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On Fractional Epidemic Model Order Shigella

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Abstract:

Introduction: We have adapted the continuous mathematical framework developed to examine the dynamics of a Shigella epidemic at a constant recruitment rate. They divided the population into seven groups in their model: susceptible (S), vaccination(V), exposé(E), infected (I), isolated (G), hospitalized(H) and recovered (R), each with its own set of parameters. We examined a mathematical model of a Shigella outbreak in a community with a constant population using the SVEIGHR compartmental nonlinear deterministic approach. The model was subjected to analytical investigations utilizing the linearized stability method. The greatest eigenvalue of the next-generation matrix yields the fundamental reproductive number R_0, which controls the spread of the disease. The generalized Routh-Hurwitz and Jacobian criterion are used to determine the threshold value R_0. The model is unstable if, $R_0 > 1$, and stable if, $R_0 < 1$. Additionally, we determine the endemic and disease-free equilibrium points, which are helpful for a faster recovery. We have created graphs in Matlab to provide a more accurate model representation.

Objectives: To describe the disease Shigella with the help of SVEIGHER compartmental nonlinear deterministic approach through factorial differential method and compare it to classical differential method and see the results.

Results: From the above studies we conclude that fractional differential method is more effective than classical differential method.

Conclusion : In this paper, we discuss SVEIGHR epidemic model for disease Shigella . This SVEIGHR model controlss the spreading of the disease in Human population. By using Routh-Hurwitz Criteria we find all the eigen values for endemic point are negative which shows that the above model SVEIGHR (Susceptible Vacation Infectious Isolated Hospitalization Recovered) is stable. Extending our work, we can also use harmonic mean type incidence rate for better stability and control the disease.

Key word : Mathematical model, stability analysis, endemic equilibrium; Routh-Hurwitz criteria, Reproduction number.

AMS Classification 2010 : 93A30, 92B05, 92D30

1. Introduction

Gram-negative, non-spore-forming Shigella bacteria are the main cause of Shigella infection, commonly referred to as Shigellosis, a diarrheal illness. It is via fecal-oral matter, which is typically brought on by contaminated food, water, or close contact with infected people. These bacteria infect the intestinal lining. Tenesmus, fever, abdominal pain, and diarrhea-often bloody-are among the symptoms that start to show up 1-2 days after exposure and can linger for up to a week. Serious side effects include hemolytic uremic syndrome, sepsis, and reactive arthritis could happen. The majority of cases go away without special care, but rest and hydration are crucial. Antibiotics such as azithromycin or ciprofloxacin may be necessary in severe cases, though resistance is becoming a bigger problem. Although there isn't a licensed vaccine at the moment, a number of possibilities are being developed, such as conjugate and live attenuated vaccines. Reducing transmission still requires preventive actions like better sanitation and hand hygiene.

Fractional calculus investigates function derivatives and integrals. However, in this area of mathematics, we are examining integrals and derivatives of non-integer order rather than the typical integer order. These might be of real or complex order and are referred to as fractional-order derivatives and fractional integrals. A mathematical tool that unifies and generalizes the derivative and integral of integer order to any arbitrary order is generally referred to as fractional calculus.

Leibniz initially addressed the theory of fractional derivative in 1695. Because of this, mathematicians from the 18th to the 19th centuries were interested in studying this field. A well known scientist, Abel, in 1823 was the first scientist to apply Fractional calculus for investigating tautochrone problems. Fractional calculus has been the subject of worldwide attention in the last decades due to its broad range in biology, physics, chemistry, engineering, modeling etc.

