{2} & \frac{p-\sqrt{(p-2)^2+8}}{2} \\ 2p-3 & 1 & 1 & 1 \end{Bmatrix}$$

Now,

$$\mathfrak{PE}_A(B_{p,p}) = (2p-3)|-1| + |p-1| + \left|\frac{p+\sqrt{(p-2)^2+8}}{2}\right| + \left|\frac{p-\sqrt{(p-2)^2+8}}{2}\right|$$

Thus,

$$\mathfrak{PE}_A(B_{p,p}) = 3p - 4 + \sqrt{(p-2)^2+8}$$

3.5 Theorem: Let G be obtained by removing an edge $'e'$ from K_p, $p > 4$. Then G has the spectrum$p-1$, -1 and 0 with multiplicities 1,$p-2$ and 1 respectively. And hence $\mathfrak{PE}_A(G) = 2p-3$.

Proof: Let $G = K_p - e$. Let v_1, v_p be the non-adjacent vertices of G. Then the vertices other than v_1, v_p would be act as a PD-set. Since $p \geq 5$, $\deg(v_1) = \deg(v_p) = p-2$, any PD-set will satisfy the constrain of RD-set. Let $A = \{v_2\}$ is the PRD-set of G. Then the PRD matrix is,

$$\mathfrak{P}_A(G) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$; $B = \begin{pmatrix} J_{1\times p-3} & 0 \\ J_{1\times p-3} & 1 \end{pmatrix}$

$C = \begin{pmatrix} J_{p-3\times 1} & J_{p-3\times 1} \\ 0 & 1 \end{pmatrix}$; $D = \left(J_{p-3} - I_{p-3}\right)$

where J and I represents the matrix of 1's and identity matrix.

Now, $|\mathfrak{P}_A(G) - \lambda I| = (-1)^p \lambda(\lambda+1)^{p-4}(\lambda^3 - (p-3)\lambda^2 - (2p-3)\lambda - 2)$

$$\Rightarrow \varphi(G,\lambda) \approx (-1)^p \lambda(\lambda+1)^{p-4}\left(\lambda^3 - (p-3)\lambda^2 - (2p-3)\lambda - (p-1)\right)$$

$$\Rightarrow \varphi(G,\lambda) \approx (-1)^p \lambda(\lambda+1)^{p-2}\left(\lambda - (p-1)\right)$$

Therefore, $\mathrm{Spec}_A(G) = \begin{Bmatrix} p-1 & -1 & 0 \\ 1 & p-2 & 1 \end{Bmatrix}$

Thus, $\mathfrak{PE}_A(G) = |p-1| + (p-2)|-1| = p-1+p-2$

$$\mathfrak{PE}_A(G) = 2p-3$$

4. PRD ENERGY OF LINE GRAPHS OF STAR GRAPHS

4.1 Theorem: For the line graph of a star graph $L(K_{1,p-1})$ with$p > 4$,

$$\mathfrak{PE}_A\left(L(K_{1,p-1})\right) = p - 3 + \sqrt{(p-2)^2+4}$$

Proof: Let $E(K_{1,p-1}) = \{e_1, \ldots, e_{p-1}\}$. And it's known that $L(K_{1,p-1}) \cong K_{p-1}$.

Therefore $L(K_{1,p-1})$ and K_{p-1} are co-spectral and hence equi-energetic. Thus $\mathfrak{PE}_A\left(L(K_{1,p-1})\right) \cong \mathfrak{PE}_A(K_{p-1}) = p - 3 + \sqrt{(p-2)^2+4}$. Hence

$$\mathfrak{PE}_A\left(L(K_{1,p-1})\right) = p - 3 + \sqrt{(p-2)^2+4}$$

4.2 Theorem: For the line graph of a double star graph $L(S_{p,p})$ with $p > 3$,

$$\mathfrak{PE}\left(L(S_{p,p})\right) = 4p-5$$

Proof: Let $V(L(S_{p,p})) = \{e_0, e_1, \ldots, e_{p-1}, e_1', \ldots, e_{p-1}'\}$, where e_0 is the bridge connecting two star graphs. Then the unique PRD set of $L(S_{p,p})$ is $A = \{e_0\}$. Therefore

