Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

Equitable Total Coloring of Line Graph of Certain Graphs

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract: An equitable total-coloring of a graph G is a proper total-coloring such that the number of vertices and edges in any two color classes differ by at most one. In this paper, we determined the equitable total chromatic number for line graph of ladder, slanting ladder, triangular snake, alternate triangular snake, quadrilateral snake and alternate quadrilateral snake

Introduction: Graph coloring is a fundamental problem in graph theory with applications in scheduling, networking, and resource allocation. A **total-coloring** of a graph G is an assignment of colors to both vertices and edges such that adjacent vertices, adjacent edges, and incident vertex-edge pairs receive distinct colors. An **equitable total-coloring** is a special type of total-coloring where the sizes of any two color classes differ by at most one. The **equitable total chromatic number**, denoted as $\chi_e''(G)$ is the minimum number of colors required for such a coloring.

Objectives: To establish the equitable total chromatic number for the line graph of specific families of structured graphs. To develop systematic coloring techniques for achieving an equitable total- coloring of these graphs. To contribute to the broader study of equitable colorings in graph theory and expand the known results in this domain.

Methods: To determine the equitable total chromatic number for the line graphs of the given graph families, we employ the following methodology: **Graph Construction:** We formally define the structure of the ladder, slanting ladder, triangular snake, alternate triangular snake, quadrilateral snake, and alternate quadrilateral snake, along with their corresponding line graphs. **Coloring Strategy:** We apply systematic coloring techniques ensuring that adjacent vertices, adjacent edges, and incident vertex-edge pairs receive different colors while maintaining equitable distribution of color classes. **Mathematical Analysis:** We derive lower bounds for $\chi_e''(G)$ and establish its exact value using combinatorial and structural properties of the graphs. **Verification and Proof:** We validate the obtained chromatic numbers through case-based analysis and, where applicable, provide rigorous proofs for correctness.

Results: The study successfully determines the exact value of the equitable total chromatic number for the line graphs of the considered structured graphs. The results provide new insights into the equitable total-coloring of line graphs of ladder-based and snake-like structures, which are commonly encountered in chemical graph theory and network design problems.

Conclusions: This paper establishes the equitable total chromatic number for the line graphs of several structured graphs, contributing to the ongoing research in equitable colorings. The findings demonstrate that the structural properties of the base graphs significantly influence their equitable total chromatic numbers. These results can be extended to other classes of graphs, and future research may explore algorithmic approaches for efficient equitable total-coloring in larger and more complex graph families

Keywords: Total coloring, equitable total coloring, line graph, ladder graphs, snake graphs.

1. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let G = (V(G), E(G)) be a graph with vertex set V (G) and edge set E(G). The total coloring of a graph G is an assignment of colors to both the vertices and edges of G, such that no two adjacent or incident vertices and edges of G are received the same colors. They both conjectured that for any graph G the following inequality holds: $\Delta(G)+1 \le \chi''(G) \le \Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G. It is clear that $\Delta(G)+1$ is the possible lower bound. In 1994, Fu [4] first introduced the concept of equitable total coloring and the equitable total chromatic number of a graph. Gong Kun et.al [3] proved some results on the equitable total chromatic number of $W_n \vee K_n, F_m \vee K_n$ and $S_m \vee K_n$. Jayaraman et al.[5, 6] determined the equitable total chromatic number for the splitting midde, total graph of paths, cycles and splitting graph of star graphs. Veninstine vivik et.al [8] determining the equitable total chromatic number for wheel, gear, helm, and sunlet graphs. Gong Kun et.al [2] derived several findings regarding the equitable total chromatic number for the graphs $W_n \vee K_n, F_n \vee K_n$ and $S_m \vee K_n$. Wang et.al [9] addressed the equitable total coloring for the graphs with a maximum degree of 3, while Zhang Zhong-fu [10] investigated the equitable total coloring of certain join graphs.

2. Preliminaries

Definition 2.1. The line graph *L(G)* is defined such that its vertices correspond to the edges of *G*, and two vertices in *L(G)* are adjacent if their corresponding edges in *G* share a common vertex.

Definition 2.2. The ladder graph L_n formed by taking two parallel paths of length $'n'$ and connecting corresponding vertices with additional edges.

Definition 2.3. The slanting ladder SL_n is a graph that consists of two copies of P_n having vertex set $\{u_i : 1 \le i \le n\} \cup \{v_i : 1 \le i \le n\}$ and the edge set is formed by adjoining u_i and v_{i+1} for all $1 \le i \le n-1$.

Definition 2.4. A Triangular snake T_n [1] is obtained from th $v_1, v_2, ..., v_n$ by joining v_i and v_{i+1} to a new vertex u_i for $i = 1, 2, 3, ..., n-1$.

Definition 2.5. An alternate *triangular snake* AT_n [1] is a graph having vertex and edge set

$V(AT_m) = \{v_l : 1 \le l \le m\} \bigcup \left\{ u_l : 1 \le l \le \left\lfloor \frac{m}{2} \right\rfloor \right\}$ and

$$E(AT_m)=\{v_l v_{l+1}:1\le l\le m-1\}\cup\left\{v_{2l-1}u_l:1\le l\le\left\lfloor\frac{m}{2}\right\rfloor\right\}\cup\left\{v_{2l}u_l:1\le l\le\left\lfloor\frac{m}{2}\right\rfloor\right\}$$

Definition 2.6. A *Quadrilateral snake* Q_η [7] is obtained from a path by replacing every edge by a cycle C_4.