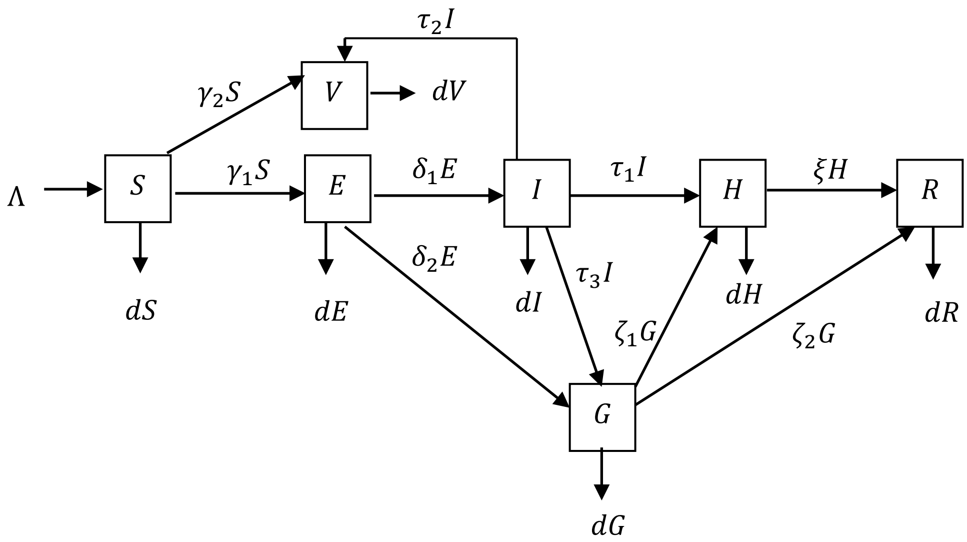
Fractional differential equations system allows more degree of freedom, memory effect, more adequate, more realistic interpretation of natural phenomena than integer-order derivatives in the model. They have become an excellent tool in modeling epidemiology. Hence, fractional derivatives based on epidemic system have also been used to deal with some epidemic behaviors.

The statically data gathered during an actual disease epidemic cannot be adequately reproduced by traditional first-order differential equations, and they also do not yield satisfactory results. The research has examined a more specific and complex set of differential equations in order to provide better conclusions that are more in line with reality. We suggest a significantly altered system of equations that makes use of the fractional order differential equation.

Hypotheses of the model
i. The recruitment process is constant and solely based on birth.
ii. Every person is susceptible from birth.

iii. Contact with the excrement of diseased people and tainted food or water can infect a person.
iv. Infected people can pass either naturally or as a result of the illness.
v. Only susceptible adults and children between the ages of 0 and 6 months are eligible for immunization.
vi. No long-term recovery is possible.
vii. The population is homogeneously mixed.
viii. People engage with one another in a panmictic manner

2. Model Derivation :



$$\left.\begin{aligned}
\frac{dS}{dt} &= \Lambda - \gamma_1 S - \gamma_2 S - dS \\
\frac{dV}{dt} &= \gamma_2 S + \tau_2 I - dV \\
\frac{dE}{dt} &= \gamma_1 S - \delta_1 E - \delta_2 E - dE \\
\frac{dI}{dt} &= \delta_1 E - \tau_1 I - \tau_2 I - \tau_3 I - dI \\
\frac{dG}{dt} &= \delta_2 E + \tau_3 I - \zeta_1 G - \zeta_2 G - dG \\
\frac{dH}{dt} &= \tau_1 I + \zeta_1 G - \xi H - dH \\
\frac{dR}{dt} &= \xi H + \zeta_2 G - dR
\end{aligned}\right\} \quad .... \quad (1)$$

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Condition : $S + V + E + I + G + H + R = N$

where,

Variables	Description
S	The total number of susceptible population at time (t)
V	The total number of vaccination population at time (t)
E	The total number of exposed population at time (t)
I	The total number of infective population at time (t)
G	The total number of isolated population at time (t)
H	The total number of hospitalized population at time (t)
R	The total number of recovered population at time (t)

and

Parameters	Description	Values
Λ	All the new immigrants are susceptible and join the group at a constant rate	
N	The human population	10000
γ_1	The exposed rate by susceptible population	0.25
γ_2	The vaccination rate by susceptible population	0.75
δ_1	The infected rate by exposed population	0.5
δ_2	The isolated rate by exposed population	0.2
τ_1	The hospitable rate by infected population	0.15
τ_2	The vaccination rate by infected population	0.25
τ_3	The isolated rate by infected population	0.2
ζ_1	The hospitable rate by isolated population	0.25
ζ_2	The recovery rate by isolated population	0.20
ξ	The recovery rate by hospitable population	0.25
d	The natural death rate	0.000457
β	The order of fractional	$0 < \beta \leq 1$

