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Recurrence Relations for Moments of Generalized Order Statistics from Area-biased Rayleigh Distribution and its Characterization

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Abstract:

In this paper, the derivation of recurrence relations for single and product moments from generalized order statistics using the Area-biased Rayleigh distribution is presented. This includes specific cases for order statistics and records. Additionally, the Area-biased Rayleigh distribution is characterized through a recurrence relation for single moments.

Keywords: Generalized order statistics -Single and product moments - Recurrence relations - Area biased Rayleigh distribution - Characterization.

1. Introduction

Recurrence relations are mathematical formulas that link each term in a series to the preceding terms, thereby simplifying calculations and revealing data patterns. These relations find applications in moments of order statistics and generalized order statistics, such as parameter estimation, hypothesis testing, and deriving conclusions about the underlying distribution. Many researchers have employed the generalized order statistics (GOS) in their research, such as Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000), Pawlas and Szynal (2001), Ahmed (2007), Ahmed and Fawzy (2003), Khan et al. (2007), AL-Hussaini et al. (2005), Kumar (2011), Mahmoud and Ghazal (2012). Recently, Abdul-Moniem [(2014, 2019, 2022)] has contributed significantly to this field, presenting various studies on recurrence relations for moments of generalized order statistics from different distributions, extending well-known life distribution families, and exploring new characterizations of these extended distributions. These studies have resulted in deriving new recurrence relations for moments, particularly in the context of weighted distributions such as the length-biased Maxwell distribution and extended distributions like those based on the Marshall–Olkin family. Additionally, these works have enhanced the understanding of life distribution properties and provided novel characterizations that improve the applicability of these statistical models in various practical scenarios. Mohsin et al. (2010) further contributed by deriving a recurrence relation for single moments of generalized order statistics from the Rayleigh distribution, enhancing understanding of distribution properties and offering new characterizations for practical applications.

A random variable (RV) Z is considered to have Area-biased Rayleigh distribution (ABRD) if its probability density function (PDF) has the following form:

$$f(z) = \frac{2z^3}{\beta^4}\mathbf{exp}\left(-\frac{z^2}{\beta^2}\right);\ \beta > 0\ , z \geq 0 \tag{1}$$

The survival function (SF) corresponding to equation (1) is expressed as:

$$\bar{F}(z) = \left(1+\frac{z^2}{\beta^2}\right)\mathbf{exp}\left(-\frac{z^2}{\beta^2}\right);\ \beta > 0\ , z \geq 0 \tag{2}$$

Substituting from (2) in (1), we get:

$$\bar{F}(z) = \left(\frac{\beta^2}{2z}+\frac{\beta^4}{2z^3}\right)f(z) \tag{3}$$

Bashir and Rasul, (2018) provided further details on this distribution and its applications. Kamps (1995) developed the concept of generalized order statistics (GOS), which encompasses various order models of random variables. for simplicity, Let, F throughout represent a continuous distribution function with a density function f. The random variables $Z(1,n,\tilde{m},k),\dots,Z(n,n,\tilde{m},k)$ are referred to as generalized order statistics based on F, if their joint probability density function takes the following form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[\bar{F}(z_i)]^{m_i}f(z_i)\right)[\bar{F}(z_n)]^{k-1}f(z_n)\ ,$$

$for\ \ \bar{F}^{-1}(1) > z_1 \geq z_2 \geq \dots \geq z_n > \bar{F}^{-1}(0)$, with parameters $n \in N, n \geq 2, k > 0\ , \tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in R^{n-1}, M_r = \sum_{i=r}^{n=1} m_i$, such that $\gamma_r = k+n-r+M_r > 0$, for all $r \in \{1,2,\dots,n-1\}$.For $\gamma_i \neq \gamma_j\ , i \neq j\ for\ all\ i,j\ \in (1,2,\dots,n-1)$ the probability density function (PDF) of $Z(r,n,\tilde{m},k)$ is expressed (by Cramer and Kamps.(2000)) as follows

$$f_{Z(r,n,\tilde{m},k)}(z) = C_{r-1}f(z)\sum_{i=1}^{r}a_i(r)\,[\bar{F}(z)]^{\gamma_i-1} \tag{4}$$

The joint PDF for $Z(r,n,\tilde{m},k)$ and $Z(s,n,\tilde{m},k), 1 \leq r < s \leq n$ is given as

$$f_{Z(r,n,\tilde{m},k),Z(s,n,\tilde{m},k)}(z,t) = C_{s-1}\left(\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\bar{F}(t)}{\bar{F}(z)}\right]^{\gamma_i}\right)\left(\sum_{i=1}^{r}a_i(r)\ [\bar{F}(z)]^{\gamma_i}\right)\frac{f(z)f(t)}{\bar{F}(z)\bar{F}(t)}\ , \tag{5}$$

where $z < t$
and

$$a_i(r) = \prod_{\substack{j=1 \\ j\neq i}}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \le i \le r \le n\,,$$

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j\neq i}}^{s} \frac{1}{\gamma_j - \gamma_i}, \qquad r+1 \le i \le s \le n\,.$$

It should be noted that when $m_1 = m_2 = \ldots = m_{n-1} = m \neq -1$,

$$a_i(r) = \frac{(-1)^{r-i}}{(m+1)^{r-1}(r-1)!}\binom{r-1}{r-i}, \tag{6}$$

and

$$a_i^{(r)}(s) = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!}\binom{s-r-1}{s-i} \tag{7}$$

Therefore, the PDF of $Z(r,n,\widetilde{m},k)$ given in (4) reduces to

$$f_{Z(r,n,m,k)}(z) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(z)]^{\gamma_r-1} f(z) g_m^{r-1}\,[F(z)] \tag{8}$$

and joint PDF of $Z(r,n,\widetilde{m},k)$ and $Z(s,n,\widetilde{m},k)$ given in (5) reduces to

$$f_{Z(r,n,m,k),Z(s,n,m,k)}(z,t) = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}[\bar{F}(z)]^m f(z) g_m^{r-1}[F(z)]$$
$$\times \{h_m[F(t)] - h_m[F(z)]\}^{s-r-1}[\bar{F}(t)]^{\gamma_s-1} f(t), \qquad z < t \tag{9}$$

