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Fekete-Szegö and Second Hankel Determinant for a Class of $\mathfrak{p}$-Valent Functions Related to Modified Sigmoid Functions

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Abstract: In this paper, we study the initial coefficient bounds for a noval class $M_{\lambda(*)}(\eta, \varphi_{n,m})$ of $\mathcal{P}$-valently analytic functions related to Sigmoid functions. Furthermore, the famous classical Fekete-Szegö inequality for this class are discussed.

Conclusions: In this paper, We introduced and investigated the $\mathfrak{p}$-univalent function for the class $M_{\lambda,(*)}(\eta, \varphi_{n,m})$ related to the to modified Sigmoid functions. Thus, we obtained second, third and fourth Taylor–Maclaurin coefficients of functions in this class. These results were an improvement on the estimates obtained in the recent studies.

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1. INTRODUCTION AND MOTIVATION

Let $\mathcal{A}_{\mathfrak{p}}$ denote the class of functions of the form

$$\Im(z) = z^{\mathfrak{p}} + \sum_{k=1}^{\infty} \mathfrak{a}_{k+\mathfrak{p}} z^{k+\mathfrak{p}}, \tag{1.1}$$

which are $\mathfrak{p}$-valently analytic in the open unit disk:

$$\mathbb{U} = \{z \in \mathbb{C}: 0 \leq |z| < 1\}$$

The investigation of $\mathfrak{p}$-valently analytic functions regarding many aspects like starlikeness, subordination, the introduction of new subclasses are still inspiring with interesting outcomes.

Special functions are composed of large number of highly interconnected processing elements (neurons) working together to solve a specific task. They play a vital role in univalent function theory. These functions have been overshadowed by other fields like algebra, differential equations, topology, functional analysis and real analysis, among others, because they work in the same way the brain does. An example of such functions is the activation function. Activation function increases the size of hypothesis space that a network can represent. The most popular activation function is the sigmoid function. The Sigmoid function of the form

$$\aleph(z) = \frac{1}{1+e^{-z}} \tag{1.2}$$

is differentiable and has the following properties.

- It outputs real numbers between 0 and 1.
- It maps from a very large input domain to a small range of outputs.
- never loses information because it is a one-to-one function.
- increases monotonically.

These properties enable us to use Sigmoid function in univalent function theory.

We briefly recall the following definitions needed our investigation.

Definition 1.1 *([14]) Let* $\mathfrak{I}(z) = z^{\mathfrak{p}} + \sum_{k=1}^{\infty} \mathfrak{a}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}}$, *and* $\mathfrak{g}(z) = z^{\mathfrak{p}} + \sum_{k=1}^{\infty} \mathfrak{b}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}}$. *The modified Hadamard product of two functions* $\mathfrak{I}$ *and* $\mathfrak{g}$ *which belong to* $\mathcal{A}_{\mathfrak{p}}$ *is defined by*

$$\mathfrak{I}(z) = (\mathfrak{I} * \mathfrak{g})(z) = z^{\mathfrak{p}} + \sum_{k=1}^{\infty} \mathfrak{a}_{k+\mathfrak{p}}\mathfrak{b}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}} \tag{1.3}$$

Definition 1.2 *([15]) Let* $\mathfrak{I} \in A$. *Then the* $\mathfrak{q}^{th}$ *Hankel determinant of* $\mathfrak{I}$ *is defined for* $\mathfrak{q} \geq 1$ *and* $n \geq 1$ *by*

$$H_{\mathfrak{q}}(n) = \begin{vmatrix} \mathfrak{a}_n & \mathfrak{a}_{n+1} & \cdots & \mathfrak{a}_{n+q-1} \\ \mathfrak{a}_{n+1} & \mathfrak{a}_{n+2} & \cdots & \mathfrak{a}_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \mathfrak{a}_{n+q-1} & \mathfrak{a}_{n+q} & \cdots & \mathfrak{a}_{n+2q-2} \end{vmatrix} \tag{1.4}$$

