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Space-Time Caputo Fabrizio Fractional Differentiation through a Proposed Fixed Transform-Based Domain (Ftd) Approach for Groundwater Model

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Abstract:
Mathematical models of groundwater flow have been used since the late 1800s. A set of differential equations known to control groundwater flow make up a mathematical model. We suggest the Fixed Transform Domain (FTD) approach to address the Ground Water Flow Problem's Time-space Fractional derivative. The fractional differential equation can represent groundwater flow since it is not static and can relate to initial and boundary circumstances. Utilizing the Darcy law and the conservation of mass law, this model considers the water flow to the piezometric head as a function of the derivative of the fractional arrangement. The primary goal of this study was to provide a novel idea for simulating issues with groundwater flow. Finally, two cases and simulations for these examples show that the proposed method works. Empirical instances demonstrate that the suggested approach is highly efficient and applicable to related issues. A numerical scheme stability analysis is provided to confirm the effectiveness of the proposed framework. By comparing the results of the suggested method with those of Barker's fractal radial flow framework, an association between the non-integral dimension of the flow and the fractional order of the derivative is illustrated. Additionally, the proposed method is highly effective and applicable to similar problems.

Keywords: fractional derivative; time derivative; differential equation; difference method; groundwater.

1. Introduction

The advantageous properties of fractional calculus, including non-locality, great dependability, inheritance, and analyticity, have made it an extremely esteemed framework for administering complex and sustainable phenomena for over 30 years. The modified fractional concept addressed issues related to processes, including homogeneities. The fundamental foundation was established by several pioneers, such as Hadamard [2], Liouville and Caputo [1], Davison and Essex and Jumarie [5], Grunwald and Letnikov [3], Riez, Coimbra [4], and others. Fractional-order differential equations provide valuable insights into various fundamental interactions in electrochemistry, acoustics, electromagnetics, viscoelasticity, and material science [6, 7, 8, 9]. Many new ideas about derivatives of fractional order have been created. These comprise the following: the generalized Riemann-Liouville and generalized Caputo derivative by Atangana [12], the Losada and Nieto [11], the Yang-Srivastava-Machado derivative by Yang [11], and the Caputo-Fabrizio derived form by Caputo and Fabrizio [10]. The Caputo-Fabrizio fractional derivative for the mean square displacement exhibits crossover behaviours, whereas the Riemann-Liouville is scale invariant.

While a Gaussian to non-Gaussian crossover dispersion is possible, the Caputo-Fabrizio kernel remains constant throughout the transition. More favourable attributes are demonstrated by the new Caputo-Fabrizio fractional derivative than its antecedent. It can disclose distinctions and patterns in substances on many levels that are not attainable through conventional local theories or popular fractional order derivatives. In addition, these varieties of materials are used to study the large-scale behaviours of some materials linked to atom-to-atom nonlocal communications, which are known to be important material properties [13]. In recent years, many studies have been conducted on fractional-order ordinary differential equations, partial differential equations, and partial integral differential equations. Numerous fields use these equations, including finance, economics, engineering, and biology [14].

Consequently, much emphasis has been placed on the solutions to these equations of physical importance [15, 16, 17, 18]. It is widely known that most of these non-linear fractional order differential equations have analytical solutions; however, they are either extremely complex to use or completely missing in closed-form solutions. Approximate and computational strategies are introduced in this case. Many techniques were employed, including the spectral methodology [20], the variational iteration methodology [21], the Adomian decomposition methodology [22, 23], and the finite difference method [19]. To promote worldwide research and useful applications of related approaches, the Regional Groundwater Flow Commission (RGFC) was established in 2011 following the International Association of Hydrogeologists (IAH)'s recognition of the issue of "groundwater flow." According to an evaluation of the future of groundwater science and research, new instruments for contemporary hydrogeology and basin-wide investigations of regional groundwater flow systems are required. This study will concentrate on the suggested Fixed Transform Domain (FTD) approach to solving fractional differential equations based on groundwater flow. This transformation can be helpful in differential equations since it can make the equations easier to solve. Additionally, it creates additional complications when boundary conditions are added to fractional differential equations.