Whereas

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i\ ,\gamma_i = k + (n-i)(m+1),$$

$$h_m(z) = \begin{cases} \dfrac{-1}{m+1}(1-z)^{m+1}, & m \neq -1 \\ -\ln(1-z) & , \ m = -1 \end{cases}$$

and $$g_m(z) = h_m(z) - h_m(0)\,,\ \ z \in [0,1).$$

We also set $Z(0,n,m,k) = 0$. If $m = 0, k = 1$ then $Z(r,n,m,k)$ simplifies to the $(n-r+1)^{th}$ order statistics, $Z_{n-r+1:n}$ from the sample $Z_1, Z_2, \ldots, Z_n$. When $m = -1$ then $Z(r,n,m,k)$ simplifies to the k^{th} record values as noted (by Pawlas. and Szynal. (2001)).

In this research, we provide explicit expressions and some recurrence relations for single and product moments of GOS from ABRD. Its many deductions and special cases are also studied. A recurrence relation for single moments was utilized to characterize ABRD.

2. Recurrence relation for single moments of GOS

The single moments of GOS for ABRD are as the follows

$$E\left[Z^j(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!}\int_0^\infty Z^j\,[\bar{F}(z)]^{\gamma_r-1} f(z) g_m^{r-1}\,[F(z)]\,dz$$

$$= \frac{C_{r-1}}{(m+1)^{r-1}(r-1)!}\int_0^\infty Z^j\,[\bar{F}(z)]^{\gamma_r-1} f(z)\,\left[1-\left(\bar{F}(z)\right)^{m+1}\right]^{r-1}\,dz$$

$$= \frac{C_{r-1}\,\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^w}{(m+1)^{r-1}(r-1)!}\int_0^\infty Z^j\,[\bar{F}(z)]^{\gamma_r+w(m+1)-1} f(z)\,dz$$

$$= \frac{j\,C_{r-1}}{(r-1)!\,(m+1)^{r-1}}\sum_{w=0}^{r-1}\frac{\binom{r-1}{w}(-1)^w}{[\gamma_r+w(m+1)]}\int_0^\infty Z^{j-1}\,[\bar{F}(z)]^{\gamma_r+w(m+1)}\,dz$$

using equation (2), we get

$$E\left[Z^j(r,n,m,k)\right] = \frac{j\,C_{r-1}}{(r-1)!\,(m+1)^{r-1}}\sum_{w=0}^{r-1}\frac{\binom{r-1}{w}(-1)^w}{[\gamma_r+w(m+1)]}\int_0^\infty Z^{j-1}\left[\left(1+\frac{z^2}{\beta^2}\right)\right]^{\gamma_r+w(m+1)} \boldsymbol{exp}\left(-\left(\gamma_r+w(m+1)\right)\frac{z^2}{\beta^2}\right)\,dz$$

By applying the binomial expansion, we get

$$E\left[Z^j(r,n,m,k)\right] = \frac{j\,C_{r-1}}{(r-1)!\,(m+1)^{r-1}}\sum_{w=0}^{r-1}\sum_{v=0}^{\gamma_r+w(m+1)}\frac{\binom{r-1}{w}\binom{\gamma_r+w(m+1)}{v}(-1)^w}{\beta^{2v}[\gamma_r+w(m+1)]}\int_0^\infty (z^2)^{v+\frac{j-1}{2}}\,\boldsymbol{exp}\left(-\left(\gamma_r+w(m+1)\right)\frac{z^2}{\beta^2}\right)\,dz$$

By substitution, $y=\frac{\gamma_r+w(m+1)}{\beta^2}z^2$ in the integration process, we obtain

$$E\left[Z^j(r,n,m,k)\right] = \frac{j\beta^j\,C_{r-1}}{2(r-1)!\,(m+1)^{r-1}}\sum_{w=0}^{r-1}\sum_{v=0}^{\gamma_r+w(m+1)}\frac{\binom{r-1}{w}\binom{\gamma_r+w(m+1)}{v}(-1)^w}{[\gamma_r+w(m+1)]^{v+\frac{j}{2}+1}}\int_0^\infty (y)^{v+\frac{j}{2}-1}\,\boldsymbol{exp}(-y)\,dy$$

Therefore

$$E\left[Z^j(r,n,m,k)\right] = \frac{j\beta^j\,C_{r-1}}{2(r-1)!\,(m+1)^{r-1}}\sum_{w=0}^{r-1}\sum_{v=0}^{\gamma_r+w(m+1)}\frac{\binom{r-1}{w}\binom{\gamma_r+w(m+1)}{v}(-1)^w}{[\gamma_r+w(m+1)]^{v+\frac{j}{2}+1}}\Gamma\left(v+\frac{j}{2}\right) \qquad (10)$$

Remark 2.1 Setting $m=0, k=1$ in equation (10), we derive the single moments of order statistics for ABRD as following form

$$E\left[Z_{r:n}^{j}\right]=\frac{jn!\ \beta^{j}}{2(r-1)!\,(n-r)!}\sum_{w=0}^{r-1}\sum_{v=0}^{n-r+w+1}\frac{\binom{r-1}{w}\binom{n-r+w+1}{v}(-1)^{w}}{[n-r+w+1]^{v+\frac{j}{2}+1}}\Gamma\left(v+\frac{j}{2}\right)$$

Then

$$E\left[Z_{r:n}^{j}\right]=\frac{jn!\,\beta^{j}}{2(n-r)!}\sum_{w=0}^{r-1}\sum_{v=0}^{n-r+w+1}\frac{(n-r+w+1)^{-v-\frac{j}{2}}.(n-r+w)!(-1)^{w}}{w!(r-1-w)!(n-r+w+1-v)!v!}\Gamma\left(v+\frac{j}{2}\right) \qquad (11)$$

Using equation (11), some numerical results for the mean and variance of order statistics from ABRD, as presented in the following tables.