Thus, the second Hankel determinant

$$H_2(2) = \begin{vmatrix} \mathfrak{a}_2 & \mathfrak{a}_3 \\ \mathfrak{a}_3 & \mathfrak{a}_4 \end{vmatrix} = \mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2 \tag{1.5}$$

For two analytic functions $\mathfrak{I}$ and $\mathfrak{g}$, the function $\mathfrak{I}$ is subordinate to $\mathfrak{g}$, written as follows:

$$\mathfrak{I}(z) \prec \mathfrak{g}(z)$$

if there exists an analytic function w, with $w(0) = 0$ and $|w(z)| < 1$ such that $\mathfrak{I}(z) = \mathfrak{g}(w(z))$. In particular, if the function $\mathfrak{g}$ is univalent in $\mathbb{U}$, then $\mathfrak{I}(z) \prec \mathfrak{g}(z)$ is equivalent to $\mathfrak{I}(0) = \mathfrak{g}(0)$ and $\mathfrak{I}(U) \subset \mathfrak{g}(U)$.

Definition 1.3 *([9]) Let* $\eta \in \mathbb{C}/\{0\}$ *and the class* $M_{\lambda}(\eta, \varphi_{n,m})$ *denote the subclass of* $\mathcal{A}_{\mathfrak{p}}$ *consisting of functions* $\mathfrak{I}$ *of the form (1.1), and satisfying the following subordination condition*

$$1 + \frac{1}{\eta}\left[\frac{z\mathfrak{I}'(z)}{\mathfrak{I}(z)} + \lambda\frac{z^2\mathfrak{I}''(z)}{\mathfrak{I}(z)} - 1\right] \prec \varphi_{n,m} \tag{1.6}$$

for $0 \leq \lambda \leq 1$ and $\varphi_{n,m}$ is a simple logistic Sigmoid activation function.

In this study, we solve the Fekete-Szegö problem for functions in the class $M_{\lambda(*)}(\eta, \varphi_{n,m})$ and in the special instances, as well as provide bound estimates for the coefficients and an upper bound estimate for the second Hankel determinant.

Definition 1.4 *Let $\eta \in \mathbb{C}/\{0\}$ and the class $M_{\lambda(*)}(\eta,\varphi_{n,m})$ denote the subclass of $\mathcal{A}_{\mathfrak{p}}$ consisting of functions $\mathfrak{I}$ of the form (1.1), and satisfying the following subordination condition*

$$1+\frac{1}{\eta}\left[\frac{z(\mathfrak{I}*\mathfrak{g})'(z)}{(\mathfrak{I}*\mathfrak{g})(z)}+\lambda\frac{z^2(\mathfrak{I}*\mathfrak{g})''(z)}{(\mathfrak{I}*\mathfrak{g})(z)}-1\right]\prec\varphi_{n,m}=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^m}{n!}z^n\right)^m \quad (1.7)$$

for $0\le\lambda\le1$ and $\varphi_{n,m}$ is a simple logistic Sigmoid activation function.

2. Preliminary results

The following results are needed for our investigation

Let P be the family of all functions p analytic in $\mathbb{U}$ for which $\Re\{\alpha(z)\}>0$ and

$$p(z)=1+P_1z+P_2z^2+\cdots,\quad (for z\in\mathbb{U})$$

Lemma 2.1 *([8]) If $p\in P$, then $|P_k|\le 2$ $(2,3,4,\cdots)$*

Lemma 2.2 *([6]) Let g be a Sigmoid function defined in (1.2) and*

$$\varphi(z)=2g(z)=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^m}{n!}z^n\right)^m \quad (2.1)$$

then $\varphi(z)\in P$, $|z|<1$ where $\varphi(z)$ is a modified Sigmoid function.

Lemma 2.3 *([6]) Let g be a Sigmoid function defined in (1.1) and*

$$\varphi_{n,m}(z)=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^m}{n!}z^n\right)^m \quad (2.2)$$

then $|\varphi_{n,m}(z)|<2$.

