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Optimal Replacement Policies for a Multistate Degenerative System using Partial Sum Process

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Abstract:

This paper investigates the maintenance problem of a multistate degenerative system with k -working states and l -failure states. The long-run average cost for a multistate degenerative system under univariate replacement policy: T-policy, T-policy with NONN repair times are derived**.** Additionally, optimality conditions are established to guide the determination of cost-effective replacement strategies.

Keywords: Partial Sum Process (PSP), Geometric Process (GP), Renewal Process, Optimal Replacement Policy.

1. Introduction

Degradation and failure are inherent characteristics of many real-world systems, resulting from the cumulative effects of aging, wear, and environmental factors. This deterioration manifests as decreasing operating times between failures and increasing repair times after failures. The operating times may eventually become negligible, while the repair times may tend towards infinity. To capture this phenomenon, Lam (1988) introduced the geometric process (GP) repair model, which provides a mathematical framework for analyzing and predicting the behaviour of deteriorating systems. In engineering and reliability applications, system failures can be categorized based on their severity, ranging from minor, correctable faults to catastrophic, system-critical failures. Furthermore, systems often exhibit multiple functional states, representing varying levels of performance, capacity, or efficiency. Conversely, systems can also exist in multiple failed states, characterized by distinct fault modes, error conditions, or failure mechanisms. In this paper, we study a monotone process model for a one-component multistate system with $(k + l)$ states, namely k-working states and l-failure states. By making different assumptions, the model can be applied to a multistate degenerative system.

We first give the definition of NONN repair times.

Definition 1.1: *If a repair to a system after failure is done in negligible or non-negligible time, then it will be called a model with* NONN repair times.

Upon system failure, two repair scenarios arise: negligible repair time with probability p, or non-negligible repair time with probability $1 - p$. This NONN repair times model, introduced by Thangaraj and Rizwan (2001) and further studied by Govindaraju et al. (2010), accounts for uncertainty in repair durations, enhancing maintenance modelling accuracy.

The rest of the Chapter is arranged as follows: In Section 2, explicit expression for the long-run average cost of the policy T is derived, also the conditions for the existence of univariate optimal replacement policy T^* are derived. In Section 3, using an alternate repair model, we derive an expression for the long-run average cost for this model under T -policy. The necessary conditions for the existence of the optimal replacement policy are also derived in this section. Finally, conclusion is given Section 4.

2. The Replacement Policy *T*

This section examines a *T*-policy for optimizing replacement in a multistate, single-component deteriorating system. The objective is to identify the optimal replacement policy, denoted as *T**, minimizes the long-run average cost per unit time.

Consider a single-component multistate system comprising (k + l) distinct states, where k represents functional working states and l denotes failure states.

Definition 2.1: *Let* $\{X_n, n=1,2,3,\ldots\}$ *be a sequence of independent non negative random variables and let* $F(x)$ *be the distribution of* X_1*. Then* $\{X_n, n=1,2,3,\ldots\}$ *is called partial sum process, if the distribution of* X_{i+1} *is* $F(\beta_i x)\,(i=1,2,3,\ldots)$*, where* β_i *>0 are constants with* $\beta_i = \beta_0+\beta_1+\beta_2+\ldots\beta_{i-1}$ *and* $\beta_0 = \beta > 0$.

Assumption 2.1: At time t = 0, a new system is deployed in the field. Each time the system fails, it will be replaced by an identical new unit after a certain period.

Assumption 2.2: Let $S(t) \in \Omega$, where Ω is the state space, represent the system state at time t, characterizing the system's dynamic behaviour.

$$S(t) = \begin{cases} 0, & \text{if the state is working at time } t; \\ i, & \text{if the system is in the } i \text{ th type of failure state at time } t, i = 1,2,3,\ldots.k \end{cases}$$

when the system fails, it transitions to one of k possible states, with probabilities p_i, i =1, 2, ..., k, representing the likelihood of entering each failed state and $\sum_{i=1}^{k} p_i = 1$.