Given that the suggested approach is limited to handling the fractional space operator and the beginning conditions. These characteristics allow for reduced computing time and rapid method execution. Numerical experiments show the precision and efficacy of the proposed methodology.

2. RELATED WORK

Many scientific and engineering investigations regarding numerical simulations and mathematical modelling of dynamical systems have been implemented. Fractional order operators are a well-researched and practical approach to enhancing the capabilities of traditional models. The history of fractional order operators includes solitary and non-singular kernels and local and nonlocal kernels. A wide range of fractional differential equations (FDEs) have been studied by specialists, including sequential, hybrid, mixed, and many other classes that have not yet been fully investigated in this field. Plot perturbation techniques are extremely helpful in non-linear analysis because they facilitate comprehension of system dynamics represented by a diverse array of mathematical approaches. Solving or evaluating a differential equation that characterizes a certain dynamical system can often be difficult. To determine whether any unique answers (EUS) existed in the Riemann-Liouville meaning of the derivatives, Zhao et al. [24] examined the second kind of quadratic perturbation problem. In the study of Sitho et al. [25], fractional integro-differential equations for the EUS were investigated along with possible applications in situations where the derivative was in the R-L sense. In work [26], Awadalla and Abuasbeh investigated a second-class perturbed sequential FDE for the EUS for Caputo-Hadamard operators. A group of hybridized FDEs were studied by Gul et al. [27], who applied their results to dynamical issues. Their function was the derivative of Caputo. Khan et al. [28] used the Leray-Schauder and Banach alternative theorems to analyze several mixed FDEs for EUS and Hyers-Ulam

stability. They employed two distinct forms of fractional ordering: the AB-fractional operator and Caputo's derivative of fractions. Losada and Nieto [29] looked into sequential FDEs for the EUS with a non-singular kernel. The description of a fractional derivative with a kernel that is not a singular number is provided by Caputo, M. Fabrizio [30]. Atangana and Baleanu [31] discussed the idea of the AB-fractional derivative and how it can be used. The concept in [31] was expanded upon by Al-Refai and Baleanu [32], who also resolved the AB-operator setup issue. Dhage [34] and Dhage et al. [33] looked into mixed classes of fractional differential equations to see if there were any solutions and only one. The AB operator for the derivative was modified by the concept of the inverse operator of fractional order, which was first presented by Al-Refai [35] and demonstrated to have a wide range of applications. Khan et al. [36] used the problem of imperfect tests to show some simulations for a disease model. Shi and Cui [37] picked up Hepatitis C and were given the conditions they needed to stay stable. Subramanian [38, 39] discussed the existence of a solution for a connected system with integro-differential equations and sequential fractional operators. A family of fractional order rapid integro-differential equations with boundary conditions was examined by Zada et al. [40] in terms of Ulam-Hyers stability.

3. PROBLEM STATEMENT

The groundwater problem is perhaps the hardest to represent mathematically among the real-world issues. The behaviour of the medium through which the water travels must be precisely understood to model this problem accurately. However, the medium in which the flux happens might change throughout time and from one era to the next. For instance, an aquifer's hydraulic conductivity might change in different directions. Many academics have collaborated to produce a more precise framework that can be used to predict the amount of water that will flow through the aquifer. Nevertheless, the findings continue to exhibit certain deficiencies. By utilizing the proposed Fixed Transform Domain approach, we demonstrate that the fractionally ordered derivative model of real problems yields superior results to experimental data in its mathematical representation.

4. PROPOSED METHODOLOGY

The groundwater flow model in this study will be established using the Caputo-Fabrizio derivative. For this, the issue of water flow inside a finite aquifer is used.

4.1. Caputo Derivative:

Definition -4.1.1: The following is the definition of the Caputo derivative of order $\propto$ for the function f(t), if $\propto > 0$:

$$D^{\propto} f(t) = \frac{1}{\Gamma(\mathrm{n} - \propto)} \int_0^x \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau \qquad (4.1)$$

Where in $-1 < \propto < n$, $n \in N$, & $\Gamma(.)$ id represent the gamma function. It has linear property.

$$D^{\propto}\big(k_1 g_1(t) + k_1 g_1(t)\big) = k_1 D^{\propto} g_1(t) + k_2 D^{\propto} g_2(t),$$

Where k_1 and k_2 in this case are constants.