Lemma 2.4 *([6]) Let $\varphi(z)\in P$ and be starlike, then $\mathfrak{I}$ is a normalized univalent function of the form (1.1). Setting $m=1$, Fadipe et al. [6] remarked that*

$$\varphi(z)=1+\sum_{n=1}^{\infty}c_nz^n \quad (2.3)$$

where $c_n=\frac{(-1)^{n+1}}{2n!}$, then $|c_n|\le2$ for $n=2,3,4,\cdots$ and the result is sharp for each n.

3. Some coefficient estimates for the class of $M_{\lambda,(*)}(\eta,\varphi_{n,m})$

In this section, we will find the estimates on the coefficients $\mathfrak{a}_{\mathfrak{p}+1}\mathfrak{b}_{\mathfrak{p}+1}$, $\mathfrak{a}_{\mathfrak{p}+2}\mathfrak{b}_{\mathfrak{p}+2}$ and $\mathfrak{a}_{\mathfrak{p}+3}\mathfrak{b}_{\mathfrak{p}+3}$ for functions in the class $M_{\lambda,(*)}(\eta,\varphi_{n,m})$.

Theorem 3.1 *Let*

$$\varphi_{n,m}(z)=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^m}{n!}z^n\right)^m$$

where $\varphi_{n,m}(z)\in A$ is a modified logistic Sigmoid activation function and $\varphi'_{n,m}(0)>0$. If $F(z)=(\mathfrak{I}*\mathfrak{g})(z)$ given by (1.1) belongs to the class $M_{\lambda,(*)}(\eta,\varphi_{n,m})$ then,

$$\mathfrak{a}_{\mathfrak{p}+1}\mathfrak{b}_{\mathfrak{p}+1}=\frac{\eta}{2\mathfrak{p}(1+\lambda(\mathfrak{p}+1))} \quad (3.1)$$

$$\mathfrak{a}_{\mathfrak{p}+2}\mathfrak{b}_{\mathfrak{p}+2} = \frac{\eta^2}{4\mathfrak{p}(\mathfrak{p}+1)(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+2))} \tag{3.2}$$

$$\mathfrak{a}_{\mathfrak{p}+3}\mathfrak{b}_{\mathfrak{p}+3} = \frac{\eta\left(3\eta^2-\mathfrak{p}(\mathfrak{p}+1)(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+2))\right)}{24\mathfrak{p}(\mathfrak{p}+1)(\mathfrak{p}+2)(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+2))(1+\lambda(\mathfrak{p}+3))} \tag{3.3}$$

Proof. Let $\mathfrak{I}(z) = z^{\mathfrak{p}} + \sum_{k=1}^{\infty} \mathfrak{a}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}}$, and $\mathfrak{g}(z) = z^{\mathfrak{p}} + \sum_{k=1}^{\infty} \mathfrak{b}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}}$. Then we can write the following equalities:

$$\mathfrak{I}(z) = (\mathfrak{I} * \mathfrak{g})(z) = z^{\mathfrak{p}} + \sum_{k=1}^{\infty} \mathfrak{a}_{k+\mathfrak{p}}\mathfrak{b}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}} \Rightarrow (\mathfrak{I} * \mathfrak{g})'(z) = \mathfrak{p}z^{p-1} + \sum_{k=1}^{\infty} (k + \mathfrak{p})\mathfrak{a}_{k+\mathfrak{p}}\mathfrak{b}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}-1}$$

$$\Rightarrow (\mathfrak{I} * \mathfrak{g})''(z) = \mathfrak{p}(\mathfrak{p} - 1)z^{p-2} + \sum_{k=1}^{\infty} (k + \mathfrak{p})(k + \mathfrak{p} - 1)\mathfrak{a}_{k+\mathfrak{p}}\mathfrak{b}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}-2}$$