Assumption 2.3: To model system behaviour, we define four key time variables: X_1 be the first operating time, X_n be the operating time after $(n-1)$ repairs, Y_n be the repair time after n failures, and Z be the replacement time, which capture the dynamics of system operation, maintenance, and replacement. Let t_n denote the time of the *n-th* failure, representing the moment when the system fails for the nth time. Assume that there exist a life-time distribution *U(t)* and a_i > 0, $i = 1,2,3,\ldots.k$ such that ,

$$P(X_1 \le t) = U(t)$$

and $$P(X_2 \le t | S(t_1) = i) = U(a_i t)\,, i = 1,2,3,\ldots.k.$$

In general for $j = 1,2,3,\ldots.n-1$; $i_j = 1,2,3,\ldots.k$,

$$P(X_n \le t | S(t_1) = i_1, \ldots, S(t_{n-1}) = i_{n-1}) = U(a_{i_1} a_{i_{2.}} \ldots a_{i_{n-1}} t)\,,$$

where $1 \le a_1 \le a_2 \le \cdots \le a_k$.

Similarly, assume that there exist a life-time distribution $V(t)$ and b_i > 0, $i = 1,2,3,\ldots.k$. such that,

$$P(Y_1 \le t) = V(t)$$

and $P(Y_2 \le t | S(t_1) = i) = V(b_i t)\,, i = 1,2,3,\ldots.k.$

In general for $j = 1,2,3, \ldots . n$; $i_j = 1,2,3, \ldots . k$,

$P(Y_n \leq t | S(t_1) = i_1, \ldots, S(t_n) = i_n) = V(b_{i_1} b_{i_{2.}} \ldots b_{i_n} t)$,

where $1 \geq b_1 \geq b_2 \geq \cdots \geq b_k \geq 0$.

Assumption 2.4: The Working age T of the system at time t is the cumulative life-time given by

$$T(t) = \begin{cases} t - V_n , & U_n + V_n \leq t \leq U_{n+1} + V_n \\ U_{n+1} , & U_{n+1} + V_n \leq t \leq U_{n+1} + V_{n+1} \end{cases}$$

where, $U_n = \sum_{k=1}^{n} X_k$ and $V_n = \sum_{k=1}^{n} Y_k$ and $U_0 = V_0 = 0$.

Assumption 2.5: Let 'r' be the reward rate per unit time of the system when it is operating and 'c' be the repair cost per unit time of the system. Assume further that the replacement cost comprises of two parts: one part is the basic replacement cost R and the other part is the cost.

Assumption 2.6: The replacement policy $\boldsymbol{T}$ is adapted.

Following Lam, Zhang and Zheng [2002], we have if $E[X_1] = \mu > 0$ the mean survival time is $E[X_n] = \frac{\mu}{2^{i-1}\beta}$ $where\ \beta = \left(\sum_{i=1}^{k} \frac{p_i}{a_i}\right)^{-1}$ and if $E[Y_1] = \gamma$, then the mean repair time is $E[Y_i] = \frac{\gamma}{a^{i-1}}$ $where\ a = \left(\sum_{j=1}^{k} \frac{q_j}{b_j}\right)^{-1}$.

We consider a T -policy under which the system will be replaced whenever its working age reaches T. Under the replacement policy T, the problem is to determine an optimal T such that the long-run average cost per unit time is minimized.

The replacement times are defined as follows: T_1 represents the first replacement time, and for n = 2, 3, ..., T_n denotes the time between the (n-1)-st replacement and the n-th replacement. Consequently, the sequence $\{T_n$, n = 1, 2, ...$\}$ constitutes a renewal process. Cycle are defined by the time between consecutive system replacements, spanning from initial installation to first replacement, and subsequent replacement intervals. Each replacement concludes one cycle and begins another. By combining the successive cycles with the costs associated with each, we establish a renewal reward process, facilitating the assessment of cumulative costs, rewards, and system efficiency.

The cost structure for this model is,

$$C(T) = \frac{the\ expected\ cost\ incurred\ in\ a\ cycle}{the\ expected\ length\ of\ a\ cycle}$$

$$= \frac{cE\left[\sum_{k=1}^{\eta-1} Y_k\right] - r\, E\left[\sum_{k=1}^{\eta} X_k\right] + R\ +\ c_p E[\,Z\,]}{E\left[\sum_{k=1}^{\eta} X_k\right] + E\left[\sum_{k=1}^{\eta-1} Y_k\right] + E[Z]}, \tag{2.1}$$

where η is a random variable denoting the number of failures before the working age of the system reaches T.