Table 1: Mean of order statistics for ABRD

n	parameter										
	R	β=0.4	β=0.6	β=0.8	β=1	β=1.3	β=1.5	β=2	β=2.5	β=3	β=3.5
1	1	0.532	0.798	1.063	1.329	1.728	1.994	2.659	3.323	3.988	4.653
2	1	0.423	0.634	0.846	1.057	1.375	1.586	2.115	2.644	3.172	3.701
	2	0.64	0.961	1.281	1.601	2.082	2.402	3.202	4.003	4.804	5.604
3	1	0.372	0.559	0.745	0.931	1.21	1.396	1.862	2.327	2.793	3.258
	2	0.524	0.786	1.048	1.311	1.704	1.966	2.621	3.276	3.932	4.587
	3	0.699	1.048	1.397	1.747	2.27	2.62	3.493	4.366	5.24	6.113
4	1	0.341	0.512	0.682	0.853	1.108	1.279	1.705	2.131	2.558	2.984
	2	0.466	0.7	0.933	1.166	1.516	1.749	2.332	2.915	3.498	4.081
	3	0.582	0.873	1.164	1.455	1.892	2.183	2.91	3.638	4.365	5.093
	4	0.737	1.106	1.475	1.844	2.397	2.765	3.687	4.609	5.531	6.453
5	1	0.319	0.478	0.638	0.797	1.037	1.196	1.595	1.994	2.392	2.791
	2	0.429	0.644	0.859	1.073	1.395	1.61	2.147	2.683	3.22	3.757
	3	0.522	0.783	1.044	1.305	1.697	1.958	2.61	3.263	3.915	4.568
	4	0.622	0.933	1.244	1.555	2.022	2.333	3.11	3.888	4.665	5.443
	5	0.766	1.149	1.533	1.916	2.491	2.874	3.832	4.789	5.747	6.705
6	1	0.302	0.453	0.604	0.756	0.982	1.133	1.511	1.889	2.267	2.644
	2	0.403	0.604	0.805	1.007	1.309	1.51	2.013	2.517	3.02	3.523
	3	0.483	0.724	0.965	1.207	1.569	1.81	2.413	3.017	3.62	4.223
	4	0.561	0.842	1.123	1.404	1.825	2.105	2.807	3.509	4.211	4.912
	5	0.652	0.979	1.305	1.631	2.12	2.446	3.262	4.077	4.893	5.708
	6	0.789	1.184	1.578	1.973	2.565	2.959	3.946	4.932	5.918	6.905

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Note that: the results presented in Table1 are consistent with the properties of order statistics described by David and Nagaraja (2004), specifically $\sum_{i=1}^{n}\mu_{i:n}=n\mu_{1:1}$

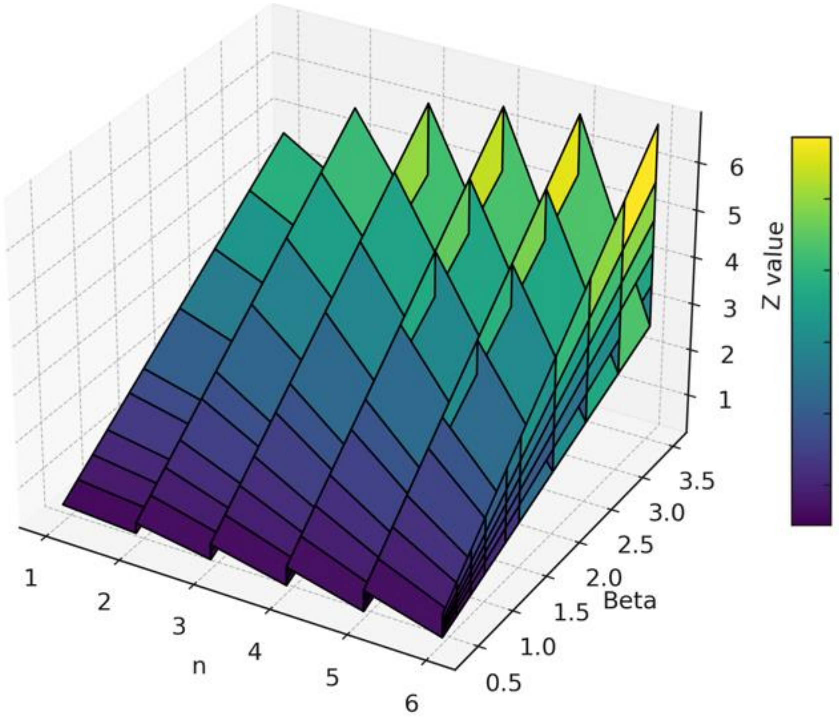
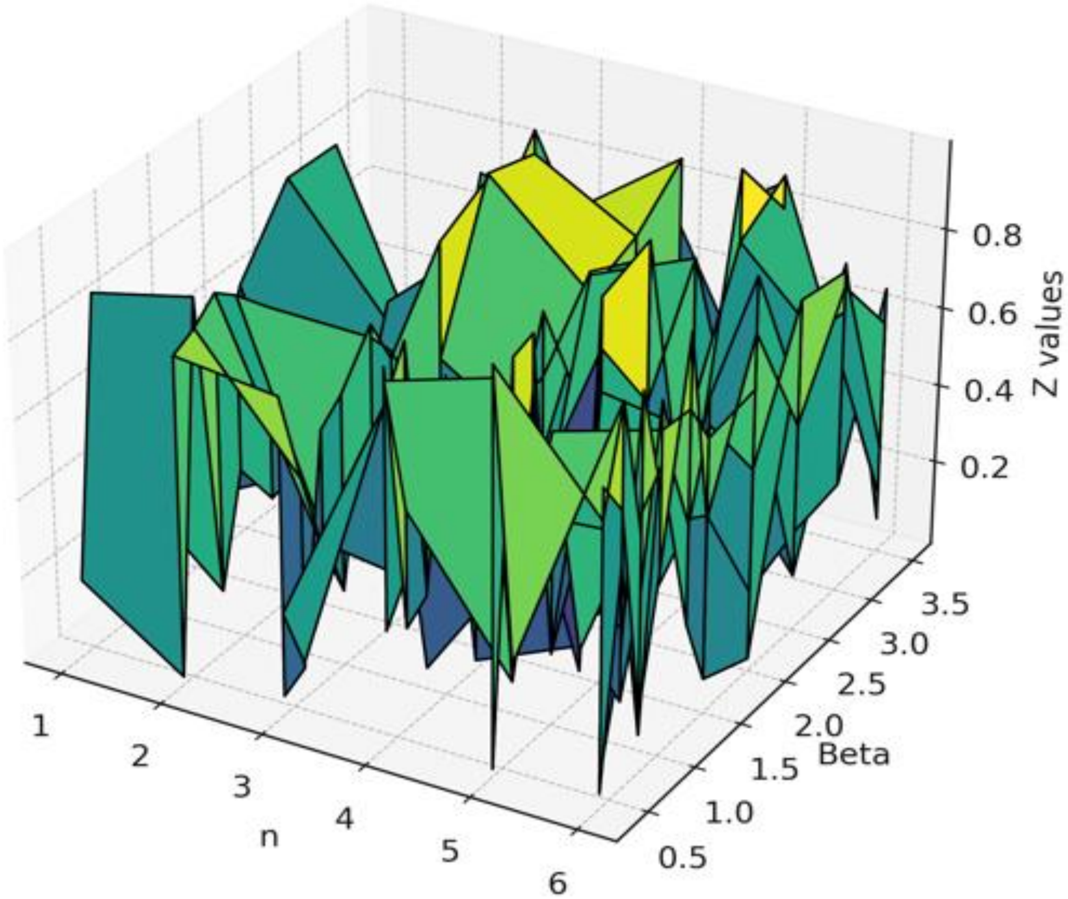