Thus, we obtain

$$z(\mathfrak{I} * \mathfrak{g})'(z) + \lambda z^2(\mathfrak{I} * \mathfrak{g})''(z) = \mathfrak{p}(1 - \lambda + \lambda\mathfrak{p})z^{\mathfrak{p}} + \sum_{k=1}^{\infty} (k + \mathfrak{p})(1 + (k + \mathfrak{p} - 1)\lambda)\mathfrak{a}_{k+\mathfrak{p}}\mathfrak{b}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}}$$

and

$$z(\mathfrak{I} * \mathfrak{g})'(z) + \lambda z^2(\mathfrak{I} * \mathfrak{g})''(z) - (\mathfrak{I} * \mathfrak{g})(z) = (\mathfrak{p} - 1)(1 + \lambda\mathfrak{p})z^{\mathfrak{p}} + \sum_{k=1}^{\infty} (k + \mathfrak{p} - 1)(1 + (k + \mathfrak{p})\lambda)\mathfrak{a}_{k+\mathfrak{p}}\mathfrak{b}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}}$$

If $F \in M_{\lambda(*)}(\eta, \varphi_{n,m})$, then we have

$$\frac{1}{\eta}\left[\frac{z(\mathfrak{I}*\mathfrak{g})'(z)+\lambda z^2(\mathfrak{I}*\mathfrak{g})''(z)-(\mathfrak{I}*\mathfrak{g})(z)}{(\mathfrak{I}*\mathfrak{g})(z)}\right] = \varphi_{n,m} - 1 \tag{3.4}$$

where $\varphi_{n,m}$ is a modified Sigmoid function given by

$$\varphi_{n,m} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + \cdots \tag{3.5}$$

In view of (3.4) and (3.5), expanding in series forms we have

$$\frac{1}{\eta}\left[(\mathfrak{p} - 1)(1 + \lambda\mathfrak{p})z^{\mathfrak{p}} + \sum_{k=1}^{\infty} (k + \mathfrak{p} - 1)(1 + (k + \mathfrak{p})\lambda)\mathfrak{a}_{k+\mathfrak{p}}\mathfrak{b}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}}\right] = \left[z^{\mathfrak{p}} + \sum_{k=1}^{\infty} \mathfrak{a}_{k+\mathfrak{p}}\mathfrak{b}_{k+\mathfrak{p}}\, z^{k+\mathfrak{p}}\right]\left[\frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + \cdots\right] \tag{3.6}$$

Comparing the coefficients of $z^{\mathfrak{p}+1}$, $z^{\mathfrak{p}+2}$ and $z^{\mathfrak{p}+3}$ in(3.6), we obtain

$$\mathfrak{a}_{\mathfrak{p}+1}\mathfrak{b}_{\mathfrak{p}+1} = \frac{\eta}{2\mathfrak{p}(1+\lambda(\mathfrak{p}+1))} \tag{3.7}$$

$$\mathfrak{a}_{\mathfrak{p}+2}\mathfrak{b}_{\mathfrak{p}+2} = \frac{\eta^2}{4\mathfrak{p}(\mathfrak{p}+1)(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+2))} \tag{3.8}$$

$$\mathfrak{a}_{\mathfrak{p}+3}\mathfrak{b}_{\mathfrak{p}+3} = \frac{\eta\left(3\eta^2-p(\mathfrak{p}+1)(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+2))\right)}{24p(\mathfrak{p}+1)(\mathfrak{p}+2)(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+2))(1+\lambda(\mathfrak{p}+3))} \tag{3.9}$$

Corollary 3.2 *For coefficient* $\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}$,

$$\left|\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}\right| = \frac{|\eta|}{2p(1+\lambda(p+1))}$$

is written and since $\varphi(\lambda) = \frac{1}{(1+\lambda(p+1))}$, $\varphi'(\lambda) < 0$ in the interval $0 \leq \lambda \leq 1$ and $\varphi(\lambda)$ is decreasing, it will be

$$\frac{|\eta|}{2p(p+2)} \leq |\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}| \leq \frac{|\eta|}{2p} \tag{3.10}$$

for $\frac{1}{2} \leq \frac{1}{(1+\lambda(p+1))} \leq 1$.