$$\mathfrak{P}_A(L(S_{p,p})) = \begin{pmatrix} 1 & J_{1\times p-1} & J_{1\times p-1} \\ J_{p-1\times 1} & J_{p-1} - I_{p-1} & O_{p-1} \\ J_{p-1\times 1} & O_{p-1} & J_{p-1} - I_{p-1} \end{pmatrix}$$

where J, O and I represents the matrix of 1's, 0's and identity matrix.
And the corresponding characteristic polynomial is,

$$\varphi(L(S_{p,p}),\lambda) = |\mathfrak{P}_A(L(S_{p,p})) - \lambda I)|$$
$$\varphi(L(S_{p,p}),\lambda) = -(\lambda+1)^{2p-3}(\lambda-(p-2))(\lambda-p)$$

Hence

$$Spec_A(L(S_{p,p})) = \begin{Bmatrix} 1 & 2-p & -p \\ 2p-3 & 1 & 1 \end{Bmatrix}$$

Now,

$$\mathfrak{PE}\left(L(S_{p,p})\right) = (2p-3)|-1| + |2-p| + |-p|$$
$$\Rightarrow \mathfrak{PE}\left(L(S_{p,p})\right) = 4p-5$$

5. CHARACTERISTICS OF PRD SPECTRUM

5.1 Theorem: Consider graph G. If $\lambda_1, \lambda_2, \ldots \lambda_p$ are the $\mathfrak{P}$-spectrum of $\mathfrak{P}_A(G)$, then the following condition holds.

(i) $\sum_{i=1}^{p} \lambda_i = \mathfrak{p}$.
(ii) $\sum_{i=1}^{p} {\lambda_i}^2 = 2q + \mathfrak{p}$.

Proof: As sum of spectrum of $\mathfrak{P}_A(G)$ is same as its trace,

$$\sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{p} \mathfrak{p}_{ii} = |A| = \mathfrak{p}$$

(i) As summation of squares of the spectrum of $\mathfrak{P}_A(G)$ is the trace of $[\mathfrak{P}_A(G)^2]$,

$$\sum_{i=1}^{p} {\lambda_i}^2 = \sum_{i=1}^{p}\sum_{i=1}^{p} \mathfrak{p}_{ij}\,\mathfrak{p}_{ji}$$
$$= \sum_{i=1}^{p} (\mathfrak{p}_{ii})^2 + \sum_{i\neq j} \mathfrak{p}_{ij}\,\mathfrak{p}_{ji}$$
$$= \sum_{i=1}^{p} (\mathfrak{p}_{ii})^2 + 2\sum_{i<j} (\mathfrak{p}_{ij})^2$$
$$= \mathfrak{p} + 2q$$
$$\sum_{i=1}^{p} {\lambda_i}^2 = 2q + \mathfrak{p}.$$

6. BOUNDS ON PRD ENERGY

6.1 Theorem: Let G be a (p,q) simple graph. Let $\Delta = |\det G_S(G)|$. Then

$$\sqrt{(2q+\mathfrak{p}) + p(p-1)\Delta^{\left(2/p\right)}} \leq \mathfrak{PE}(G) \leq \sqrt{p(2q+\mathfrak{p})}$$

where $\mathfrak{p}$ is the PRD number of G.
Proof: Consider Cauchy Schwarz inequality:

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} {a_i}^2\right)\left(\sum_{i=1}^{p} {b_i}^2\right)$$

Take $a_i = 1,\ b_i = |\lambda_i|$, gives

$$\left(\sum_{i=1}^{p} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{p} 1\right)\left(\sum_{i=1}^{p} {\lambda_i}^2\right)$$
$$\Rightarrow [\mathfrak{PE}(G)]^2 \leq p \sum_{i=1}^{p} {\lambda_i}^2$$

$$= p\,(2q + \mathfrak{p})$$

$$\Rightarrow \mathfrak{PE}(G) \leq \sqrt{p\,(2q + \mathfrak{p})} \qquad \dots\dots\dots (I)$$

Consider $[\mathfrak{PE}(G)\,]^2 = \left(\sum_{i=1}^{p} |\lambda_i|^2\right)$

$$= \left(\sum_{i=1}^{p} |\lambda_i|^2\right) + \sum_{i \neq j} |\lambda_i||\lambda_j|$$

$$[\mathfrak{PE}(G)\,]^2 = (2q + \mathfrak{p}) + \sum_{i \neq j} |\lambda_i||\lambda_j| \qquad \dots\dots\dots (II)$$

where $2q + \mathfrak{p} = \sum_{i=1}^{p} |\lambda_i|^2$.