Definition 2.7. An *Alternate quadrilateral snake* AQ_η [7] is a graph constructed from a path P_η by replacing every alternate edges with a quadrilateral cycle C_4, forming a sequence of interspersed C_4 cycles along the path.

Conjecture 2.8([4]). For any simple graph G, $\chi_e''(G)\le\Delta(G)+2.$

Conjecture 2.9([9]). For every graph G, G has an equitable total *k*-coloring for each $k\ge\max\{\chi''(G),\ \Delta(G)+2\}$.

3. Results and Discussion

Theorem 3.1: Let $L(L_n)$ represent the line graph of a ladder graph, then $\chi_e''\left(L(L_n)\right)=5.$

Proof: Let $V(L(L_n))=\left\{u_\tau,v_\tau:1\le\tau\le n-1\right\}\cup\left\{u_\tau':1\le\tau\le n\right\}$ and

$E(L(L_n))=\left\{e_\tau,\ f_\tau:1\le\tau\le n-2\right\}\cup\left\{e_\tau',e_\tau'',\ f_\tau',\ f_\tau'':1\le\tau\le n-1\right\}$, where $e_\tau=u_\tau u_{\tau+1},\ e_\tau'=u_\tau u_\tau',\ e_\tau''=u_\tau u_{\tau+1}',$ $f_\tau=v_\tau v_{\tau+1},\ f_\tau'=u_\tau' v_\tau,\ f_\tau''=v_\tau u_{\tau+1}'.$

We divide the vertex and edge set of $L(L_n)$ into distinct partition as described below.

For $1\le\tau\le n-1.$

$$T_1=\begin{cases}\{u_4,u_9,...,u_{5\tau-1}\}\cup\{v_2,v_7,...,v_{5\tau-3}\}\cup\{u_1',u_6',...,u_{5\tau-4}'\}\cup\{e_5,e_{10},...,e_{5\tau}\}\cup\{e_2',e_7',...,e_{5\tau-3}'\}\\ \{e_3'',e_8'',...,e_{5\tau-2}''\}\cup\{f_5,f_{10},...,f_{5\tau}\}\cup\{f_3',f_8',...,f_{5\tau-2}'\}\cup\{f_4'',f_9'',...,f_{5\tau-1}''\}\end{cases}$$

$$T_2=\begin{cases}\{u_5,u_{10},...,u_{5\tau}\}\cup\{v_3,v_8,...,v_{5\tau-2}\}\cup\{u_2',u_7',...,u_{5\tau-3}'\}\cup\{e_6,e_{11},...,e_{5\tau+1}\}\cup\{e_3',e_8',...,e_{5\tau-2}'\}\\ \{e_4'',e_9'',...,e_{5\tau-1}''\}\cup\{f_1,f_6,...,f_{5\tau-4}\}\cup\{f_4',f_9',...,f_{5\tau-1}'\}\cup\{f_5'',f_{10}'',...,f_{5\tau}''\}\end{cases}$$

$$T_3=\begin{cases}\{u_1,u_6,...,u_{5\tau-4}\}\cup\{v_4,v_9,...,v_{5\tau-1}\}\cup\{u_3',u_8',...,u_{5\tau-2}'\}\cup\{e_2,e_7,...,e_{5\tau-3}\}\cup\{e_4',e_9',...,e_{5\tau-1}'\}\\ \{e_5'',e_{10}'',...,e_{5\tau}''\}\cup\{f_2,f_7,...,f_{5\tau-3}\}\cup\{f_5',f_{10}',...,f_{5\tau}'\}\cup\{f_1'',f_6'',...,f_{5\tau-4}''\}\end{cases}$$

$$T_4=\begin{cases}\{u_2,u_7,...,u_{5\tau-3}\}\cup\{v_5,v_{10},...,v_{5\tau}\}\cup\{u_4',u_9',...,u_{5\tau-1}'\}\cup\{e_3,e_8,...,e_{5\tau-2}\}\cup\{e_5',e_{10}',...,e_{5\tau}'\}\\ \{e_1'',e_6'',...,e_{5\tau-4}''\}\cup\{f_3,f_8,...,f_{5\tau-2}\}\cup\{f_1',f_6',...,f_{5\tau-4}'\}\cup\{f_2'',f_7'',...,f_{5\tau-3}''\}\end{cases}$$

$$T_5=\begin{cases}\{u_3,u_8,...,u_{5\tau-2}\}\cup\{v_1,v_6,...,v_{5\tau-4}\}\cup\{u_5',u_{10}',...,u_{5\tau}'\}\cup\{e_4,e_9,...,e_{5\tau-1}\}\cup\{e_1',e_6',...,e_{5\tau-4}'\}\\ \{e_2'',e_7'',...,e_{5\tau-3}''\}\cup\{f_4,f_9,...,f_{5\tau-1}\}\cup\{f_2',f_7',...,f_{5\tau-3}'\}\cup\{f_3'',f_8'',...,f_{5\tau-2}''\}\end{cases}$$

Based on the above procedure of coloring, it is evident that the color classes T_1,T_2,T_3,T_4 and T_5 are independent sets of $L(L_n)$ and it holds inequality $\|T_a|-|T_b\|\le1$ for $a\ne b$. This implies that

$\chi_e^{''}\left(L(L_n)\right)\le 5$. Further, since $\Delta=4$, we have $\chi_e^{''}\left(L(L_n)\right)=\chi^{''}\left(L(L_n)\right)\ge\Delta+1\ge 4+1\ge 5$. Hence $\chi_e^{''}\left(L(L_n)\right)=5.$

Example 3.1: The graph $L(L_7)$ and its equitable total coloring is shown in figure 1.