Remark 1: Assume that ($0 < \alpha \leq 1$). The formula that follows can be obtained from Definition 4.1.1.

$$D^{\alpha} t^{\gamma} = \begin{cases} 0 & \gamma = 0 \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha} & \gamma = 1,2,3 \ldots\ldots\ldots \end{cases} \qquad (4.2)$$

Definition -4.1. 2: Let

$$A = \{ f(t);\ \exists\, \tau_{1,}\tau_2 > 0. \text{ such that, } |f(z)| < M e^{\frac{|t|}{\tau_j}}, if\ t \in\ (-1)^j \times [0,\infty]\}$$

F(u)= ~[f(t)] is the suggested transform of $f \in A$ and has the following definition:

$$F(u) = \sim [f(t)] = \int_0^{\infty} f(ut) e^{-t} dt, u \in (\tau_1, \tau_2) \qquad (4.3)$$

When u is a parameter, it might be real or complex and is not dependent on t. The connection [41], states that G(1/j)=JL(j) and L(1/u)=u G (u) represents the relationship between the sumudu transform and Laplace transformation, respectively.

$$J\,[D^{\alpha} f(t)] = \frac{F(u)}{u^{\alpha}} - \sum_{k=0}^{n-1} \frac{f^k(0)}{u^{\alpha-k}}, n-1 < \propto \leq n, n \in N \qquad (4.4)$$

Where $F(u) = \sim [f(t)]$

As part of their groundwater studies, numerous academics are looking into and solving the standard equation for shallow water [42]. The equations that were examined and resolved in this investigation regulate groundwater flow. This derivative uses the concept of continuity low without recourse and the Darcy law.

$$S_0 \partial_t \varphi\,(r,t) = \frac{k}{r^{n-1}} \partial_r [r^{n-1} \partial_r \varphi\,(r,t)] \qquad (4.5)$$

$$\varphi\,(r,0) = ar,$$

$$Q = \frac{2\pi^{n/2}}{\Gamma(n/2)} r_b^{n-1} k d^{3-n} \partial_r \varphi\,(r_b,t) \qquad (4.6)$$

where n denotes the radial dimension for n = 1, 2, or 3. The scalar quantities represent the hydraulic conductivity and the recommended storage capacity S_0 and K, respectively. Furthermore, φ(r, 0) = ar suggests that the aquifer's primary hydraulic head, with a positive gradient a, is a linear function of space, with $\lim_{r\to\infty} \varphi(r,t) = ar$ meaning that the water level of the aquifer is either at or at the start of the pumping test.

$$S_0 \partial_t \varphi\,(r,t) = \frac{k}{r^{n-1}} \partial_r\, [r^{n-1} \partial_r^{\propto} \varphi\,(r,t)] \qquad (4.7)$$

This physical parameter, which describes the flow across geological formations, is contained in the binary integral equation [43]. The boundary condition is given a more general form at the well by utilizing the same transformation in (4.6):

$$Q = (-1)^{m+1} \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)\Gamma(p)} r_b^{n-1}\, k d^{3-n+\propto} \partial_r^{\alpha} \varphi\,(r_b,t) \qquad (4.8)$$

The initial condition in (4.6) and the relationships in (4.7) and (4.8) represent an entire set of equations for an existential solution. The groundwater model equation can be solved in various ways, including the Frobenius and finite element methods. The method suggested in this paper will be applied to solve the fractional groundwater model equation.

4.2. The Generalised Flow Equation's Solution

Examine the Caputo derivation of an order α regulated groundwater flow, where α is larger than or equal to 1 and lies between 0 and 1.