The geometric mean cannot exceed the arithmetic mean, so

$$\frac{1}{p(p-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| \geq \left[\prod_{i \neq j} |\lambda_i||\lambda_j|\right]^{\frac{1}{p(p-1)}}$$

$$= \left[\prod_{i=1}^{p} |\lambda_i|^{2(p-1)}\right]^{\frac{1}{p(p-1)}}$$

$$= \left[\prod_{i=1}^{p} |\lambda_i|\right]^{2/p}$$

$$= \left|\prod_{i=1}^{p} \lambda_i\right|^{2/p}$$

$$= \left|\det(\mathfrak{PE}(G))\right|^{2/p}$$

$$= \Delta^{(2/p)}$$

i.e, $\frac{1}{p(p-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| \geq \Delta^{(2/p)}$

$$\Rightarrow \sum_{i \neq j} |\lambda_i||\lambda_j| \geq p(p-1)\,\Delta^{(2/p)} \qquad \dots\dots\dots (III)$$

Put (III) in (II)

$$[\mathfrak{PE}(G)\,]^2 \geq (2q + \mathfrak{p}) + p(p-1)\,\Delta^{2/p}$$

$$\mathfrak{PE}(G) \geq \sqrt{(2q + \mathfrak{p}) + p(p-1)\,\Delta^{2/p}} \qquad \dots\dots\dots (IV)$$

From (I) and (IV)

$$\sqrt{(2q + \mathfrak{p}) + p(p-1)\Delta^{2/p}} \leq \mathfrak{PE}(G) \leq \sqrt{p(2q + \mathfrak{p})}$$

6.2 Theorem: For any graph G $\sqrt{2q + \mathfrak{p}} \leq \mathfrak{PE}(G) \leq \sqrt{p(2q + \mathfrak{p})}$.

Proof: Consider Cauchy Schwarz inequality,

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} {a_i}^2\right)\left(\sum_{i=1}^{p} {b_i}^2\right)$$

Take $a_i = 1, b_i = |\lambda_i|$, gives

$$\left(\sum_{i=1}^{p} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{p} 1\right)\left(\sum_{i=1}^{p} |\lambda_i|^2\right)$$

$$\Rightarrow [\mathfrak{PE}(G)]^2 \leq p \sum_{i=1}^{p} {\lambda_i}^2$$

$$= \mathrm{p}(2\mathrm{q} + \mathfrak{p})$$

$$\Rightarrow \mathfrak{PE}(G) \leq \sqrt{\mathrm{p}(2\mathrm{q} + \mathfrak{p})} \qquad \ldots\ldots\ldots \text{(I)}$$

Consider $\left(\mathfrak{PE}(G)\right)^2 = \left(\sum_{i=1}^{P} |\lambda_i|\right)^2$

$$\geq \sum_{i=1}^{P} |\lambda_i|^2$$

$$= 2\mathrm{q} + \mathfrak{p}$$

$$\Rightarrow \mathfrak{PE}(G) \geq \sqrt{2\mathrm{q} + \mathfrak{p}} \qquad \ldots\ldots\ldots \text{(II)}$$

From (I) and (II), $\sqrt{2\mathrm{q} + \mathfrak{p}} \leq \mathfrak{PE}(G) \leq \sqrt{\mathrm{p}(2\mathrm{q} + \mathfrak{p})}$

7. OPEN PROBLEMS

1. Determine the class of graphs whose PE is same as to number of vertices.
2. Determine the class of graphs whose PE is same as usual energy.
3. Construct the PE equi-energetic graphs.

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