Many definitions of fractional derivatives [3,4] are used. The Riemann –Liouville derivatives of ith order is defined as

$$_{RL}\mathcal{D}_{0+}^{\beta}f(t)=\frac{1}{\Gamma(n-i)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}\frac{f(s)}{(t-s)^{i-n+1}}ds \text{ where}, n=[i]+1 \quad \dots \quad (2)$$

An alternative definition by Caputo was as follows

$$\mathcal{D}_{t}^{\beta}f(t)=\frac{1}{\Gamma(n-i)}\int_{0}^{t}\frac{f^{(n)}(s)}{(t-s)^{i-n+1}}ds \text{ where}, n=[i]+1 \quad \dots \quad (3)$$

In this paper we will use Caputo fractional derivative for the system (1)

$$\left.\begin{aligned}
\mathcal{D}_{t}^{\beta}S&=\Lambda-(\gamma_1+\gamma_2+d)S\\
\mathcal{D}_{t}^{\beta}V&=\gamma_2 S+\tau_2 I-dV\\
\mathcal{D}_{t}^{\beta}E&=\gamma_1 S-(\delta_1+\delta_2+d)E\\
\mathcal{D}_{t}^{\beta}I&=\delta_1 E-(\tau_1+\tau_2+\tau_3+d)I\\
\mathcal{D}_{t}^{\beta}G&=\delta_2 E+\tau_3 I-(\zeta_1+\zeta_2+d)G\\
\mathcal{D}_{t}^{\beta}H&=\tau_1 I+\zeta_1 G-(\xi+d)H\\
\mathcal{D}_{t}^{\beta}R&=\xi H+\zeta_2 G-dR
\end{aligned}\right\} \quad \dots. \quad (4)$$

All parameters are assumed to be non-negative from system (4), where $t\geq 0$

$$\mathcal{D}_{t}^{\beta}N=\Lambda-dN \quad \dots \quad (5)$$

We solve the equation (5) by using Mittage-Leffler function, which is defined as

$$E_{\beta,\gamma}(z)\sim-\sum_{i=1}^{n}\frac{z^{i}}{\Gamma(\gamma-\beta i)}+O(|z|^{-1-n}),\left(|z|\to\infty,\ \frac{\beta\pi}{2}<|\arg(z)|\leq\pi\right)$$

Let $\Omega=\{(S,V,E,I,G,H,R)\in R_{7}^{+}: S,V,E,I,G,H,R\geq 0,\ S+V+E+I+G+H+R=\Lambda/N\}$

where Ω is a closed set.

After evaluating the equation (5), we get $N(t)=N(0)E_{\beta,1}\left(-Nt^{\beta}\right)+\Lambda t^{\beta}E_{\beta,\beta+1}\left(-Nt^{\beta}\right)$.

3. Equilibrium Points :

Let as assume that all equilibrium points are as follows:

$$\mathcal{D}_{t}^{\beta}S=0,\ \mathcal{D}_{t}^{\beta}V=0,\ \mathcal{D}_{t}^{\beta}E=0,\ \mathcal{D}_{t}^{\beta}I=0,\ \mathcal{D}_{t}^{\beta}G=0,\ \mathcal{D}_{t}^{\beta}H=0,\ \mathcal{D}_{t}^{\beta}R=0 \quad \dots (6)$$

The disease free equilibrium $E_0=(\Lambda/N,0,0,0,0,0,0)$

The basic reproduction number of the disease is R_0 which is defined as

$$R_0 = \frac{\Lambda}{(\delta_1 + \delta_2 + d)(\tau_1 + \tau_2 + \tau_3 + d)}$$