$$E\left[\sum_{k=1}^{\eta} X_k\right] = E\left[E\left[\sum_{k=1}^{n-1} X_k \mid = n\right]\right] = \mu \sum_{n=1}^{\infty} \frac{F_n(T)}{2^{n-1}\beta} \tag{2.2}$$

where $F_n\ (\cdot)$ is the n -fold convolution of $F(\cdot)$ with itself and

$$E\left[\sum_{k=1}^{\eta-1} Y_k\right] = \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n-1} E[Y_k]\right] P(= n-1) = \gamma \sum_{n=1}^{\infty} \frac{G_n(T)}{a^{n-1}} \tag{2.3}$$

where $G_n\ (\cdot)$ is the n -fold convolution of $G(\cdot)$ with itself.
Using Equations (2.2) and (2.3), the equation (2.1) becomes

$$C(T) = \frac{c\gamma \sum_{n=1}^{\infty} \frac{G_n(T)}{a^{n-1}} - r\,\mu \sum_{n=1}^{\infty} \frac{F_n(T)}{2^{n-1}\beta} + R\ +\ c_p\,\tau}{\mu \sum_{n=1}^{\infty} \frac{F_n(T)}{2^{n-1}\beta} + \gamma \sum_{n=1}^{\infty} \frac{G_n(T)}{a^{n-1}} +\ \tau} \tag{2.4}$$

Further

$$C(T) = \frac{(c+r)\sum_{i=1}^{n-1}\left(\frac{\gamma}{a^{n-1}}\right) G_n(T) +\ R_1}{\mu \sum_{n=1}^{\infty} \frac{F_n(T)}{2^{n-1}\beta} + \gamma \sum_{n=1}^{\infty} \frac{G_n(T)}{a^{n-1}} + \tau} - r$$

where $R_1 = R + \left(c_p + r\right)\tau$,

$$C_1(T) = \frac{(c+r)\sum_{i=1}^{n-1}\left(\frac{\gamma}{a^{n-1}}\right) G_n(T) +\ R_1}{\mu \sum_{n=1}^{\infty} \frac{F_n(T)}{2^{n-1}\beta} + \gamma \sum_{n=1}^{\infty} \frac{G_n(T)}{a^{n-1}} + \tau} \tag{2.5}$$

On differentiating (2.5) we get

$$C_1'(T) = \frac{\left[\begin{array}{c}\left[\mu \sum_{n=1}^{\infty}\ \frac{F_n(T)}{2^{n-1}\beta} + \tau\right]\left[(c+r)\sum_{n=1}^{n-1}\ \left(\frac{\gamma}{a^{n-1}}\right) G_n{}'(T)\right] - \\ \left[(c+r)\gamma \sum_{n=1}^{n-1}\ \left(\frac{\gamma}{a^{n-1}}\right) G_n(T) + R_1\right]\left[\left(\mu \sum_{n=1}^{\infty}\ \frac{F_n{}'(T)}{2^{n-1}\beta}\right)\right]\end{array}\right]}{\left(\mu \sum_{n=1}^{\infty}\ \frac{F_n(T)}{2^{n-1}\beta} + \sum_{i=1}^{n-1}\ \left(\frac{\gamma}{a^{n-1}}\right) G_n(T) +\ \tau\right)^2} \tag{2.6}$$

If $C_1'(T) = 0$, then

$$\left[\left[\mu \sum_{n=1}^{\infty}\ \frac{F_n(T)}{2^{n-1}\beta} + \tau\right]\left[(c+r)\sum_{n=1}^{n-1}\ \left(\frac{\gamma}{a^{n-1}}\right) G_n{}'(T)\right] - \left[(c+r)\sum_{n=1}^{n-1}\ \left(\frac{\gamma}{a^{n-1}}\right) G_n(T) + R_1\right]\left[\left(\mu \sum_{n=1}^{\infty}\ \frac{F_n{}'(T)}{2^{n-1}\beta}\right)\right]\right] = 0$$

which implies

$$\left[\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\tau\right]\left[(c+r)\sum_{n=1}^{n-1}\left(\frac{\gamma}{a^{n-1}}\right)G_n{}'(T)\right]$$

$$=\left[(c+r)\sum_{n=1}^{n-1}\left(\frac{\gamma}{a^{n-1}}\right)G_n(T)+R_1\right]\left[\left(\mu\sum_{n=1}^{\infty}\frac{F_n{}'(T)}{2^{n-1}\beta}\right)\right] \qquad (2.7)$$