Table 2: Variance of order statistics for ABRD

n	parameter										
	R	β=0.4	β=0.6	β=0.8	β=1	β=1.3	β=1.5	β=2	β=2.5	β=3	β=3.5
1	1	0.037	0.084	0.149	0.233	0.394	0.524	0.931	1.455	2.096	2.852
2	1	0.021	0.047	0.084	0.132	0.223	0.296	0.527	0.823	1.186	1.614
	2	0.03	0.067	0.119	0.186	0.315	0.419	0.745	1.164	1.676	2.281
3	1	0.015	0.035	0.062	0.096	0.163	0.217	0.385	0.602	0.867	1.18
	2	0.017	0.038	0.068	0.106	0.18	0.24	0.426	0.666	0.958	1.305
	3	0.026	0.059	0.104	0.163	0.275	0.366	0.651	1.017	1.464	1.993
4	1	0.012	0.023	0.05	0.078	0.131	0.175	0.311	0.486	0.7	0.953
	2	0.013	0.028	0.05	0.078	0.132	0.176	0.313	0.489	0.704	0.958
	3	0.015	0.033	0.06	0.093	0.157	0.209	0.372	0.581	0.837	1.14
	4	0.024	0.053	0.095	0.148	0.25	0.333	0.592	0.926	1.333	1.815
5	1	0.011	0.024	0.042	0.066	0.112	0.149	0.265	0.414	0.596	0.811
	2	0.01	0.023	0.04	0.063	0.107	0.142	0.253	0.395	0.568	0.773
	3	0.011	0.025	0.044	0.068	0.116	0.154	0.274	0.428	0.616	0.839
	4	0.014	0.03	0.054	0.084	0.143	0.19	0.338	0.527	0.759	1.034
	5	0.022	0.05	0.088	0.138	0.233	0.311	0.552	0.863	1.242	1.691
6	1	0.0093	0.021	0.037	0.058	0.098	0.131	0.233	0.364	0.524	0.713
	2	0.0086	0.019	0.034	0.054	0.091	0.121	0.215	0.335	0.483	0.657
	3	0.0089	0.02	0.035	0.055	0.094	0.125	0.222	0.346	0.499	0.679
	4	0.0099	0.022	0.04	0.062	0.105	0.14	0.249	0.389	0.56	0.762

	5	0.013	0.028	0.05	0.078	0.132	0.176	0.313	0.489	0.704	0.958
	6	0.021	0.047	0.084	0.131	0.221	0.294	0.522	0.816	1.175	1.599



2.1 The r^th TL-moments and r^th L-moments

The r^{th} TL-moments are given in the following formula (by Elamir. and Seheult. (2003)).

$$L_r^{(s,t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E(Z_{r+s-k:r+s+t}) , \tag{12}$$

Whereas *r, s* and *t* take the values 1,2,3, ...We have noted that the r^{th} L-moments can be obtained by taking $s = t = 0$.

Using (12) and (11) with j=1, $n = r+s+t$ and $r = r+s-k$ the r^{th} TL-moments can be obtained as follows

$$L_r^{(s,t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} \sum_{w=0}^{r+s-k-1}\sum_{v=0}^{t+k+w+1} \frac{\beta.(r+s+t)!\ (t+k+w+1)^{-v-\frac{1}{2}}.(t+k+w)!(-1)^w}{2w!(r+s-k-1-w)!(t+k+w+1-v)!v!(t+k)!} \times \Gamma\left(v+\frac{1}{2}\right)$$

Then

$$L_r^{(s,t)} = \frac{1}{r}\sum_{k=0}^{r-1}\sum_{w=0}^{r+s-k-1}\sum_{v=0}^{t+k+w+1} \frac{\beta.(r+s+t)!(r-1)!\ (t+k+w+1)^{-v-\frac{1}{2}}.(t+k+w)!(-1)^{w+k}}{2w!k!(r-1-k)!(r+s-k-1-w)!(t+k+w+1-v)!v!(t+k)!} \times \Gamma\left(v+\frac{1}{2}\right) \quad (13$$

The initial four TL-moments can be obtained from (13) by setting r = 1,2,3 and 4 respectively.