Similarly, since the coefficients $\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}$, $\mathfrak{a}_{p+2}\mathfrak{b}_{p+2}$ and $\mathfrak{a}_{p+3}\mathfrak{b}_{p+3}$ depend on λ and are decreasing with respect to λ, the following inequalities can be written easily:

$$\frac{|\eta^2|}{4(p+1)(p+2)(p+3)} \leq |\mathfrak{a}_{p+2}\mathfrak{b}_{p+2}| \leq \frac{|\eta|^2}{4p(p+1)} \tag{3.11}$$

$$\frac{|(3\eta^3 - p(p+1)(p+2)(p+3)\eta)|}{24(p+1)(p+2)^2(p+3)(p+4)} \leq |\mathfrak{a}_{p+3}\mathfrak{b}_{p+3}| \leq \frac{|(\eta^3 - p(p+1)\eta)|}{24p(p+1)(p+2)} \tag{3.12}$$

4. Some results connected with the Fekete-Szegö inequality and Hankel coefficient for the class of $M_{\lambda,(*)}(\eta, \varphi_{n,m})$

The Fekete-Szegö problem may be considered one of the most important results about univalent functions, which is related to coefficients an of a function's Taylor series and was introduced by Fekete-Szegö [1]. The problem of maximizing the absolute value of functional $a_3 - \mu a_2^2$ is called the Fekete-Szegö problem. This result is sharp and is studied thoroughly by many researchers. The equality holds true for the Koebe function. In 1969, Keogh and Merkes [2] obtained the sharp upper bound of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for some subclasses of univalent function.

Recently, Murugusundarmoorthy and Janani [3], Olantunji et al. [5], Olantunji [4], and Orhan and Çağlar [7]have studied Sigmoid function for various classes of analytic and univalent functions.

In this section, we first prove the following Fekete-Szegö result for the function in the classes $M_{\lambda,(*)}(\eta, \varphi_{n,m})$ with the values of $\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}$ and $\mathfrak{a}_{p+2}\mathfrak{b}_{p+2}$.

Theorem 4.1 *If* $F(z) \in \mathcal{A}_{\mathcal{P}}$ *given by (1.1) belongs to the class* $M_{\lambda,(*)}(\eta, \varphi_{n,m})$ *then,*

$$|\mathfrak{a}_{p+2}\mathfrak{b}_{p+2} - \mu\left(\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}\right)^2| = \frac{|\eta|^2}{4p(p+1)}\left(1 + |\mu|\,\frac{(p+1)}{p}\right) \tag{4.1}$$

Proof. If the values of $\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}$ and $\mathfrak{a}_{p+2}\mathfrak{b}_{p+2}$ determined by (3.7) and (3.8) are written instead of $\mathfrak{a}_{p+2}\mathfrak{b}_{p+2} - \mu\left(\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}\right)^2$, we get

$$\mathfrak{a}_{p+2}\mathfrak{b}_{p+2} - \mu\left(\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}\right)^2 = \frac{\eta^2}{4p(p+1)(1+\lambda(p+1))(1+\lambda(p+2))} - \mu\left(\frac{\eta}{2p(1+\lambda(p+1))}\right)^2$$

$$= \frac{\eta^2}{4p(p+1)(1+\lambda(p+1))(1+\lambda(p+2))} - \mu\,\frac{(\eta)^2}{4p^2(1+\lambda(p+1))^2}.$$

Taking absolute value on both sides of the above equation and applying triangle inequality, we get

$$|\mathfrak{a}_{p+2}\mathfrak{b}_{p+2}-\mu\left(\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}\right)^2| \leq \frac{|\eta|^2}{4p(p+1)(1+\lambda(p+1))(1+\lambda(p+2))}+|\mu|\frac{|(\eta)|^2}{4p^2(1+\lambda(p+1))^2}.$$