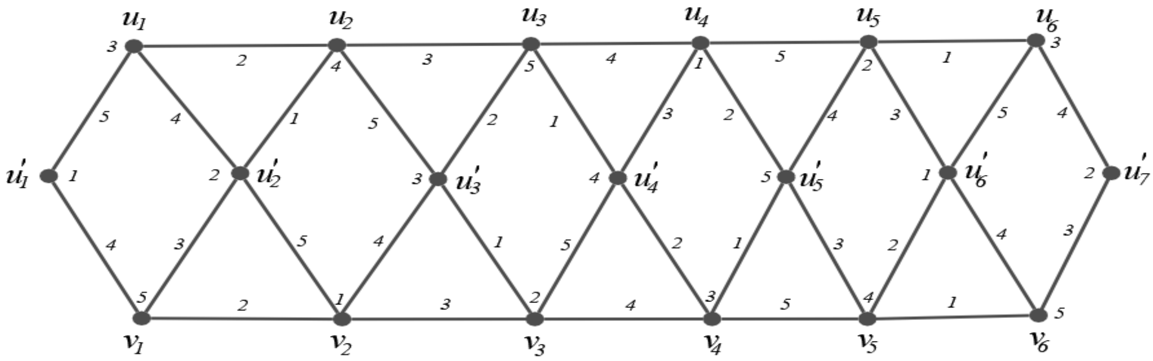

Figure 1: $L(L_7)$ and its equitable total coloring

It is clear that the color classes T_1,T_2,T_3,T_4 and T_5 are independent sets of $L(L_7)$. For which the color classes are partition into $|T_1|=|T_2|=|T_4|=10$ and $|T_3|=|T_5|=11$ (see Figure 1) and it holds the inequality $\|T_i\|-\|T_j\|\le 1,$ for every pair of (i,j). This implies that $\chi_e^{''}\left(L(L_7)\right)\le 5$. Moreover, $\Delta=4$. We have $\chi_e^{''}\left(L(L_7)\right)\ge\ \Delta+1\ge 4+1\ge 5$. Hence $\chi_e^{''}\left(L(L_7)\right)=5$.

Theorem 3.2: Let $L(SL_n)$ represent the line graph of a slanting ladder, then $\chi_e^{''}\left(L(SL_n)\right)=5.$

P***roof:*** $V(L(SL_n))=\{u_\tau,v_\tau,u_\tau^{'}:1\le\tau\le n\}$

$E(L(SL_n))=\left\{e_\tau,\ f_\tau:1\le\tau\le n-2\right\}\cup\left\{e_\tau^{'},e_\tau^{''},\ f_\tau^{'},\ f_\tau^{''}:1\le\tau\le n-1\right\}$, where $e_\tau=u_\tau u_{\tau+1},\ e_\tau^{'}=u_\tau u_\tau^{'},$ $e_\tau^{'}=u_\tau u_\tau^{'},\ e_\tau^{''}=u_\tau u_{\tau+1}^{'},\ \ f_\tau=v_\tau v_{\tau+1},\ f_\tau^{'}=u_\tau^{'}v_\tau,\ f_\tau^{''}=v_\tau u_{\tau+1}^{'}.$

We divide the vertex and edge set of $L(SL_n)$ into distinct partition as described below.

$$T_1=\begin{cases}\{u_4,u_9,...,u_{5\tau-1}\}\cup\{v_2,v_7,...,v_{5\tau-3}\}\cup\{u_5^{'},u_{10}^{'},...,u_{5\tau}^{'}\}\cup\{e_1,e_6,...,e_{5\tau-4}\}\cup\{e_3^{'},e_8^{'},...,e_{5\tau-2}^{'}\}\\ \{e_5^{''},e_{10}^{''},...,e_{5\tau}^{''}\}\cup\{f_5,\ f_{10},...,\ f_{5\tau}\}\cup\{f_4^{'},\ f_9^{'},...,\ f_{5\tau-1}^{'}\}\cup\{f_2^{''},\ f_7^{''},...,\ f_{5\tau-3}^{''}\}\end{cases}$$

$$T_2=\begin{cases}\{u_5,u_{10},...,u_{5\tau}\}\cup\{v_3,v_8,...,v_{5\tau-2}\}\cup\{u_1^{'},u_6^{'},...,u_{5\tau-4}^{'}\}\cup\{e_2,e_7,...,e_{5\tau-2}\}\cup\{e_4^{'},e_9^{'},...,e_{5\tau-1}^{'}\}\\ \{e_1^{''},e_6^{''},...,e_{5\tau-4}^{''}\}\cup\{f_1,\ f_6,...,\ f_{5\tau-4}\}\cup\{f_5^{'},\ f_{10}^{'},...,\ f_{5\tau}^{'}\}\cup\{f_3^{''},\ f_8^{''},...,\ f_{5\tau-2}^{''}\}\end{cases}$$