The generalized TL-moments ratios, such as the coefficient of variation, coefficient of skewness and coefficient of kurtosis, are calculated from the initial four generalized TL-moments. These ratios are defined as $\tau_1^{(s,t)} = \frac{L_1^{(s,t)}}{L_2^{(s,t)}}, \tau_3^{(s,t)} = \frac{L_3^{(s,t)}}{L_2^{(s,t)}}\ and\ \tau_4^{(s,t)} = \frac{L_4^{(s,t)}}{L_2^{(s,t)}}$ respectively. The r^{th} L-moments can be obtained by taking s= t= 0 in equation (13) as

$$L_r = \frac{1}{r}\sum_{k=0}^{r-1}\sum_{w=0}^{r-k-1}\sum_{v=0}^{k+w+1} \frac{\beta.(r)!(r-1)!\ (k+w+1)^{-v-\frac{1}{2}}.(k+w)!(-1)^{w+k}}{2w!k!(r-1-k)!(r-k-1-w)!(k+w+1-v)!v!(k)!}\Gamma\left(v+\frac{1}{2}\right) \quad (14)$$

The initial four L-moments can be obtained from equation (14) by setting r =1,2,3 and 4 respectively. Using equation (13), some numerical results for $L_1^{(s,t)}, L_2^{(s,t)}, L_3^{(s,t)}\ L_4^{(s,t)}, L_1, L_2, L_3, L_4, \tau_1^{(s,t)}, \tau_1^{(s,t)}, \tau_3^{(s,t)}, \tau_4^{(s,t)}, \tau_{1,}\tau_2 and\ \tau_3$ are obtained in Table.3.

Table 3

parameter's	(s,t)	(1,1)	(2,2)	(0,1)	(0,2)	(1,0)	(2,0)	(0,0)
$\beta = 0.4$	$L_1^{(s,t)}$	0.524	0.522	0.423	0.372	0.64	0.699	0.532
	$L_2^{(s,t)}$	0.058	0.039	0.076	0.063	0.087	0.078	0.109
	$L_3^{(s,t)}$	0.002	0.001	-0.003	-0.006	0.013	0.015	0.008
	$L_4^{(s,t)}$	0.003	0.001	0.006	0.005	0.009	0.008	0.012
	τ_1	9.067	13.263	5.571	5.94	7.346	8.99	4.89
	τ_3	0.042	0.03	-0.043	-0.094	0.152	0.19	0.069

	τ_4	0.059	0.036	0.082	0.076	0.106	0.108	0.114
$\beta = 0.6$	$L_1^{(s,t)}$	0.786	0.783	0.634	0.559	0.961	1.048	0.798
	$L_2^{(s,t)}$	0.087	0.059	0.114	0.094	0.131	0.117	0.163
	$L_3^{(s,t)}$	0.004	0.002	-0.005	-0.009	0.02	0.022	0.011
	$L_4^{(s,t)}$	0.005	0.002	0.009	0.007	0.014	0.013	0.019
	τ_1	9.067	13.263	5.571	5.94	7.346	8.99	4.89
	τ_3	0.042	0.03	-0.043	-0.094	0.152	0.19	0.069
	τ_4	0.059	0.036	0.082	0.076	0.106	0.108	0.114
$\beta = 0.8$	$L_1^{(s,t)}$	1.048	1.044	0.846	0.745	1.281	1.397	1.063
	$L_2^{(s,t)}$	0.116	0.079	0.152	0.125	0.174	0.155	0.217
	$L_3^{(s,t)}$	0.005	0.002	-0.006	-0.012	0.027	0.03	0.015
	$L_4^{(s,t)}$	0.007	0.003	0.012	0.0095	0.018	0.017	0.025
	τ_1	9.067	13.263	5.571	5.94	7.346	8.99	4.89
	τ_3	0.042	0.03	-0.043	-0.094	0.152	0.19	0.069
	τ_4	0.05	0.036	0.082	0.076	0.106	0.108	0.114
$\beta = 1$	$L_1^{(s,t)}$	1.311	1.305	1.057	0.931	1.601	1.747	1.329
	$L_2^{(s,t)}$	0.145	0.098	0.19	0.157	0.218	0.194	0.272
	$L_3^{(s,t)}$	0.006	0.003	-0.008	-0.015	0.033	0.037	0.019
	$L_4^{(s,t)}$	0.008	0.004	0.016	20.01	0.023	0.021	0.031
	τ_1	9.067	13.263	5.571	5.94	7.346	8.99	4.89
	τ_3	0.042	0.03	-0.043	-0.094	0.152	0.19	0.069
	τ_4	0.059	0.036	0.082	0.076	0.106	0.108	0.114
$\beta = 1.3$	$L_1^{(s,t)}$	1.704	1.697	1.375	1.21	2.082	2.27	1.728
	$L_2^{(s,t)}$	0.188	0.128	0.247	0.204	0.283	0.253	0.353
	$L_3^{(s,t)}$	0.008	0.004	-0.011	-0.019	0.043	0.048	0.024
	$L_4^{(s,t)}$	0.011	0.005	0.02	50.01	0.03	0.027	0.04
	τ_1	9.067	13.263	5.571	5.94	7.346	8.99	4.89
	τ_3	0.042	0.03	-0.043	-0.094	0.152	0.19	0.069
	τ_4	0.059	0.036	0.082	0.076	0.106	0.108	0.114
$\beta = 1.5$	$L_1^{(s,t)}$	1.966	1.958	1.586	1.396	2.402	2.62	1.994
	$L_2^{(s,t)}$	0.217	0.148	0.285	0.235	0.327	0.291	0.408
	$L_3^{(s,t)}$	0.009	0.004	-0.012	-0.022	0.05	0.055	0.028
	$L_4^{(s,t)}$	0.013	0.005	0.023	0.018	0.035	0.031	0.046
	τ_1	9.067	13.263	5.571	5.94	7.346	8.99	4.89
	τ_3	0.042	0.03	-0.043	-0.094	0.152	0.19	0.069
	τ_4	0.059	0.036	0.082	0.076	0.106	0.108	0.114