If $R_0 > 1$, then model contains a unique positive equilibrium point $E_1 = (S', V', E', I', G', H', R')$ where,

$$S' = \frac{\Lambda}{\gamma_1 + \gamma_2 + d}$$

$$V' = \frac{\Lambda\{\gamma_2(\delta_1 + \delta_2 + d)(\tau_1 + \tau_2 + \tau_3 + d) + \tau_2\gamma_1\delta_1\}}{(\gamma_1 + \gamma_2 + d)(\delta_1 + \delta_2 + d)(\tau_1 + \tau_2 + \tau_3 + d)}$$

$$E' = \frac{\gamma_1\Lambda}{(\gamma_1 + \gamma_2 + d)(\delta_1 + \delta_2 + d)}$$

$$I' = \frac{\delta_1\gamma_1\Lambda}{(\gamma_1 + \gamma_2 + d)(\delta_1 + \delta_2 + d)(\tau_1 + \tau_2 + \tau_3 + d)}$$

$$G' = \frac{\gamma_1\Lambda\{\delta_2(\tau_1 + \tau_2 + \tau_3 + d) + \tau_3\delta_1\}}{(\gamma_1 + \gamma_2 + d)(\delta_1 + \delta_2 + d)(\tau_1 + \tau_2 + \tau_3 + d)}$$

$$H' = \frac{\{\gamma_1\delta_1(\zeta_1 + \zeta_2 + d) + \zeta_1[\delta_2(\tau_1 + \tau_2 + \tau_3 + d) + \tau_3\delta_1]\}}{(\gamma_1 + \gamma_2 + d)(\delta_1 + \delta_2 + d)(\tau_1 + \tau_2 + \tau_3 + d)}$$

$$R' = \frac{\zeta_1 H' + \zeta_2 G'}{d}$$

If $R_0 < 1$, then model is stable.

The Jacobian matrix $J(E_0)$ for system (4) for diseases free equilibrium point is as follows:

$$J(E_0) = \begin{bmatrix} -(\gamma_1 + \gamma_2 + d) & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_2 & -d & 0 & \tau_2 & 0 & 0 & 0 \\ \gamma_1 & 0 & -(\delta_1 + \delta_2 + d) & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_1 & -(\tau_1 + \tau_2 + \tau_3 + d) & 0 & 0 & 0 \\ 0 & 0 & \delta_2 & \tau_3 & -(\zeta_1 + \zeta_2 + d) & 0 & 0 \\ 0 & 0 & 0 & \tau_1 & \zeta_1 & -(\xi + d) & 0 \\ 0 & 0 & 0 & 0 & \zeta_2 & \xi & -d \end{bmatrix}$$

The characteristic equation $|J(E_0) - \lambda I| = 0$

$$|J(E_0) - \lambda I| = (\gamma_1 + \gamma_2 + d + \lambda)(d + \lambda)(\delta_1 + \delta_2 + d + \lambda)(\tau_1 + \tau_2 + \tau_3 + d + \lambda)(\zeta_1 + \zeta_2 + d + \lambda)(\xi + d + \lambda)(d + \lambda) = 0$$

The characteristic root for $|J(E_0) - \lambda I| = 0$ are

$\lambda_1 = -(\gamma_1 + \gamma_2 + d), \lambda_2 = -d, \lambda_3 = -(\delta_1 + \delta_2 + d), \lambda_4 = -(\tau_1 + \tau_2 + \tau_3 + d), \lambda_5 = -(\zeta_1 + \zeta_2 + d), \lambda_6 = -(\xi + d), \lambda_7 = -d$

According to the generalized Routh-Hurwitz and Jacobian criterion when all the roots are negative then

point E_0 is disease free equilibrium point, which is stable.