Again differentiating equation (2.6) using the equation (2.7) we obtain

$$C_1''(T)=\frac{\left[\begin{array}{c}\left[\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\tau\right]\left[(c+r)\sum_{i=1}^{n-1}\left(\frac{\gamma}{a^{n-1}}\right)G_n{}''(T)\right]-\\ \left[(c+r)\sum_{i=1}^{n-1}\left(\frac{\gamma}{a^{n-1}}\right)G_n(T)+R_1\right]\left[\left(\mu\sum_{n=1}^{\infty}\frac{F_n{}''(T)}{2^{n-1}\beta}\right)\right]\end{array}\right]}{\left(\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\sum_{i=1}^{n-1}\left(\frac{\gamma}{a^{n-1}}\right)G_n(T)+\tau\right)^2}$$

If $C_1{}''(T)>0$, then

$$\left[\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\tau\right]\left[(c+r)\sum_{n=1}^{n-1}\left(\frac{\gamma}{a^{i-1}}\right)G_n{}''(T)\right]$$

$$>\left[(c+r)\sum_{n=1}^{n-1}\left(\frac{\gamma}{a^{i-1}}\right)G_n(T)+R_1\right]\left[\left(\mu\sum_{n=1}^{\infty}\frac{F_n{}''(T)}{2^{n-1}\beta}\right)\right] \qquad (2.8)$$

The equation (2.7) gives *T** for which *C(T*) is* minimum. Based on the preceding discussion, we obtain the following.

Theorem 2.1: *For the model described in Section 2, under the assumptions 2.1 to 2.6, the long-run average cost per unit time $C(T)$ is given by*

$$C(T)=\frac{c\gamma\sum_{n=1}^{\infty}\frac{G_n(T)}{a^{n-1}}-r\,\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+R+c_p\,\tau}{\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\gamma\sum_{n=1}^{\infty}\frac{G_n(T)}{a^{n-1}}+\tau}$$

for a multistate degenerative system under the replacement policy T is minimum, if

$$\left[\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\tau\right]\left[(c+r)\sum_{n=1}^{n-1}\left(\frac{\gamma}{a^{n-1}}\right)G_n{}'(T)\right]=\left[(c+r)\sum_{n=1}^{n-1}\left(\frac{\gamma}{a^{n-1}}\right)G_n(T)+R_1\right]\left[\left(\mu\sum_{n=1}^{\infty}\frac{F_n{}'(T)}{2^{n-1}\beta}\right)\right]$$

and

$$\left[\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\tau\right]\left[(c+r)\sum_{n=1}^{n-1}\left(\frac{\gamma}{a^{i-1}}\right)G_n{}''(T)\right]>\left[(c+r)\sum_{n=1}^{n-1}\left(\frac{\gamma}{a^{i-1}}\right)G_n(T)+R_1\right]\left[\left(\mu\sum_{n=1}^{\infty}\frac{F_n{}''(T)}{2^{n-1}\beta}\right)\right]$$

hold.

3. The Replacement Policy *T* with NONN repair times

In this section, we introduce and study a T-policy for a multistate one-component degenerative system with NONN repair times. Under the replacement policy T, the problem is to determine an optimal replacement policy T^* such that the long-run average cost per unit time is minimized. We consider a monotone process model for a multistate one-component degenerative system and make the following assumptions:
Assumptions **3.1, 3.2, 3.4, 3.5** are the same as assumptions **2.1,2.2,2.4,2.5.**

Assumption 3.3: Let X_1 be the first operating time. For n ≥ 2, let X_n be the operating time of the system after *(n − 1)*-st repair, let ξ_n be the repair time after the *n-th* failure and *Z* be the replacement time. Now, denote the time of the *n-th* failure by t_n

Assumption 3.6:. The survival times (X_i) and the NONN repair times (ξ_i) and the replacement time *Z,* for i = 1, 2, ... are independent.

Assumption 3.7: The replacement policy T with NONN repair times is adapted under which the system will be replaced whenever its working age reaches T.

By assumption

$$\xi_n = \begin{cases} Y_n & if\ \ Y_n > 0 \\ 1 & if\ \ Y_n = 0 \end{cases}$$

for n=1,2,3….