$\beta = 2$	$L_1^{(s,t)}$	2.621	2.61	2.115	1.862	3.202	3.493	2.659
	$L_2^{(s,t)}$	0.289	0.197	0.38	0.313	0.436	0.389	0.544
	$L_3^{(s,t)}$	0.012	0.006	-0.016	-0.029	0.066	0.074	0.038
	$L_4^{(s,t)}$	0.017	0.007	0.031	0.024	0.046	0.042	0.062
	τ_1	9.067	13.263	5.571	5.94	7.346	8.99	4.89
	τ_3	0.042	0.03	-0.043	-0.094	0.152	0.19	0.069
	τ_4	0.059	0.036	0.082	0.076	0.106	0.108	0.114

Theorem 2.1. let *Z* be a random variable has a probability density function given by equation (1). For an integer *j* such that *j* > 0, the moments of Z satisfy the following recurrence relation:

$$E\left[Z^j(r,n,\tilde{m},k)\right] - E\left[Z^j(r-1,n,\tilde{m},k)\right] = \frac{j\beta^2}{2\gamma_r}\left\{E\left[Z^{j-2}(r,n,\tilde{m},k)\right] + \beta^2 E\left[Z^{j-4}(r,n,\tilde{m},k)\right]\right\} \tag{15}$$

Proof. We have from Lemma 2.3 (by Athar. and Islam. (2004)) that

$$E[\xi\{Z(r,n,\tilde{m},k)\}] - E[\xi\{Z(r-1,n,\tilde{m},k)\}] = C_{r-2}\int_{\theta}^{\beta} \xi'(z) \sum_{i=1}^{r} a_i(r)\,[\bar{F}(z)]^{\gamma_i}\ dz$$

If we let $\xi(z) = z^j$, then

$$E\left[Z^j(r,n,\tilde{m},k)\right] - E\left[Z^j(r-1,n,\tilde{m},k)\right] = jC_{r-2}\int_{\theta}^{\beta} z^{j-1} \sum_{i=1}^{r} a_i(r)\,[\bar{F}(z)]^{\gamma_i}\ dz \tag{16}$$

On using (3) in (16), we get

$$E\left[Z^j(r,n,\tilde{m},k)\right] - E\left[Z^j(r-1,n,\tilde{m},k)\right] = \frac{jC_{r-1}}{\gamma_r}\int_0^{\infty} z^{j-1}\left(\frac{\beta^2}{2z} + \frac{\beta^4}{2z^3}\right)\sum_{i=1}^{r} a_i(r)\,[\bar{F}(z)]^{\gamma_i-1} f(z)\,dz$$

$$E\left[Z^j(r,n,\tilde{m},k)\right] - E\left[Z^j(r-1,n,\tilde{m},k)\right] = \frac{j\beta^2 C_{r-1}}{2\gamma_r}\int_0^{\infty} \left(\frac{1}{z} + \frac{\beta^2}{z^3}\right) z^{j-1}\, C_{r-1}\quad f(z)\sum_{i=1}^{r} a_i(r)\,[\bar{F}(z)]^{\gamma_i-1}\,dz$$

After simplification, the recurrence relation in equation (15) is derived.

Corollary 2.2 When $m_1 = m_2 = \cdots = m_{n-1} = m \neq -1$ the recurrence relations for single moment of GOS for ABRD are provided as follows

$$E\left[Z^{j}(r,n,m,k)\right]-E\left[Z^{j}(r-1,n,m,k)\right]=\frac{j\beta^{2}}{2\gamma_{r}}\left\{E\left[Z^{j-2}(r,n,m,k)\right]+\beta^{2}E\left[Z^{j-4}(r,n,m,k)\right]\right\} \qquad (17)$$

Proof. This can easily be deduced from (15) given the relation (6).

Remark 2.3 By setting $m=0, k=1$ in Theorem 2.1., we derive the recurrence relations for a single moment of order statistics from ABRD

$$E\left(Z_{r:n}^{j}\right)-E\left(Z_{r-1:n}^{j}\right)=\frac{j\beta^{2}}{2(n-r+1)}\left[E\left(Z_{r:n}^{j-2}\right)+\beta^{2}E\left(Z_{r:n}^{j-4}\right)\right] \qquad (18)$$

Remark 2.4 Setting $m=-1, k=1$ in Theorem 2.1., we derive the recurrence relations for upper record values as

$$E\left[Z^{j}(r,n,-1,1)\right]-E\left[Z^{j}(r-1,n,-1,1)\right]=\frac{j\beta^{2}}{2}\left\{E\left[Z^{j-2}(r,n,-1,1)\right]+\beta^{2}E\left[Z^{j-4}(r,n,-1,1)\right]\right\} \qquad (19)$$

3. Recurrence relation for product moments of GOS

Theorem 3.1 let *Z* be a random variable has a probability density function given by equation (1). For integer *i*, *j* such that *i*, *j* >0, the product moments of *Z* satisfy the following recurrence relation:

$$E\left[Z^{i}(r,n,\widetilde{m},k).Z^{j}(s,n,\widetilde{m},k)\right]-E\left[Z^{i}(r,n,\widetilde{m},k).Z^{j}(s-1,n,\widetilde{m},k)\right]=\frac{j\beta^{2}}{2\gamma_{s}}\left\{E\left[Z^{i}(r,n,\widetilde{m},k).Z^{j-2}(s,n,\widetilde{m},k)\right]+\beta^{2}E\left[Z^{i}(r,n,\widetilde{m},k).Z^{j-4}(s,n,\widetilde{m},k)\right]\right\} \qquad (20)$$