Example 1: To determine the following time derivatives in fractions, the flow of groundwater equation needs to be examined.

$$K\,\partial_r^2\,\varphi\,(r,t) + \frac{k}{r} \partial_r \varphi(r,t) - S_0 \partial_t^{\alpha} \varphi\,(r,t) = 0 \qquad (4.9)$$

Subject Eq. (4.6). Here, the approach relies on using transform (4.9) to obtain

$$s\,[\varphi(r,t)] = \varphi(r,0) + \vartheta^{\alpha} J \left[\frac{k}{S_0}\partial_r^2 \varphi(r,t) + \frac{k}{S_0 r}\partial_r \varphi(r,t)\right] (4.10)$$

Applying the inverse operator yields

$$\varphi(r,t) = \varphi\,(r,0) + S^{-1}[v^{\alpha} J \left[\frac{k}{S_0}\partial_r^2 \varphi(r,t) + \frac{k}{S_0 r}\partial_r \varphi\,(r,t)\right] \quad (4.11)$$

Assuming the Adomian decomposition approach for the series solution, (4.11) is

$$\varphi\,(r,t) = \sum_{i=0}^{\infty} \varphi_i\,(r,t) \qquad (4.12)$$

Often, the elements ψi (r, t) are chosen. Substitute (4.12) into both sides of (4.11) to reach the desired result.

$$\sum_{i=0}^{\infty} \varphi_i\,(r,t) = \varphi(r,0) + S^{-1}[v^{\alpha} S[\frac{k}{S_0}\partial_r^2 \sum\nolimits_{i=0}^{\infty} \varphi_i\,(r,t) + \frac{k}{S_0 r}\ \partial r \sum_{i=0}^{\infty} \varphi_i(r,t)]]\ (4\cdot 13)$$

Next, recursive relationships are shown as

$$\varphi_0(r,t) = pr,$$

$$\varphi_{i+1}(r,t) = J^{-1}\left[v^{\alpha} S\left[\frac{k}{S_0}\partial_r^2 \varphi_i(r,t) + \frac{k}{S_0 r}\partial_r \varphi_i(r,t)\right]\right] i \geq 1 \qquad (4.14)$$

The remaining components, i > 1, are determined by applying the preconditions because component $\varphi_0(r, t)$ is defined.

$$\varphi_0(r,t) = \varphi\,(r,0) = pr,$$

$$\varphi_1(r,t) = J^{-1}[v^{\alpha} S\left[\frac{k}{S_0}\partial_r^2 \varphi_0(r,t) + \frac{k}{S_0 r}\partial_r \varphi_0(r,t)\right]]$$

$$= S^{-1}[\vartheta^{\propto} S\left[\frac{pk}{S_0 r}\right] = \frac{pkt^{\propto}}{rS_0\Gamma(1+\propto)}$$

$$\varphi_2(r,t) = J^{-1}[\vartheta^{\alpha} S\left[\frac{k}{S_0}\partial_r^2 \varphi_1(r,t) + \frac{k}{S_0 r}\partial_r \varphi_1(r,t)\right]]$$

$$= J^{-1}[v^{\propto} J\left[\frac{k}{S_0}\left(\frac{2pkt^{\infty}}{r^3 S_0\Gamma(1+\propto)}\right) + \frac{k}{S_0 r}\left(\frac{-pkt^{\propto}}{r^2 S_0\Gamma(1+\propto)}\right)\right]]$$

$$= J^{-1}[\vartheta^{\propto} J\left[\frac{pk^2 t^{\alpha}}{r^3 S_0^2\Gamma(1+\alpha)}\right]]$$

$$= J^{-1}[\frac{pk^2\vartheta^{2\propto}}{r^3 S_0^2}] = \frac{pk^2 t^{2\propto}}{r^3 S_0^2\Gamma(1+2\propto)} \qquad (4.15)$$

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In this instance, we calculated the values of the four components of Eq. (4.12) to obtain ψ(r, t), resulting in the following expansion.

$$\varphi\,(r,t) = pr + \frac{pkt^{\alpha}}{rS_0\Gamma(1+\alpha)} + \frac{pk^2t^{2\alpha}}{r^3S_0^2\Gamma(1+2\alpha)} + \frac{9pk^3\vartheta^{3\alpha}}{r^5S_0^3\Gamma(1+3\,\alpha)} \quad (4.16)$$

Additionally, by using the boundary condition on (4.16), we may determine a constant number (a) to acquire.

$$p = \frac{Q}{2\pi r_b k d f(r_b,t)} \quad (4.17)$$

where

$$f(r_b,t) = 1 - \frac{kt^{\propto}}{r_b^2S_0\Gamma(\propto+1)} + \frac{3kt^{2\propto}}{r_b^4S_0^2\Gamma(2\propto+1)} + \cdots. \quad (4.18)$$