$$T_3=\begin{cases}\{u_1,u_6,...,u_{5\tau-4}\}\cup\{v_4,v_9,...,v_{5\tau-1}\}\cup\{u_2^{'},u_7^{'},...,u_{5\tau-3}^{'}\}\cup\{e_3,e_8,...,e_{5\tau-2}\}\cup\{e_5^{'},e_{10}^{'},...,e_{5\tau}^{'}\}\\ \{e_2^{''},e_7^{''},...,e_{5\tau-3}^{''}\}\cup\{f_2,\ f_7,...,\ f_{5\tau-3}\}\cup\{f_1^{'},\ f_6^{'},...,\ f_{5\tau-4}^{'}\}\cup\{f_4^{''},\ f_9^{''},...,\ f_{5\tau-1}^{''}\}\end{cases}$$

$$T_4=\begin{cases}\{u_2,u_7,...,u_{5\tau-3}\}\cup\{v_5,v_{10},...,v_{5\tau}\}\cup\{u_3^{'},u_8^{'},...,u_{5\tau-2}^{'}\}\cup\{e_4,e_9,...,e_{5\tau-1}\}\cup\{e_1^{'},e_6^{'},...,e_{5\tau-4}^{'}\}\\ \{e_3^{''},e_8^{''},...,e_{5\tau-2}^{''}\}\cup\{f_3,\ f_8,...,\ f_{5\tau-2}\}\cup\{f_2^{'},\ f_7^{'},...,\ f_{5\tau-3}^{'}\}\cup\{f_5^{''},\ f_{10}^{''},...,\ f_{5\tau}^{''}\}\end{cases}$$

$$T_5 = \begin{cases} \{u_3, u_8, ..., u_{5\tau-2}\} \cup \{v_1, v_6, ..., v_{5\tau-4}\} \cup \{u_4^{'}, u_9^{'}, ..., u_{5\tau-1}^{'}\} \cup \{e_5, e_{10}, ..., e_{5\tau}\} \cup \{e_2^{'}, e_7^{'}, ..., e_{5\tau-3}^{'}\} \\ \{e_4^{"}, e_9^{"}, ..., e_{5\tau-1}^{"}\} \cup \{f_4, f_9, ..., f_{5\tau-1}\} \cup \{f_3^{'}, f_8^{'}, ..., f_{5\tau-2}^{'}\} \cup \{f_1^{"}, f_6^{"}, ..., f_{5\tau-4}^{"}\} \end{cases}$$

Based on the coloring technique described above, it is evident that the color classes T_1, T_2, T_3, T_4 and T_5 are independent sets of $L(SL_n)$ and it holds inequality $\| T_a | - | T_b \| \le 1$ for $a \ne b$. This implies that $\chi_e^{"}\left(L(SL_n)\right) \le 5$. Further, since $\Delta = 4$, we have $\chi_e^{"}\left(L(SL_n)\right) = \chi^{"}\left(L(SL_n)\right) \ge \Delta + 1 \ge 4 + 1 \ge 5$. Hence $\chi_e^{"}\left(L(SL_n)\right) = 5$.

Example 3.2: The graph $L(SL_7)$ and its equitable total coloring is shown in figure 2.

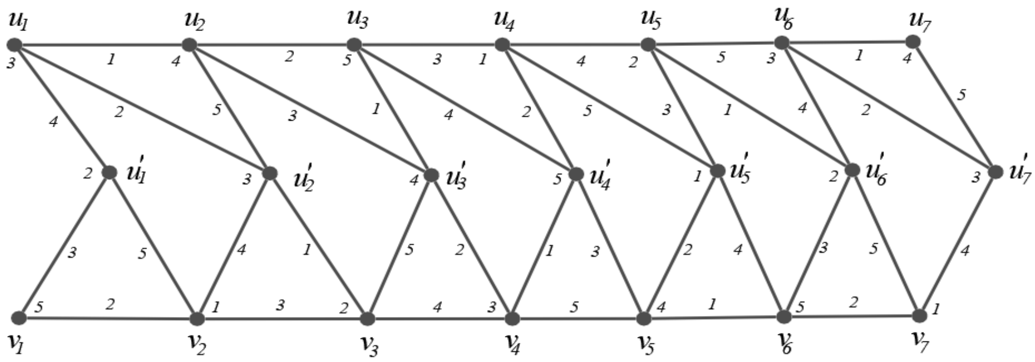

Figure 2: $L(SL_7)$ and its equitable total coloring

The color classes are $T\left(L(SL_7)\right) = \left\{T_1, T_2, T_3, T_4, T_5\right\}$. It is clear that the color classes T_1, T_2, T_3, T_4 and T_5 are independent sets of $L(SL_7)$. For which the color classes are partition into $\left|T_2\right| = \left|T_3\right| = \left|T_5\right| = 12$ and $\left|T_1\right| = \left|T_4\right| = 11$ (see Figure 2) and it holds the inequality $\| T_i \| - \| T_j \| \le 1,$ for every pair of (i, j). This implies that $\chi_e^{"}\left(L(SL_7)\right) \le 5$. Moreover, $\Delta = 4$. We have $\chi_e^{"}\left(L(SL_7)\right) \ge \Delta + 1 \ge 4 + 1 \ge 5$. Hence $\chi_e^{"}\left(L(SL_7)\right) = 5$.