Proof. We have from Lemma 3.2 (by Athar. and Islam. (2004)) that

$$E[\xi\{Z\,(r,n,\widetilde{m},k).Z(s,n,\widetilde{m},k)\}]-E[\xi\{Z(r,n,\widetilde{m},k).Z(s-1,n,\widetilde{m},k)\}]=C_{s-2}\int_{\theta}^{\beta}\int_{z}^{\beta}\frac{\partial}{\partial t}\xi(z,t)\sum_{l=r+1}^{s}a_{l}^{(r)}(s)\left[\frac{\bar{F}(t)}{\bar{F}(z)}\right]^{\gamma_{l}}\sum_{l=1}^{r}a_{l}(r)[\bar{F}(z)]^{\gamma_{l}}\frac{f(z)}{\bar{F}(z)}\,dtdz$$

If we let $\xi(z,t)=z^{i}t^{j}$, then

$$E\left[Z^i(r,n,\tilde{m},k).Z^j(s,n,\tilde{m},k)\right]-E\left[Z^i(r,n,\tilde{m},k).Z^j(s-1,n,\tilde{m},k)\right] = \frac{jC_{s-1}}{\gamma_s}\int_0^\infty\int_z^\infty z^i t^{j-1}\sum_{l=r+1}^{s} a_l^{(r)}(s)\left[\frac{\bar{F}(t)}{\bar{F}(z)}\right]^{\gamma_l}\sum_{l=1}^{r} a_l(r)[\bar{F}(z)]^{\gamma_l}\frac{f(z)}{\bar{F}(z)}\,dtdz$$

In view of equation (3), note that

$$\frac{\bar{F}(t)}{f(t)}=\left(\frac{\beta^2}{2t}+\frac{\beta^4}{2t^3}\right)=\frac{\beta^2}{2}t^{-1}\left(1+\frac{\beta^2}{t^2}\right)$$

Therefore,

$$E\left[Z^i(r,n,\tilde{m},k).Z^j(s,n,\tilde{m},k)\right]-E\left[Z^i(r,n,\tilde{m},k).Z^j(s-1,n,\tilde{m},k)\right] = \frac{jC_{s-1}}{\gamma_s}\int_0^\infty\int_z^\infty z^i t^{j-1}\sum_{l=r+1}^{s} a_l^{(r)}(s)\left[\frac{\bar{F}(t)}{\bar{F}(z)}\right]^{\gamma_l}\sum_{l=1}^{r} a_l(r)[\bar{F}(z)]^{\gamma_l}\frac{f(z)}{\bar{F}(z)}\left(\frac{\bar{F}(t)}{f(t)}.\frac{f(t)}{\bar{F}(t)}\right)dtdz$$

$$E\left[Z^i(r,n,\tilde{m},k).Z^j(s,n,\tilde{m},k)\right]-E\left[Z^i(r,n,\tilde{m},k).Z^j(s-1,n,\tilde{m},k)\right]$$
$$=\frac{jC_{s-1}}{\gamma_s}\int_0^\infty\int_z^\infty z^i t^{j-1}\frac{\beta^2}{2}t^{-1}\left(1+\frac{\beta^2}{t^2}\right)\sum_{l=r+1}^{s} a_l^{(r)}(s)\left[\frac{\bar{F}(t)}{\bar{F}(z)}\right]^{\gamma_l}\sum_{l=1}^{r} a_l(r)[\bar{F}(z)]^{\gamma_l}\frac{f(z)}{\bar{F}(z)}.\frac{f(t)}{\bar{F}(t)}dtdz$$
$$=\frac{j\beta^2C_{s-1}}{2\gamma_s}\int_0^\infty\int_z^\infty z^i t^{j-2}\sum_{l=r+1}^{s} a_l^{(r)}(s)\left[\frac{\bar{F}(t)}{\bar{F}(z)}\right]^{\gamma_l}\sum_{l=1}^{r} a_l(r)[\bar{F}(z)]^{\gamma_l}\frac{f(z)}{\bar{F}(z)}.\frac{f(t)}{\bar{F}(t)}dtdz$$
$$+\frac{j\beta^2C_{s-1}}{2\gamma_s}\int_0^\infty\int_z^\infty z^i t^{j-4}\beta^2\sum_{l=r+1}^{s} a_l^{(r)}(s)\left[\frac{\bar{F}(t)}{\bar{F}(z)}\right]^{\gamma_l}\sum_{l=1}^{r} a_l(r)[\bar{F}(z)]^{\gamma_l}\frac{f(z)}{\bar{F}(z)}.\frac{f(t)}{\bar{F}(t)}dtdz$$

After simplification, the recurrence relation in equation (20) is derived.

Corollary 3.2 For $m_1=m_2=\cdots=m_{n-1}=m\neq -1$ the recurrence relations for product moments of the Generalized Order Statistics (GOS) for ABRD are derived from equation (20) in relation to equation (7) is given as

$$E\left[Z^i(r,n,m,k).Z^j(s,n,m,k)\right]-E\left[Z^i(r,n,m,k).Z^j(s-1,n,m,k)\right] = \frac{j\beta^2}{2\gamma_s}\left\{E\left[Z^i(r,n,m,k).Z^{j-2}(s,n,m,k)\right] + \beta^2E\left[Z^i(r,n,m,k).Z^{j-4}(s,n,m,k)\right]\right\} \quad (21)$$

Proof. This follows directly from equation (20), considering the relation given in equation (7).