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4. Numerical methods : The generalized Adams Bashforth Moulton method can be adopted to the numerical solutions of system (4). Evaluate the following nonlinear fractional differential equation to obtain the algorithm's approximate solution:

$$\left.\begin{array}{l}\mathcal{D}_t^{\beta} y(t) = f\big(t, y(t)\big),\ \ 0 \le t < b < \infty \\ y^h(0) = y_0^h,\ \ h = 0,1,2,\dots,p-1, \quad \text{where } p = [\beta]\end{array}\right\} \quad (7)$$

This equation is equivalent to Volterra integral equation:

$$y(t) = \sum_{h=0}^{p-1} y_0^h \, \frac{t^h}{h!} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \, f\big(s, y(s)\big) ds \qquad \dots (8)$$

By applying the predictor-correctors scheme to the fractional order shigella epidemic model and set $k = T/N$, (8) can be discretized as follows:

$$S_{m+1} = S_0 + \frac{k^{\beta}}{\Gamma(\beta+2)}\left(\Lambda - (\gamma_1+\gamma_2+d)S_{m+1}^q\right) + \frac{k^{\beta}}{\Gamma(\beta+2)}\sum_{i=0}^{m} b_{i,m+1}\left(\Lambda - (\gamma_1+\gamma_2+d)S_i\right)$$

$$V_{m+1} = V_0 + \frac{k^{\beta}}{\Gamma(\beta+2)}\left(\gamma_3 S_{m+1}^q + \tau_2 I_{m+1}^q - dV_{m+1}^q\right) + \frac{k^{\beta}}{\Gamma(\beta+2)}\sum_{i=0}^{m} b_{i,m+1}\left(\gamma_3 S_i + \tau_2 I_i - dV_i\right)$$

$$E_{m+1} = E_0 + \frac{k^{\beta}}{\Gamma(\beta+2)}\left(\gamma_1 S_{m+1}^q - (\delta_1+\delta_2+d)E_{m+1}^q\right) + \frac{k^{\beta}}{\Gamma(\beta+2)}\sum_{i=0}^{m} b_{i,m+1}\left(\gamma_1 S_i - (\delta_1+\delta_2+d)E_i\right)$$

$$I_{m+1} = I_0 + \frac{k^{\beta}}{\Gamma(\beta+2)}\left(\delta_1 E_{m+1}^q - (\tau_1+\tau_2+\tau_3+d)I_{m+1}^q\right) + \frac{k^{\beta}}{\Gamma(\beta+2)}\sum_{i=0}^{m} b_{i,m+1}\left(\delta_1 E_i - (\tau_1+\tau_2+\tau_3+d)I\right.$$

$$G_{m+1} = G_0 + \frac{k^{\beta}}{\Gamma(\beta+2)}\left(\delta_2 E_{m+1}^q + \tau_3 I_{m+1}^q - (\zeta_1+\zeta_2+d)G_{m+1}^q\right) + \frac{k^{\beta}}{\Gamma(\beta+2)}\sum_{i=0}^{m} b_{i,m+1}\left(\delta_2 E_i + \tau_3 I_i - (\zeta_1 +\right.$$

$$H_{m+1} = H_0 + \frac{k^{\beta}}{\Gamma(\beta+2)}\left(\tau_1 I_{m+1}^q + \zeta_1 G_{m+1}^q - (\xi+d)H_{m+1}^q\right) + \frac{k^{\beta}}{\Gamma(\beta+2)}\sum_{i=0}^{m} b_{i,m+1}\left(\tau_1 I_i + \zeta_1 G_i - (\xi+d)H_i\right)$$

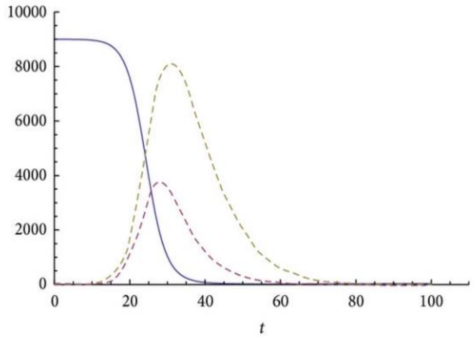
$$R_{m+1} = R_0 + \frac{k^{\beta}}{\Gamma(\beta+2)}\left(\xi H_{m+1}^q + \zeta_2 G_{m+1}^q - dR_{m+1}^q\right) + \frac{k^{\beta}}{\Gamma(\beta+2)}\sum_{i=0}^{m} b_{i,m+1}\left(\xi H_i + \zeta_2 G_i - dR_i\right)$$