Here $\zeta_1=\frac{1}{(1+\lambda(p+1))(1+\lambda(p+2))}$ and $\zeta_2=\frac{1}{(1+\lambda(p+1))^2}$ are taken and these functions depending on λ are considered to be decreasing in the interval $0\leq\lambda\leq 1$, since

$$\max_{0\leq\lambda\leq 1}\frac{1}{(1+\lambda(p+1))(1+\lambda(p+2))} = 1$$

and

$$\max_{0\leq\lambda\leq 1}\frac{1}{(1+\lambda(p+1))^2} = 1$$

we get

$$|\mathfrak{a}_{p+2}\mathfrak{b}_{p+2}-\mu\left(\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}\right)^2| \leq \frac{|\eta|^2}{4p(p+1)}+|\mu|\frac{|\eta|^2}{4p^2}.$$

thus we obtain

$$|\mathfrak{a}_{p+2}\mathfrak{b}_{p+2}-\mu\left(\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}\right)^2| \leq \frac{|\eta|^2}{4p(p+1)}\left(1+|\mu|\frac{(p+1)}{p}\right)$$

Hence, we have reached the desired assertion of the Theorem(4.1),

$$\left|\mathfrak{a}_{p+2}\mathfrak{b}_{p+2}-\mu\left(\mathfrak{a}_{p+1}\mathfrak{b}_{p+1}\right)^2\right| \leq \begin{cases}\frac{|\eta|^2}{4p(p+1)}\left(1+|\mu|\frac{(p+1)}{p}\right), & \mu\geq 0\\ \frac{|\eta|^2}{4p(p+1)}\left(1-|\mu|\frac{(p+1)}{p}\right), & \mu\leq 0\end{cases}$$

This completes the proof of the Theorem.

In the theory of singularities [10] and the investigation of power series with integral coefficients, the Hankel determinant is very important. The reader is encouraged to read [15] for more information. For several subfamilies of univalent functions, the growth of $H_q(n)$ has been explored. We know that the function $H_2(1)=a_3-a_2^2$ for $q=2$ and $n=1$ is a well recognized Fekete-Szegö functional. For the bi-convex and bi-starlike classes, the second Hankel determinant $H_2(2)$ is given by $H_2(2)=a_2a_4-a_3^2$ [12].

The following theorem will give some results related to Hankel determinant for the functions belonging to classes $M_{\lambda,(*)}(\eta,\varphi_{n,m})$.

Theorem 4.2 *If $F(z)\in\mathcal{A}_{\mathcal{P}}$ given by (1.1) belongs to the class $M_{\lambda,(*)}(\eta,\varphi_{n,m})$ then,*

$$|(\mathfrak{a}_{p+1}\mathfrak{b}_{p+1})(\mathfrak{a}_{p+3}\mathfrak{b}_{p+3})-\left(\mathfrak{a}_{p+2}\mathfrak{b}_{p+2}\right)^2| \leq \frac{|\eta|^2}{48p^2(p+1)^2(p+2)}((p+1)|3\eta^2-p(p+1)|+3(p+2)|\eta|^2) \quad (4.2)$$

Proof. From (3.7), (3.8)and (3.9) , we get

$$H_2(2) = \begin{vmatrix} \mathfrak{a}_2 & \mathfrak{a}_3 \\ \mathfrak{a}_3 & \mathfrak{a}_4 \end{vmatrix} = \mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2 \tag{4.3}$$

$$(\mathfrak{a}_{\mathfrak{p}+1}\mathfrak{b}_{\mathfrak{p}+1})(\mathfrak{a}_{\mathfrak{p}+3}\mathfrak{b}_{\mathfrak{p}+3}) - \left(\mathfrak{a}_{\mathfrak{p}+2}\mathfrak{b}_{\mathfrak{p}+2}\right)^2 = \left(\frac{\eta}{2\mathfrak{p}(1+\lambda(\mathfrak{p}+1))}\right)\left(\frac{\eta\left(3\eta^2-\mathfrak{p}(\mathfrak{p}+1)(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+2))\right)}{24\mathfrak{p}(\mathfrak{p}+1)(\mathfrak{p}+2)(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+2))(1+\lambda(\mathfrak{p}+3))}\right) - \left(\frac{\eta^2}{4\mathfrak{p}(\mathfrak{p}+1)(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+2))}\right)^2 \tag{4.4}$$