Example 2: Let us examine the groundwater equation, including fractional derivatives in both space and time.

$$S_0\partial_t\varphi\,(r,t) = \frac{k}{r}\partial_r^{\propto}\varphi(r,t) + k\partial_r^{\mu}\varphi(r,t), 0<\alpha\le 1<\mu\le 2 \quad (4.19)$$

Eq. (4.6) dictates that the gradient is the same as zero. ∂αr φ(r, 0) = constant is the expected fractional change in drag for t = 0 since the aquifer's water level is assumed to be constant before pumping. Use the Sumudu transform on equation (4.19). The result is:

$$J[\partial_t\varphi(r,t)] = J[\frac{k}{rS_0}\partial_r^{\propto}\varphi\,(r,t) + \frac{k}{S_0}\partial_r^{\mu}\varphi\,(r,t),$$

$$\vartheta^{-1}\big(J[\varphi(r,t)] - \varphi\,(r,0)\big) = J[\frac{k}{rS_0}\partial_r^{\propto}\varphi\,(r,t) + \frac{k}{S_0}\partial_r^{\mu}\varphi(r,t)$$

So

$$J[\varphi(r,t)] = \varphi(r,0) + \vartheta J\,[\frac{k}{rS_0}\partial_r^{\propto}\varphi(r,t) + \frac{k}{S_0}\partial_r^{\mu}\varphi(r,t) \quad (4.20)$$

The inverse operator is applied to Eq. (4. 20) to obtain

$$\varphi\,(r,t) = \varphi(r,0) + J^{-1}\left[\vartheta S\left[\frac{k}{rS_0}\partial_r^{\propto}\varphi(r,t) + \frac{k}{S_0}\partial_r^{\mu}\varphi(r,t)\right]\right] \quad (4.21)$$

Substituting (4.12) into both sides of (4. 21) results in

$$\sum_{i=0}^{\infty}\varphi_i(r,t) = \varphi(r,0) + J^{-1}[\vartheta J[\frac{k}{rs_0}\partial_r^{\alpha}(\sum_{i=0}^{\infty}\varphi_i(r,t)) + \frac{k}{S_0}\partial_r^{\mu}(\sum_{i=0}^{\infty}\varphi_i(r,t))]] \quad (4.22)$$

The recursive relationships are presented as

$$\varphi_0(r,t) = pr \qquad (4.23)$$

$$\varphi_{i+1}(r,t) = J\left[\vartheta J\left[\frac{k}{rS_0}\partial_r^{\infty}\varphi_1(r,t) + \frac{k}{S_0}\partial_r^{\mu}\varphi_i(r,t)\right]\right], i \geq 1 \qquad (4.24)$$

The equations below obtained by using that *∂αr ψ(r*, 0*)* = constant = *a* with (4. 24)

$$\varphi_1(r,t) = J^{-1}\left[\vartheta J\left[\frac{k}{rS_0}\partial_r^{\propto}\varphi_0(r,t) + \frac{k}{S_0}\partial_r^{\mu}\varphi_0(r,t)\right]\right]$$

$$= S^{-1}\left[\vartheta S\left[\frac{pk}{S_0 r}\right]\right] = S^{-1}\left[\frac{pk\vartheta}{S_0 r}\right] = \frac{pkt}{rs_0}$$

$$\varphi_2(r,t) = \frac{pk^2t^2}{2S_0^2}\left(\frac{\Gamma(1+\mu)}{r^{1+\mu}} + \frac{\Gamma(1+\alpha)}{\alpha r^{2+\alpha}}\right),$$

$$\varphi_3(r,t) = \frac{pk^3t^3}{6S_0^3}\left(\frac{\Gamma(1+2\mu)}{r^{1+2\mu}} + \frac{\Gamma(1+2\alpha)}{\alpha r^{2+\alpha}}\right), \qquad (4.25)$$

The approximated solution to the issue is given by the following expansion, which is obtained by combining equations (4.23) and (4.25).