Theorem 3.3: Let $L(T_n)$ represent the line graph of a triangular snake graph, then $\chi_e^{"}\left(L(T_n)\right) = \Delta(L(T_n)) + 1.$

P***roof:*** Let $V(L(T_n)) = \{u_\tau, v_\tau, u_\tau^{'} : 1 \le \tau \le n-1\}$

$E(L(SL_n)) = \left\{e_\tau, f_\tau, f_\tau^{'} : 1 \le \tau \le n-1\right\} \cup \left\{e_\tau^{'}, e_\tau^{"}, f_\tau^{"}, f_\tau^{"'} : 1 \le \tau \le n-2\right\}$, where $e_\tau = u_\tau u_\tau^{'}$, $e_\tau^{'} = u_\tau^{'} u_{\tau+1}$, $e_\tau^{"} = v_\tau v_{\tau+1}$, $f_\tau = u_\tau v_\tau$, $f_\tau^{'} = u_\tau^{'} v_\tau$, $f_\tau^{"} = u_\tau^{'} v_{\tau+1}$, $f_\tau^{"} = v_\tau u_{\tau+1}$

We divide the vertex and edge set of $L(T_n)$ into distinct partition as described below.

For $1 \le \tau \le n-1$.

$$T_1 = \{v_1, v_4, ..., v_{3\tau-2}\} \cup \{e_3, e_6, ..., e_{3\tau}\} \cup \{e_1^{'}, e_4^{'}, ..., e_{3\tau-2}^{'}\} \cup \{e_2^{"}, e_5^{"}, ..., e_{3\tau-1}^{"}\} \cup \{f_{n-1}^{'}, n \equiv 0 (\text{mod}\, 3)\}$$

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 9s (2025)

$T_2 = \{v_2, v_5, ..., v_{3\tau-1}\} \cup \{e_2, e_5, ..., e_{3\tau-1}\} \cup \{e_3^{'}, e_6^{'}, ..., e_{3\tau}^{'}\} \cup \{e_3^{''}, e_6^{''}, ..., e_{3\tau}^{''}\} \cup \{f_1\} \cup \{f_{n-1}^{'}, n \equiv 1(\text{mod}\,3)\}$

$T_3 = \begin{cases} \{v_3, v_6, ..., v_{3\tau}\} \cup \{e_4, e_7, ..., e_{3\tau+1}\} \cup \{e_2^{'}, e_5^{'}, ..., e_{3\tau-1}^{'}\} \cup \{e_4^{''}, e_7^{''}, ..., e_{3\tau+1}^{''}\} \cup \\ \{u_1\} \cup \{f_2\} \cup \{f_1^{'}, n \equiv 0, 2(\text{mod}\,3)\} \end{cases}$

$T_4 = \{u_1^{'}, u_3^{'}, ..., u_{2\tau-1}^{'}\} \cup \{f_3, f_4, ..., f_{n-1}\} \cup \{f_2^{'}\}$

$T_5 = \{u_2, u_4, ..., u_{n-1}\} \cup \{f_2^{'''}\} \cup \{f_3^{'}, f_4^{'}, ..., f_{n-2}^{'}\} \cup \{f_{n-1}^{'}, n \equiv 2(\text{mod}\,3) \text{ and } n = 6\} \cup \{f_1^{'}, n \equiv 1(\text{mod}\,3)\}$

$T_6 = \{u_2^{'}, u_4^{'}, ..., u_{n-1}^{'}\} \cup \{f_3^{'''}, f_4^{'''}, ..., f_{n-2}^{'''}\} \cup \{f_1^{''}\} \cup \{f_1^{'''}\}$

$T_7 = \{u_3, u_7, ..., u_{n-2}\} \cup \{f_2^{''}, f_3^{''}, ..., f_{n-2}^{''}\} \cup \{e_1\} \cup \{e_1^{''}\}$

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$T_{\Delta-1} = \{u_2, u_4, ..., u_{n-1}\} \cup \{f_2^{'''}\} \cup \{f_3^{'}, f_4^{'}, ..., f_{n-2}^{'}\} \cup \{f_{n-1}^{'}, n \equiv 2(\text{mod}\,3) \text{ and } n = 6\} \cup \{f_1^{'}, n \equiv 1(\text{mod}\,3)\}$

$T_{\Delta} = \{u_2^{'}, u_4^{'}, ..., u_{n-1}^{'}\} \cup \{f_3^{'''}, f_4^{'''}, ..., f_{n-2}^{'''}\} \cup \{f_1^{''}\} \cup \{f_1^{'''}\}$

$T_{\Delta+1} = \{u_3, u_7, ..., u_{n-2}\} \cup \{f_2^{''}, f_3^{''}, ..., f_{n-2}^{''}\} \cup \{e_1\} \cup \{e_1^{''}\}$

Based on the coloring technique described above, it is evident that the color classes $T_1, T_2, T_3, ...$ and $T_{\Delta+1}$ are independent sets of $L(T_n)$ and it holds inequality $\| T_a | - | T_b \| \leq 1$ for $a \neq b$. This implies that $\chi_e^{''}\left(L(T_n)\right) \leq \Delta + 1$. Further, we have $\chi_e^{''}\left(L(T_n)\right) = \chi^{''}\left(L(T_n)\right) \geq \Delta + 1$. Hence $\chi_e^{''}\left(L(T_n)\right) = \Delta + 1$.

Example 3.3: The graph $L(T_6)$ and its equitable total coloring is shown in figure 3.