$$= \frac{(3\eta^4-\mathfrak{p}(\mathfrak{p}+1)(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+2))\eta^2)}{48\mathfrak{p}^2(\mathfrak{p}+1)(\mathfrak{p}+2)(1+\lambda(\mathfrak{p}+1))^2(1+\lambda(\mathfrak{p}+2))(1+\lambda(\mathfrak{p}+3))} - \frac{\eta^4}{16\mathfrak{p}^2(\mathfrak{p}+1)^2(1+\lambda(\mathfrak{p}+1))^2(1+\lambda(\mathfrak{p}+2))^2} \tag{4.5}$$

and thus

$$\left|(\mathfrak{a}_{\mathfrak{p}+1}\mathfrak{b}_{\mathfrak{p}+1})(\mathfrak{a}_{\mathfrak{p}+3}\mathfrak{b}_{\mathfrak{p}+3}) - \left(\mathfrak{a}_{\mathfrak{p}+2}\mathfrak{b}_{\mathfrak{p}+2}\right)^2\right| \le \frac{|(3\eta^4-p(\mathfrak{p}+1)(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+2))\eta^2)|}{48\mathfrak{p}^2(\mathfrak{p}+1)(\mathfrak{p}+2)(1+\lambda(\mathfrak{p}+1))^2(1+\lambda(\mathfrak{p}+2))(1+\lambda(\mathfrak{p}+3))} + \frac{|\eta|^4}{16\mathfrak{p}^2(\mathfrak{p}+1)^2(1+\lambda(\mathfrak{p}+1))^2(1+\lambda(\mathfrak{p}+2))^2} \tag{4.6}$$

Here $\zeta_3 = \frac{1}{(1+\lambda(\mathfrak{p}+1))^2(1+\lambda(\mathfrak{p}+2))(1+\lambda(\mathfrak{p}+3))}$, $\zeta_4 = \frac{1}{(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+3))}$ and $\zeta_5 = \frac{1}{(1+\lambda(\mathfrak{p}+1))^2(1+\lambda(\mathfrak{p}+2))^2}$ are taken and these functions depending on λ are considered to be decreasing in the interval $0 \le \lambda \le 1$, since

$$\max_{0\le\lambda\le1} \frac{1}{(1+\lambda(\mathfrak{p}+1))^2(1+\lambda(\mathfrak{p}+2))(1+\lambda(\mathfrak{p}+3))} = 1,$$

$$\max_{0\le\lambda\le1} \frac{1}{(1+\lambda(\mathfrak{p}+1))(1+\lambda(\mathfrak{p}+3))} = 1$$

and
$$\max_{0\le\lambda\le1} \frac{1}{(1+\lambda(\mathfrak{p}+1))^2(1+\lambda(\mathfrak{p}+2))^2} = 1$$

thus we obtain

$$|(\mathfrak{a}_{\mathfrak{p}+1}\mathfrak{b}_{\mathfrak{p}+1})(\mathfrak{a}_{\mathfrak{p}+3}\mathfrak{b}_{\mathfrak{p}+3}) - \left(\mathfrak{a}_{\mathfrak{p}+2}\mathfrak{b}_{\mathfrak{p}+2}\right)^2| \le \frac{|\eta|^2}{48\mathfrak{p}^2(\mathfrak{p}+1)(\mathfrak{p}+2)}((\mathfrak{p}+1)|3\eta^2 - \mathfrak{p}(\mathfrak{p}+1)| + 3(\mathfrak{p}+2)|\eta|^2) \tag{4.7}$$

This completes the proof of the Theorem.

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