$$\varphi\,(r,t) = pr + \frac{pkt}{rS_0} + \frac{pk^2t^2}{2S_0^2}\left(\frac{\Gamma\,(1+\mu)}{r^{1+\mu}} + \frac{\Gamma\,(1+\alpha)}{\alpha r^{2+\alpha}}\right) + \frac{pk^3t^3}{6S_0^3}\left(\frac{\Gamma(1+2\mu)}{r^{1+2\mu}} + \frac{\Gamma(1+2\alpha)}{\alpha r^{2+2\alpha}}\right) \qquad (4.26)$$

Additionally, by using the boundary condition on (4.26), we may determine a constant value (a) to acquire.

$$p = \frac{Q}{2\pi\, r_b k d^{\alpha} f\,(r_b,t)} \qquad (4.27)$$

Where

$$f\,(r_b,t) = 1 + \frac{kt\Gamma(1+\propto)}{S_0 r_b^{1+\propto}} + \frac{k^2t^2\Gamma(1+\mu+\alpha)}{2S_0^2 r_b^{1+\alpha+\mu}} + \frac{k^2t^2\Gamma(2+2\alpha)}{2(1+\alpha)S_0^2 r_b^{2+2\alpha}} + \frac{k^3t^3}{6S_0^3}\left(\frac{\Gamma(1+2\mu+\alpha)}{r_b^{1+2\mu+\alpha}} + \frac{\Gamma(2+3\alpha)}{(1+\alpha)(1+2\alpha)r_b^{2+3\alpha}}\right) \qquad (4.28)$$

$$\varphi(r,t) = \frac{Qr^{2(1-{n}/{2})}}{4\pi^{(\frac{n}{2})}\, kd^{1-n}}\Gamma[\frac{n}{2}-1,\frac{r^3 S_0}{4kt}](4.29)$$

Gamma function is denoted by Γ. η: The special dimension of the flow, always an integer n, is the same as the special dimension.

5. Results and Discussion

The revised groundwater flow equation is precisely solved in this section by utilizing the suggested Fixed Transform Domain (FTD) technique. We show the suggested approach along with its properties for the fractional order Caputo-Fabrizio derivative.

5.1 Numerical simulations

Table 1 lists the theoretical parameters for the aquifer's properties and discharge rate used in the numerical simulation.

Table 1. Values based on theory

parameters	Values	Units
S_0	0.74	m^{-1}
Q	10.7	m^3s^{-1}
k	0.64	ms^{-1}
d	39	m
r_b	0.14	m
$\varphi_0(r,t)$	0	m
∝	0.004	None
μ	1.004	None
n	0.011	none

This section contains a numerical presentation of the new groundwater model's precise solution for various alpha values. The numerical simulations for various alpha values are displayed in Figures 1, 2, 3, 4, and 5.

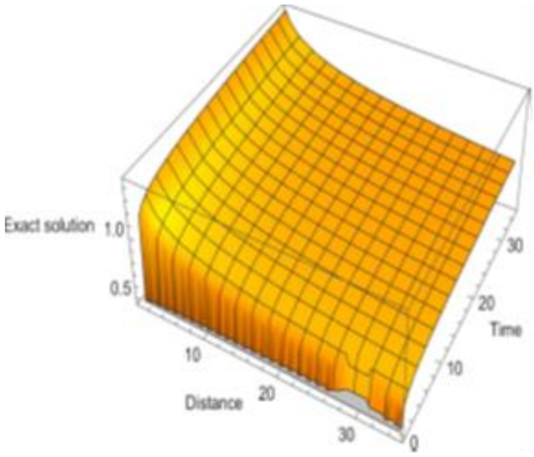

Figure. 1. Precise solution simulation for alpha = 0.95 as a function of space and time

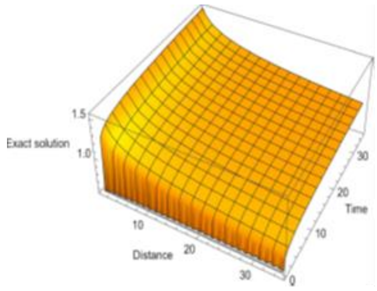

Figure. 2. Time and space-dependent simulation of the precise solution for alpha = 0.55