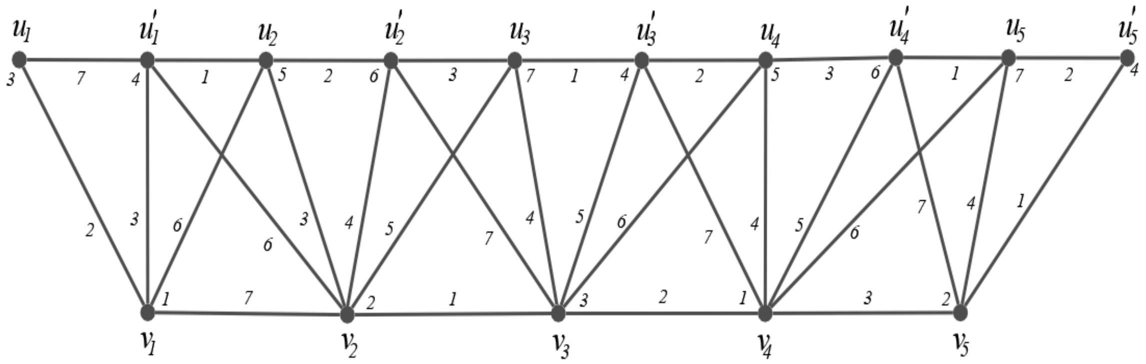

Figure 3: $L(T_6)$ and its equitable total coloring

Theorem 3.4: Let $L(AT_n)$ represent the line graph of alternate triangular snake graph, then

$\chi_e^{''}\left(L(T_n)\right) = 5,\ n \neq 7$.

P***roof:*** $V(L(AT_n)) = \{u_\tau : 1 \leq \tau \leq n\} \cup \{v_\tau : 1 \leq \tau \leq n-1\}$ and

$$E(L(AT_n)) = \left\{e_\tau : 1 \leq \tau \leq \left\lfloor \frac{n}{2} \right\rfloor\right\} \cup \left\{e_\tau^{'} : 1 \leq \tau \leq n-2\right\} \cup \{f_\tau, f_\tau^{'} : 1 \leq \tau \leq n-1\},$$

where $e_\tau = u_{2\tau-1}u_{2\tau}$, $e_\tau^{'} = v_\tau v_{\tau+1}$, $f_\tau = u_\tau v_\tau$, $f_\tau^{'} = v_\tau u_{\tau+1}$,

We divide the vertex and edge set of $L(SL_n)$ into distinct partition as described below.

For $1 \le \tau \le n-1$.

$$T_1 = \begin{cases} \{u_4, u_9, ..., u_{5\tau-1}\} \cup \{v_1, v_6, ..., v_{5\tau-4}\} \cup \{e_1, e_6, ..., e_{5\tau-4}\} \cup \\ \{e'_3, e'_8, ..., e'_{5\tau-2}\} \cup \{f_5, f_{10}, ..., f_{5\tau}\} \cup \{f'_2, f'_7, ..., f'_{5\tau-3}\} \end{cases}$$

$$T_2 = \begin{cases} \{u_2, u_7, ..., u_{5\tau-3}\} \cup \{v_4, v_9, ..., v_{5\tau-1}\} \cup \{e_5, e_{10}, ..., e_{5\tau}\} \cup \\ \{e'_1, e'_6, ..., e'_{5\tau-4}\} \cup \{f_3, f_8, ..., f_{5\tau-2}\} \cup \{f'_5, f'_{10}, ..., f'_{5\tau}\} \end{cases}$$

$$T_3 = \begin{cases} \{u_5, u_{10}, ..., u_{5\tau}\} \cup \{v_2, v_7, ..., v_{5\tau-3}\} \cup \{e_4, e_9, ..., e_{5\tau-1}\} \cup \\ \{e'_4, e'_9, ..., e'_{5\tau-1}\} \cup \{f_1, f_6, ..., f_{5\tau-4}\} \cup \{f'_3, f'_8, ..., f'_{5\tau-2}\} \end{cases}$$

$$T_4 = \begin{cases} \{u_3, u_8, ..., u_{5\tau-2}\} \cup \{v_5, v_{10}, ..., v_{5\tau}\} \cup \{e_3, e_8, ..., e_{5\tau-2}\} \cup \\ \{e'_2, e'_7, ..., e'_{5\tau-3}\} \cup \{f_4, f_9, ..., f_{5\tau-1}\} \cup \{f'_1, f'_6, ..., f'_{5\tau-4}\} \end{cases}$$

$$T_5 = \begin{cases} \{u_1, u_6, ..., u_{5\tau-4}\} \cup \{v_3, v_8, ..., v_{5\tau-2}\} \cup \{e_2, e_7, ..., e_{5\tau-3}\} \cup \\ \{e'_5, e'_{10}, ..., e'_{5\tau}\} \cup \{f_2, f_7, ..., f_{5\tau-3}\} \cup \{f'_4, f'_9, ..., f'_{5\tau-1}\} \end{cases}$$

Based on the coloring technique described above, it is evident that the color classes T_1, T_2, T_3, T_4 and T_5 are independent sets of $L(AT_n)$, and it holds the inequality $\|T_a| - |T_b\| \le 1$ for $a \ne b$. This implies that $\chi''_e(L(AT_n)) \le \Delta + 1$. Further, we have $\chi''_e(L(AT_n)) = \chi''(L(AT_n)) \ge \Delta + 1 \quad \Delta + 1 \ge 4 + 1 \ge 5$. Hence $\chi''_e(L(AT_n)) = 5$.

Example 3.4: The graph $L(AT_8)$ and its equitable total coloring is shown in figure 4.