Therefore

$$E(\xi_n) = E(Y_n)P(Y_n > 0) + 1.P(Y_n = 0)$$
$$= \left(\frac{\gamma}{a^{n-1}}\right)(1-p) + p$$

The replacement times are defined as follows: T_1 represents the first replacement time, and for n = 2, 3, ... , T_n denotes the time between the (n-1)-st replacement and the n-th replacement. Consequently, the sequence $\{T_n$, n = 1, 2, ...} constitutes a renewal process. Cycles are defined by the time between consecutive system replacements, spanning from initial installation to first replacement, and subsequent replacement intervals. Each replacement concludes one cycle and begins another. By combining the successive cycles with the costs associated with each, we establish a renewal reward process, facilitating the assessment of cumulative costs, rewards, and system efficiency.

$$C(T) = \frac{\text{the expected cost incured in a cycle}}{\text{the expected length of a cycle}}$$

$$= \frac{cE[\sum_{k=1}^{n-1} \xi_k] - rE\left[\sum_{k=1}^{\eta} X_k\right] + R + c_p E[\,Z\,]}{E\left[\sum_{k=1}^{\eta} X_k\right] + E[\sum_{k=1}^{n-1} \xi_k] + E[Z]} \tag{3.1}$$

where $\eta = 0,1,2, \dots, N-1$ is a random variable denoting the number of failures in time T.

$$E(X_n) = \sum_{n=1}^{\eta} \left(\frac{\mu}{2^{n-1}\beta}\right) F_n(T) \tag{3.2}$$

The n-fold convolution of F (.) with itself is denoted by F_N (.)

Consider

$$E(\xi_n) = E\left[E\left[\sum_{n=1}^{n-1} E\,(\xi_n|)\right]\right]$$

$$= \sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T) \tag{3.3}$$

The n-fold convolution of G (.) with itself is denoted by G_n (.)

using equations (3.2) and (3.3), the equation (3.1) becomes

$$C(T) = \frac{c\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T) - r\sum_{n=2}^{\infty}\left(\frac{\mu}{2^{n-1}\beta}\right)F_n(T) + R + c_p\tau}{\sum_{n=1}^{\infty}\left(\frac{\mu}{2^{n-1}\beta}\right)F_n(T) + \sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T) + \tau} \tag{3.4}$$

where $E[Z] = \tau$. Further,

$$C(T) = \frac{c\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T) - r\sum_{n=2}^{\infty}\left(\frac{\mu}{2^{n-1}\beta}\right)F_n(T) + R_1}{\sum_{n=1}^{\infty}\left(\frac{\mu}{2^{n-1}\beta}\right)F_n(T) + \sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T) + \tau} - r$$

where $R_1 = R + \left(c_p + r\right)\tau$,

$$C_1(T) = \frac{(c+r)\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T) + R_1}{\sum_{n=1}^{\infty}\left(\frac{\mu}{2^{n-1}\beta}\right)F_n(T) + \sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T) + \tau} \tag{3.5}$$

On differentiating (4.5) we get

$$C_1'(T) = \frac{\left[\begin{array}{l}\left[\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\tau\right]\left[(c+r)\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n{}'(T)\right] - \\ \left[(c+r)\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T)+R_1\right]\left[\left(\mu\sum_{n=1}^{\infty}\frac{F_n{}'(T)}{2^{n-1}\beta}\right)\right]\end{array}\right]}{\left(\sum_{n=1}^{\infty}\left(\frac{\mu}{2^{n-1}\beta}\right)F_n(T) + \sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T) + \tau\right)^2} \tag{3.6}$$

If $C_1'(T) = 0$, then

$$\left[\left[\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\tau\right]\left[(c+r)\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n{}'(T)\right]\right.$$
$$\left.-\left[(c+r)\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T)+R_1\right]\left[\left(\mu\sum_{n=1}^{\infty}\frac{F_n{}'(T)}{2^{n-1}\beta}\right)\right]\right]=0$$

which implies

$$\left[\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\tau\right]\left[(c+r)\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n{}'(T)\right]$$
$$=\left[(c+r)\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T)+R_1\right]\left[\left(\mu\sum_{n=1}^{\infty}\frac{F_n{}'(T)}{2^{n-1}\beta}\right)\right] \tag{3.7}$$