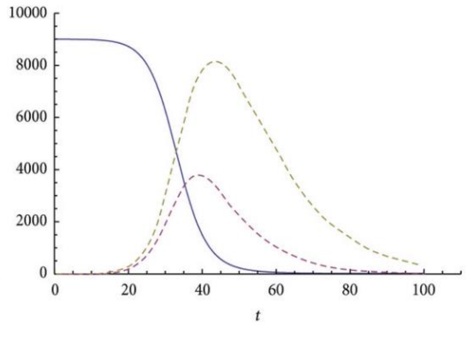
where,

$$\left.\begin{aligned}
S_{m+1}^{q} &= S_0 + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{m} c_{i,m+1}\left(\Lambda - (\gamma_1+\gamma_2+d)S_i\right)\\
V_{m+1}^{q} &= V_0 + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{m} c_{i,m+1}\left(\gamma_3 S_i + \tau_2 I_i - dV_i\right)\\
E_{m+1}^{q} &= E_0 + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{m} c_{i,m+1}\left(\gamma_1 S_i - (\delta_1+\delta_2+d)E_i\right)\\
I_{m+1}^{q} &= I_0 + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{m} c_{i,m+1}\left(\delta_1 E_i - (\tau_1+\tau_2+\tau_3+d)I_i\right)\\
G_{m+1}^{q} &= G_0 + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{m} c_{i,m+1}\left(\delta_2 E_i + \tau_3 I_i - (\zeta_1+\zeta_2+d)G_i\right)\\
H_{m+1}^{q} &= H_0 + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{m} c_{i,m+1}\left(\tau_1 I_i + \zeta_1 G_i - (\xi+d)H_i\right)\\
R_{m+1}^{q} &= R_0 + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{m} c_{i,m+1}\left(\xi H_i + \zeta_2 G_i - dR_i\right)
\end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned}
b_{i,m+1} &= \begin{cases} n^{\beta} - (m-\beta)(m+1), & i = 0\\ (n-i+2)^{\beta+1} + (n-i)^{\beta+1}, & 1 \le i \le m\\ -2(n-i+2)^{\beta+1}, & i = m+1 \end{cases}\\
c_{i,m+1} &= \frac{k^{\beta}}{\beta}\left((n-i+2)^{\beta} + (n-i)^{\beta}\right), \qquad 1 \le i \le m
\end{aligned}\right\} \quad (11)$$

5. Simulation :



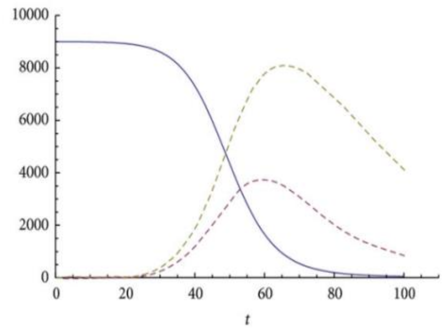


at $\boldsymbol{\beta = 1}$

Figure 1

at $\boldsymbol{\beta = 0.9}$

Figure 2



at $\boldsymbol{\beta = 0.8}$

Figure 3

In figure, blue line represents susceptible, orange dash line represents infectious and pink dash line represents hospitalization.

In all the three figures, parameters used are same except β. Figure 1 shows sharp fall in the values of S in a relatively small period of time compared to figure 2 and figure 3. While, in figure 1 both values of I and V show striking rise during the period of 30 days, there is no notable changes in the value of I and V in figure 2 and 3. Nevertheless, the conversion of a classical model into a fractional model makes it highly perceptive to the order of differentiation β *i.e*, a minute change can bring about great changes in the final result. It is evident from the numerical values from figure 2 and 3 that it depends continuously on the fractional derivative β. It is apparent that figure 2 and 3 are more realistic than figure 1 owing to the fact that it takes significant time for the rate of susceptibility or recovery to decrease or increase.

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