Remark 3.3 By setting $m=0,k=1$ in equation (21), we derive the recurrence relations for product moments of order statistics.as

$$E\left(Z_{r,s:n}^{i,j}\right) - E\left(Z_{r,s-1:n}^{i,j}\right)$$

$$= \frac{j\beta^2}{2(n-s+1)}\left[E\left(Z_{r,s:n}^{i,j-2}\right) + \beta^2 E\left(Z_{r,s:n}^{i,j-4}\right)\right] \quad (22)$$

Remark 3.4 setting $m = -1$ in equation (21), gives us the recurrence relations for product moments of k^{th} record values as

$$E\left[\left(Z_r^{(k)}\right)^i\left(Z_s^{(k)}\right)^j\right] - E\left[\left(Z_r^{(k)}\right)^i\left(Z_{s-1}^{(k)}\right)^j\right]$$

$$= \frac{j\beta^2}{2k}\left\{E\left[\left(Z_r^{(k)}\right)^i\left(Z_s^{(k)}\right)^{j-2}\right] + \beta^2 E\left[\left(Z_r^{(k)}\right)^i\left(Z_s^{(k)}\right)^{j-4}\right]\right\} \quad (23)$$

4. Characterization

Theorem 4.1 Let Z be a non-negative random variable with a continuous distribution function $F(z)$ such that

$F(0) = 0$ and $0 < F(z) < 1$ for all $z > 0$, then

$$E\left[Z^j(r,n,m,k)\right] - E\left[Z^j(r-1,n,m,k)\right] = \frac{j\beta^2}{2\gamma_r}\left\{E\left[Z^{j-2}(r,n,m,k)\right] + \beta^2 E\left[Z^{j-4}(r,n,m,k)\right]\right\}$$

if and only if

$$\bar{F}(z) = \left(1 + \frac{z^2}{\beta^2}\right)\boldsymbol{exp}\left(-\frac{z^2}{\beta^2}\right) \quad (24)$$

Proof: The necessity part follows directly from equation (17). Conversely, if the recurrence relation given in equation (24) is satisfied, then by using equation (8), we establish that

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty Z^j\,[\bar{F}(z)]^{\gamma_r-1} f(z) g_m^{r-1}[F(z)]\,dz - \frac{C_{r-2}}{(r-2)!}\int_0^\infty Z^j\,[\bar{F}(z)]^{\gamma_{r-1}-1} f(z) g_m^{r-2}[F(z)]\,dz$$

$$= \frac{j\,\beta^2 C_{r-1}}{2\gamma_r(r-1)!}\int_0^\infty Z^{j-2}\,[\bar{F}(z)]^{\gamma_r-1} f(z) g_m^{r-1}[F(z)]\,dz + \frac{j\,\beta^4 C_{r-1}}{2\gamma_r(r-1)!}\int_0^\infty Z^{j-4}\,[\bar{F}(z)]^{\gamma_r-1} f(z) g_m^{r-1}[F(z)]\,dz$$

Integrating the first term in left-hand side by parts, we get

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty Z^{j-1}\,[\bar{F}(z)]^{\gamma_r}\,g_m^{r-1}\,[F(z)]\,dz$$
$$-\frac{j\,\beta^2C_{r-1}}{2\gamma_r(r-1)!}\int_0^\infty Z^{j-2}\,[\bar{F}(z)]^{\gamma_r-1}f(z)g_m^{r-1}\,[F(z)]\,dz$$
$$-\frac{j\,\beta^4C_{r-1}}{2\gamma_r(r-1)!}\int_0^\infty Z^{j-4}\,[\bar{F}(z)]^{\gamma_r-1}f(z)g_m^{r-1}\,[F(z)]\,dz=0$$

then

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty Z^{j-1}\,[\bar{F}(z)]^{\gamma_r-1}\;\bar{F}(z)\,g_m^{r-1}\,[F(z)]\,dz$$
$$-\frac{j\,\beta^2C_{r-1}}{2\gamma_r(r-1)!}\int_0^\infty Z^{j-2}\,[\bar{F}(z)]^{\gamma_r-1}f(z)g_m^{r-1}\,[F(z)]\,dz$$

$$-\frac{j\,\beta^2C_{r-1}}{2\gamma_r(r-1)!}\int_0^\infty \beta^2\,Z^{-2}Z^{j-2}\,[\bar{F}(z)]^{\gamma_r-1}f(z)g_m^{r-1}\,[F(z)]\,dz=0$$

this implies that

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty Z^{j-1}\,[\bar{F}(z)]^{\gamma_r-1}\;\bar{F}(z)\,g_m^{r-1}\,[F(z)]\,dz$$
$$-\frac{j\,C_{r-1}}{\gamma_r(r-1)!}\int_0^\infty \frac{\beta^2}{2}[z^{-1}+\beta^2\,Z^{-3}]Z^{j-1}\,[\bar{F}(z)]^{\gamma_r-1}f(z)g_m^{r-1}\,[F(z)]\,dz=0$$

therefore,

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty Z^{j-1}\,[\bar{F}(z)]^{\gamma_r-1}\quad g_m^{r-1}\,[F(z)]\,\left\{\bar{F}(z)-\frac{\beta^2}{2}[z^{-1}+\beta^2\,Z^{-3}]f(z)\right\}dz$$
$$=0 \qquad\qquad (25)$$

Now applying a generalization of the Muntz-Szasz theorem (by Hwang. and Lin. (1984)) to equation (25), we get

$$\bar{F}(z)-\frac{\beta^2}{2}[z^{-1}+\beta^2\,Z^{-3}]f(z)=0$$

$$\frac{\bar{F}(z)}{f(z)}=\frac{\beta^2}{2}[\beta^2\,Z^{-3}+z^{-1}]$$

Hence,

$$\frac{f(z)}{\bar{F}(z)}=\left[\frac{2z^3}{\beta^2(\beta^2+z^2)}\right]$$

Integrating both sides from 0 to *t*, we get

$$\bar{F}(t) = \left(1 + \frac{t^2}{\beta^2}\right) \boldsymbol{exp}\left(-\frac{t^2}{\beta^2}\right) \qquad (26)$$

5. Conclusion

In this research, we employed the Area -biased Rayleigh distribution to derive recurrence relations for both single and product moments of generalized order statistics (GOS). Additionally, we explored specific cases related to order statistics and record values. The recurrence relation for single moments contributes to characterizing the properties of the Area -biased Rayleigh distribution.

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