Again differentiating equation (3.6) using the equation (3.7) we obtain, on simplification that

$$C_1''(T)=\frac{\left[\begin{array}{l}\left[\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\tau\right]\left[(c+r)\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n{}''(T)\right]-\\ \left[(c+r)\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T)+R_1\right]\left[\left(\mu\sum_{n=1}^{\infty}\frac{F_n{}''(T)}{2^{n-1}\beta}\right)\right]\end{array}\right]}{\left(\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T)+\tau\right)^4}$$

If $C_1{}''(T)>0$, then

$$\left[\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\tau\right]\left[(c+r)\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n{}''(T)\right]$$
$$>\left[(c+r)\sum_{n=1}^{n-1}\left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p)+p\right)G_n(T)+R_1\right]\left[\left(\mu\sum_{n=1}^{\infty}\frac{F_n{}''(T)}{2^{n-1}\beta}\right)\right] \tag{3.8}$$

For If $C_1'(T)=0$ $,C_1{}''(T)>0$, then $C_1(T)$ attain its minimum.

Remarks.

1. When p = 0, that is when the repair times non-negligible, equation (3.4) reduces to

$$C(T)=\frac{\sum_{i=1}^{n-1}c\left(\frac{\gamma}{a^{n-1}}\right)G_n(T)-\sum_{n=2}^{\infty}r\left(\frac{\mu}{2^{n-1}\beta}\right)F_n(T)+R+c_p\,\tau}{\mu\sum_{n=1}^{\infty}\frac{F_n(T)}{2^{n-1}\beta}+\gamma\sum_{n=1}^{\infty}\frac{G_n(T)}{a^{n-1}}+\tau}$$

2. When p = 1, that is, when the repair times are negligible, equation (3.4) reduces to

$$C(T) = \frac{\sum_{i=1}^{n-1} \; cG_n(T) - \sum_{n=2}^{\infty} \; r \left(\frac{\mu}{2^{n-1}\beta}\right) F_n(T) + R \; + \; c_p \, \tau}{\mu \sum_{n=1}^{\infty} \; \frac{F_n(T)}{2^{n-1}\beta} + \sum_{i=1}^{n-1} \; G_n(T) + \tau}$$

Theorem 3.1: *For the model described in Section 3, under the assumptions 3.1 to 3.7, the long-run average cost per unit time $C(T)$is given by*

$$C(T) = \frac{c \sum_{n=1}^{n-1} \; \left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p) + p\right) G_n(T) - r \sum_{n=2}^{\infty} \; \left(\frac{\mu}{2^{n-1}\beta}\right) F_n(T) + R \; + \; c_p\tau}{\sum_{n=1}^{\infty} \; \left(\frac{\mu}{2^{n-1}\beta}\right) F_n(T) + \sum_{n=1}^{n-1} \; \left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p) + p\right) G_n(T) + \tau}$$

for a multistate degenerative system under the replacement policy T is minimum, if

$$\left[\mu \sum_{n=1}^{\infty} \frac{F_n(T)}{2^{n-1}\beta} + \tau\right]\left[(c+r) \sum_{n=1}^{n-1} \; \left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p) + p\right) G_n{}'(T)\right]$$

$$= \left[(c+r) \sum_{n=1}^{n-1} \; \left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p) + p\right) G_n(T) + R_1\right]\left[\left(\mu \sum_{n=1}^{\infty} \frac{F_n{}'(T)}{2^{n-1}\beta}\right)\right]$$

and

$$\left[\mu \sum_{n=1}^{\infty} \frac{F_n(T)}{2^{n-1}\beta} + \tau\right]\left[(c+r) \sum_{n=1}^{n-1} \left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p) + p\right) G_n{}''(T)\right]$$

$$> \left[(c+r) \sum_{n=1}^{n-1} \left(\left(\frac{\gamma}{a^{n-1}}\right)(1-p) + p\right) G_n(T) + R_1\right]\left[\left(\mu \sum_{n=1}^{\infty} \frac{F_n{}''(T)}{2^{n-1}\beta}\right)\right]$$

hold.

4. Conclusion

This research undertakes a comprehensive analysis of a repairable system within a monotone process framework, tailored to a one-component multistate degenerative system. Explicit expressions for long-run average costs per unit time under univariate T -policy are derived. Furthermore, assuming an alternate repair model with non-negligible repair times (NONN), the study derives long-run average cost expressions for T-policy. Optimality existence is established for these models.

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