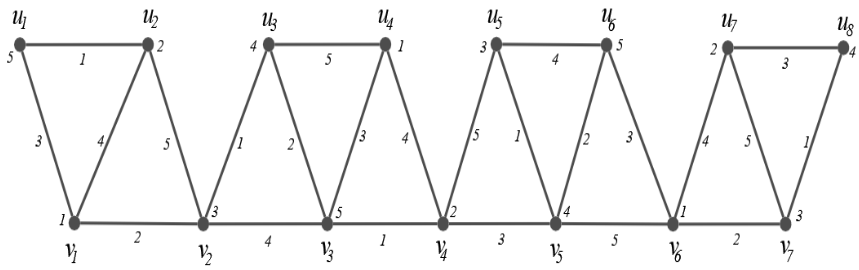

Figure 4: $L(AT_7)$ and its equitable total coloring

Theorem 3.5: Let $L(Q_n)$ represent the line graph of quadrilateral snake graph, then $\chi''_e(L(Q_n)) = 7$.

P***roof:*** $V(L(Q_n)) = \{u_\tau : 1 \le \tau \le n-1\} \cup \{v_\tau : 1 \le \tau \le 2n-2\} \cup \{z_\tau : 1 \le \tau \le n-1\}$ and $E(L(Q_n)) = \left\{e_\tau, e'_\tau, x''_\tau, x'''_\tau : 1 \le \tau \le n-2\right\} \cup \left\{x_\tau, x'_\tau, e''_\tau, e'''_\tau : 1 \le \tau \le n-1\right\}$, where $e_\tau = z_\tau z_{\tau+1}$, $e'_\tau = v_{2\tau} v_{2\tau+1}$, $e''_\tau = v_{2\tau-1} v_{2\tau}$, $e'''_\tau = z_\tau v_{2\tau}$, $x_\tau = u_\tau v_{2\tau-1}$, $x'_\tau = u_\tau v_{2\tau}$, $x''_\tau = v_{2\tau} z_{\tau+1}$, $x'''_\tau = z_\tau v_{2\tau+1}$.

We divide the vertex and edge set of $L(Q_n)$ into distinct partition as described below. There are two cases

Case (i): Suppose n is even

$T_1 = \{z_1, z_3, ..., z_{n-1}\} \cup \{u_{n-4}, u_{n-2}\} \cup \{e'_2, e'_4, ..., e'_{n-2}\} \cup \{e''_2, e''_4, ..., e''_{n-2}\}$

$T_2 = \{z_2, z_4, ..., z_{n-2}\} \cup \{u_{n-3}, u_{n-1}\} \cup \{e'_1, e'_3, ..., e'_{n-3}\} \cup \{e''_1, e''_3, ..., e''_{n-1}\} \cup \{x'_{n-2}\}$

$T_3 = \{v_1, v_3, ..., v_{2n-3}\} \cup \{x'_{n-3}, x'_{n-1}\} \cup \{e_1, e_3 ..., e_{n-3}\}$

$T_4 = \{v_2, v_4, ..., v_{2n-2}\} \cup \{x_{n-3}, x_{n-2}, x_{n-1}\} \cup \{e_2, e_4 ..., e_{n-2}\}$

$T_5 = \{x_\tau : 1 \le \tau \le n-4\} \cup \{x''_\tau : 1 \le \tau \le n-2\}$

$T_6 = \{x'_\tau : 1 \le \tau \le n-4\} \cup \{x'''_\tau : 1 \le \tau \le n-2\}$

$T_7 = \{u_\tau : 1 \le \tau \le n-5\} \cup \{e'''_\tau : 1 \le \tau \le n-1\}$

Case (ii): Suppose n is odd

$T_1 = \{z_1, z_3, ..., z_{n-2}\} \cup \{u_{n-3}, u_{n-1}\} \cup \{e'_2, e'_4, ..., e'_{n-3}\} \cup \{e''_2, e''_4, ..., e''_{n-1}\} \cup \{x'_{n-2}\}$

$T_2 = \{z_2, z_4, ..., z_{n-1}\} \cup \{u_{n-4}, u_{n-2}\} \cup \{e'_1, e'_3, ..., e'_{n-2}\} \cup \{e''_1, e''_3, ..., e''_{n-2}\}$

$T_3 = \{v_1, v_3, ..., v_{2n-3}\} \cup \{x'_{n-3}, x'_{n-1}\} \cup \{e_1, e_3 ..., e_{n-2}\}$

$T_4 = \{v_2, v_4, ..., v_{2n-2}\} \cup \{x_{n-3}, x_{n-2}, x_{n-1}\} \cup \{e_2, e_4 ..., e_{n-3}\}$

$T_5 = \{x_\tau : 1 \le \tau \le n-4\} \cup \{x''_\tau : 1 \le \tau \le n-2\}$

$T_6 = \{x'_\tau : 1 \le \tau \le n-4\} \cup \{x'''_\tau : 1 \le \tau \le n-2\}$

$T_7 = \{u_\tau : 1 \le \tau \le n-5\} \cup \{e'''_\tau : 1 \le \tau \le n-1\}$

Based on the coloring technique described above, clearly the color classes $T_1, T_2, ...,$ and T_7 are independent sets of $L(Q_n)$, and it holds the inequality $\| T_a | - | T_b \| \le 1$ for $a \ne b$. This implies that $\chi''_e(L(Q_n)) \le 7$. Further, we have $\chi''_e(L(Q_n)) = \chi''(L(Q_n)) \ge \Delta + 1 \ge 6 + 1 \ge 7$. Hence $\chi''_e(L(Q_n)) = 7$.

Example 3.5.1: The graph $L(Q_6)$ and its equitable total coloring is shown in figure 5.