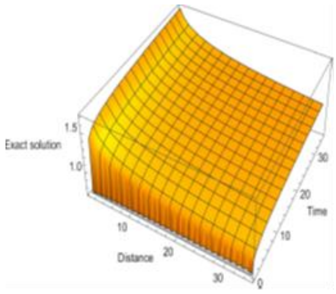

Figure. 3. Computation of the precise solution for alpha = 0.45 as a function of time and space

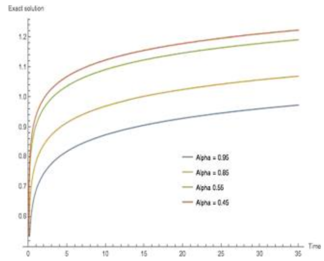

Figure. 4. Simulate the precise solution for alpha in respect to time and space.

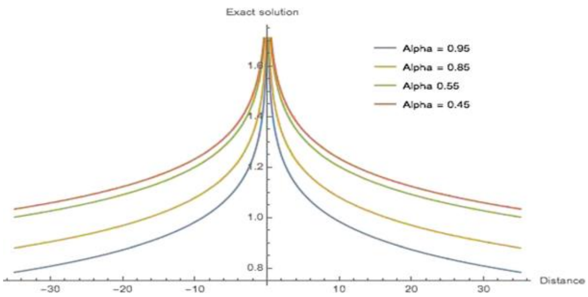

Figure. 5. Modelling of the precise response as a function of space for a range of alpha values

The above-mentioned steady-state model is precisely solved using the newly created fractional derivative. The suggested method's solution can be obtained when alpha gets 1. One additional feature of the new solution is that it may be applied to depict critical real-world situations.

Diagrams 6, 7, 8, 9, 10, 11, 12, 13, and 14 graphically depict the solutions of (4.9) and (4.19) for a range of parameter values. Additionally, these graphs demonstrate how the drawdown behaved during the pumping test.

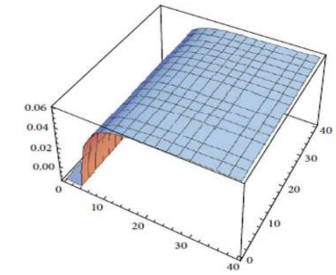

Figure 6. Solution of (4.9) (drawdown)

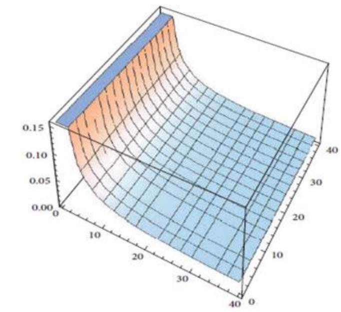

Figure. 7. Solution of (4. 9) (Piezometric head)

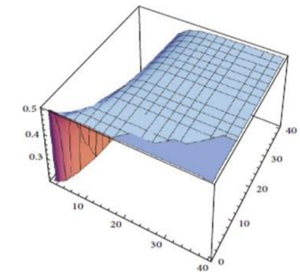

Figure. 8. Solution of (4.19) (drawdown)

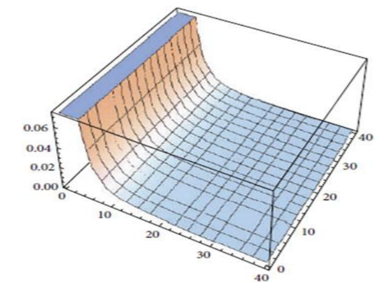

Figure. 9. Solution of (4.19) (Piezometric head)

i. Starting with the well as a function towards the perspective point of space and time.

ii. Considering a fixed distance about the well as a function of time.

iii. as a function of space for a predefined duration.

The following Barker solution for an infinite aquifer with a line source is further contrasted with them.