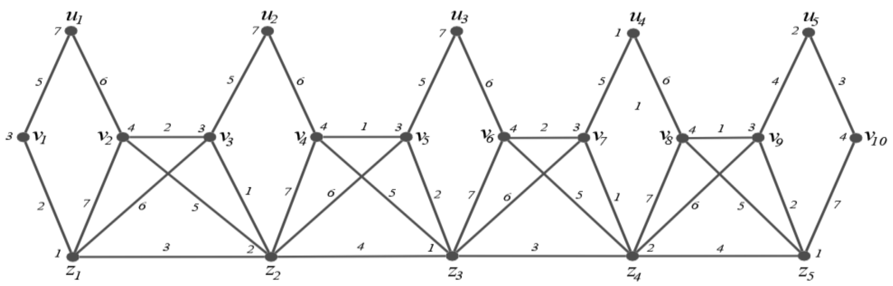

Figure 5: $L(Q_6)$ and its equitable total coloring

Example 3.5.2: The graph $L(Q_5)$ and its equitable total coloring is shown in figure 6.

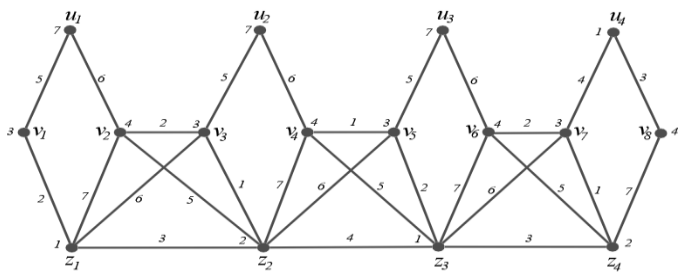

Figure 6: $L(Q_5)$ and its equitable total coloring

Theorem 3.6: Let $L(AQ_n)$ represent the line graph of alternate quadrilateral snake graph, then $\chi_e''\left(L(AQ_n)\right)=5.$

P***roof:*** $V(L(AQ_n))=\left\{u_\tau:1\le\tau\le\left\lfloor\frac{n}{2}\right\rfloor\right\}\cup\{v_\tau:1\le\tau\le n\}\cup\{z_\tau:1\le\tau\le n-1\}$ and

$$E(L(AQ_n))=\{e_\tau:1\le\tau\le n-2\}\cup\left\{e_\tau',e_\tau'':1\le\tau\le n-1\right\}\cup\left\{x_\tau,x_\tau':1\le\tau\le\left\lfloor\frac{n}{2}\right\rfloor\right\},$$

where $e_\tau=z_\tau z_{\tau+1},\ e_\tau'=v_\tau z_\tau,\ e_\tau''=z_\tau v_{\tau+1},\ x_\tau=u_\tau v_{2\tau-1},\ x_\tau'=u_\tau v_{2\tau}.$

We divide the vertex and edge set of $L(AQ_n)$ into distinct partition as described below.

$$T_1=\{z_1,z_3,...,z_{n-1}\}\cup\{e_2'',e_4'',...,e_{n-1}''\}\cup\left\{x_{2\tau-1}':1\le\tau\le\left\lfloor\frac{n}{2}\right\rfloor-3\right\}\cup\left\{u_{\frac{n}{2}}\right\}$$

$$T_2=\{z_2,z_4,...,z_{n-2}\}\cup\left\{x_{2\tau-1}:1\le\tau\le\left\lfloor\frac{n}{2}\right\rfloor-1\right\}\cup\{e_1'',e_3'',...,e_{n-2}''\}\cup\{u_{\lfloor\frac{n}{2}\rfloor}\}$$

$$T_3=\left\{x_2',x_4',,...,x_{\lfloor\frac{n}{2}\rfloor}'\right\}\cup\{v_1,v_3...,v_{n-2}\}\cup\{e_1,e_3...,e_{n-2}\}\cup\left\{x_{\lfloor\frac{n}{2}\rfloor-1}'\right\}$$

$$T_4=\{v_2,v_4,...,v_{n-1}\}\cup\{e_2,e_4...,e_{n-3}\}\cup\left\{x_2,x_4,...,x_{\lfloor\frac{n}{2}\rfloor}\right\}\cup\{e_1'\}$$

$$T_5=\{u_\tau:1\le\tau\le\lfloor\tfrac{n}{2}\rfloor-1\}\cup\{e_\tau':2\le\tau\le n-2\}$$

Based on the coloring technique described above, clearly the color classes T_1,T_2,T_3,T_4 and T_5 are independent sets of $L(AQ_n)$, and it holds the inequality $\left\||T_a|-|T_b|\right\|\le1$ for $a\ne b$. This implies that $\chi_e''\left(L(AQ_n)\right)\le5.$ Further, we have $\chi_e''\left(L(AQ_n)\right)=\chi''\left(L(AQ_n)\right)\ge\Delta+1\ge4+1\ge5$. Hence $\chi_e''\left(L(AQ_n)\right)=5.$

Example 3.6.1: The graph $L(AQ_7)$ and its equitable total coloring is shown in figure 7.

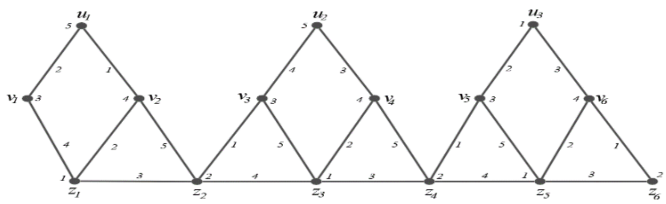

Figure 7: $L(AQ_7)$ and its equitable total coloring

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