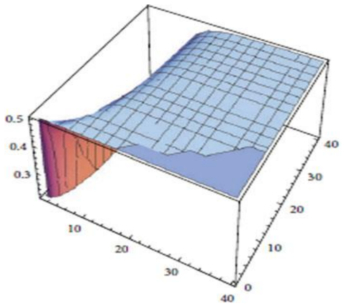

Figure. 10. Barker's solution (drawdown)

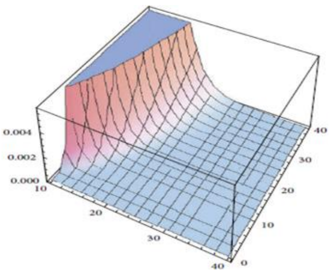

Figure. 11. Barker's solution (piezometric head)

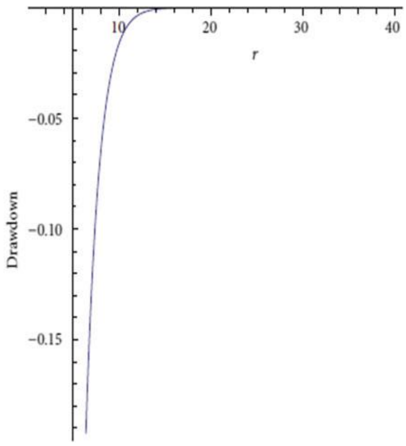

Figure. 12. Solution as a function of the base of (4.9)

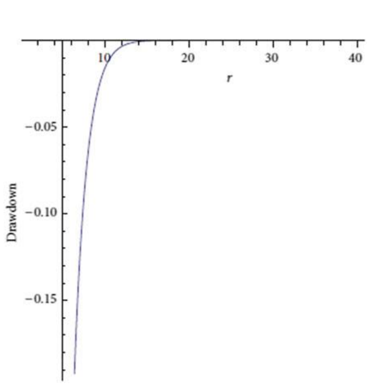

Figure. 13. Solution as the basis of (4.19) function

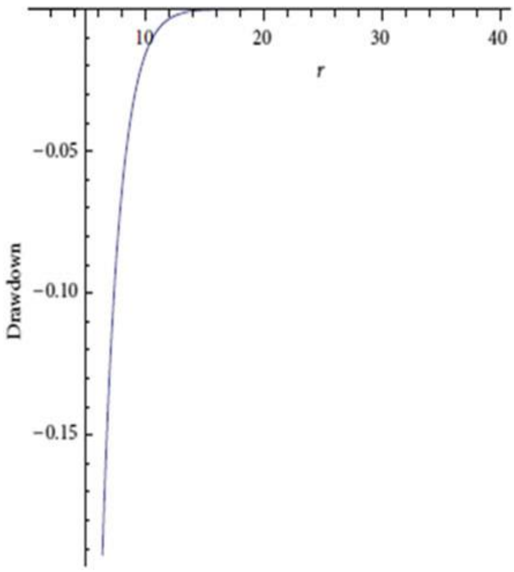

Figure. 14. Barker's solution as a function of space

Problems with groundwater flow continue to grow in importance as a global environmental concern. This is one of the main reasons that numerous scientists from different disciplines have worked very hard to create new mathematical formulas and models that can be used to illustrate and comprehend the behaviour of groundwater flow over time and space. It is necessary to compare the outcomes of the suggested method (FTD) with Barker's solution to ascertain whether there is a connection between the parameter fractal that Barker previously described and the fractional order of the derivative.

6. CONCLUSION

Numerical groundwater models are constructed by numerically approximating both space and time. Describing underground water passage across an aquifer, a diverse geological structure is perhaps the most difficult aspect of studying groundwater. The main problem is that a true mathematical formula considering the aquifer's heterogeneity is difficult to develop. Recently, a new fractional order derivative has been introduced. This derivative is derived from the convolution of an exponential function with the conventional first derivative. The derivative can be employed to characterize material diffusion on various scales. We give a way to demonstrate the uniqueness and existence of solutions to the groundwater flow issue in a confined aquifer within the context of a derivative with fractional order. The Fixed Transform based Domain (FTD) technique confirmed the exact solution.

Moreover, we can show only one solution to the changed groundwater flow equation. The local temporal and spatial derivative can be substituted with the Caputo-Fabrizio derivative to create this equation. Given this discovery, numerical simulations and stability of solution analyses were conducted for a variety of α values to illustrate the efficacy of this innovative model. A relationship between the flow's non-integral dimension and the derivative's fractional order is demonstrated by comparing the results of the proposed method with those of Barker's fractal radial flow framework. Additionally, the proposed method is highly effective and applicable to similar problems.

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