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OPTICAL SOLITONS AND LIE SYMMETRY ANALYSIS OF KAUP–NEWELL EQUATION

Arshdeep Kaur[1], Anupma Bansal[2], Rajeev Budhiraja[3]
[1,3]Department of Mathematics
MMEC, Maharishi Markandeshwar (Deemed to be University)
Mullana, Ambala-133203 (Haryana), India
E-mail: dhimanarshdeep500@gmail.com
[2]Department of Mathematics
D.A.V. College for Women, Ferozepur-152001(Punjab), India.
E-mail: anupma2512@yahoo.co.in

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Abstract

This manuscript is based on the Kaup-Newell equation, specifically contemplated to be a kind of nonlinear Schrödinger equation. One of the frameworks derived from the well-known nonlinear Schrödinger equation is the Kaup–Newell equation which is one of the three derivative forms of the nonlinear Schrödinger equation that is frequently explored in nonlinear Optics. A systematic investigation of Lie symmetry method is pertained to derive the symmetry reductions of the given equation. By utilizing these symmetry reductions, the nonlinear Partial differential equation are transformed into the Ordinary differential equations. Consequently, the solutions are obtained with arbitrary parameters through various methods which are in the form of hyperbolic functions, rational functions and trigonometric functions.

Keywords: Lie Classical method, Kaup–Newell equation, $\frac{G'}{G}$–Expansion Method, Symmetry reductions, Optical Solitons.

1 Introduction

The Schrödinger equation is a core concept in Quantum mechanics with widespread applications in different areas. This equation is referred as a partial differential equation that describes wave function of physical systems like fluid dynamics, waves in water, etc. Optical outcomes [1-5] of nonlinear Schrödinger equation are fundamental solutions to design the high-performance communication systems, understanding wave interactions in nonlinear media and pushing forward technologies such as super continuum generation and pulse compression. In this direction, Exact traveling wave solutions are key to grasping wave dynamics in nonlinear media, influencing fields such as Optics, Fluid dynamics and more. They provide insights into pulse behavior in optical fibers and other nonlinear systems.

Thereby, in this supervision, multivarious adequate methods for finding out accurate and exact solutions to the Schrödinger equation are determined and evolved in the course of the numerous former decades. Besides the assorted distinct methods, Lie group method, also stated as Lie symmetry method [11-20], is apparently a significant method to calculate the solutions of nonlinear partial differential equations. The Lie symmetry methodology is smoothly appertained to many physical and engineering methods. Under the Lie Group of point transformations when particular differential equations are invariant then reduction exists. By utilizing these symmetry reductions, the nonlinear partial differential equations are transformed into ordinary differential equations. Then employing various methods and techniques exact traveling wave solutions are accumulated.

1.1 Governing Model

The Kaup–Newell equation is given by:

$$q_t + \iota a q_{xx} + b(|q|^2 q)_x = 0, \tag{1.1}$$

where '$q(x,t)'$ is a complex-valued function, 'a' demonstrates velocity dispersion parameter, 'b' the nonlinearity coefficient. The Kaup-Newell equation is a specific type of nonlinear partial differential equation derived from the nonlinear Schrödinger equation. It describes wave phenomena and is particularly relevant in field such as plasma physics and optics. The Kaup-Newell equation can model various physical situations involving stability, interactions and soliton solutions.

2 Lie Symmetry Analysis

Recently, symmetry analysis has become popular for studying differential equations. Lie Symmetry analysis is the most effective method for evaluating the precise and exact solutions of nonlinear partial differential equations. In this document, we procure symmetry reductions, the symmetries and the solutions of Kaup-Newell equation [6-10] by emphasizing Lie classical method. To determine the symmetries of the Kaup-Newell equation, firstly we consider the complex function q(x,t) as

$$q(x,t) = u(x,t) + \iota v(x,t). \tag{2.1}$$

The equation can be expressed in terms of its real and imaginary components as-

$$u_t - a v_{xx} + 2bu^2 u_x + 2buv v_x + bu^2 u_x + bv^2 u_x = 0, \tag{2.2}$$

$$v_t + a u_{xx} + 2buv u_x + 2bv^2 v_x + bu^2 v_x + bv^2 v_x = 0. \tag{2.3}$$

The one parameter Lie group of infinitesimal transformations constitutes as aforementioned-

$$\begin{aligned}\widetilde{u} &= u + \epsilon\eta(x,t,u,v) + O(\epsilon^2),\\ \widetilde{v} &= v + \epsilon\phi(x,t,u,v) + O(\epsilon^2),\\ \widetilde{x} &= x + \epsilon\xi(x,t,u,v) + O(\epsilon^2),\\ \widetilde{t} &= t + \epsilon\tau(x,t,u,v) + O(\epsilon^2),\end{aligned} \tag{2.4}$$

where ϵ is group parameter. Applying these one-parameter Lie group of infinitesimal transformations, we obtain the invariance condition of the equation (1.1):

$$\begin{aligned}&\eta^t - a\phi^{xx} + 2bu^2\eta^x + 4buu_x\eta + 2buv_x\phi + 2buv\phi^x + 2bvv_x\eta + bu^2\eta^x + 2buu_x\eta\\ &+bv^2\eta^x + 2bvu_x\phi = 0,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\phi^t + a\eta_{xx} + 2buv\eta^x + 2buu_x\phi + 2bvu_x\eta + 2bv^2\phi^x + 4bvv_x\phi + bu^2\phi^x + 2buv_x\eta\\ &+bv^2\phi^x + 2bvv_x\phi = 0,\end{aligned} \tag{2.6}$$

The mentioned transformation abandons the entire set of resolutions of the equation (1.1) invariant and brings about an over determined linear system of equations for the infinitesimals $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\eta(x,t,u,v)$.
By using these values of the infinitesimal η^t,ϕ^t, η^x, ϕ^x, η^{xx} and ϕ^{xx} into equation (2.5) and (2.6), we equalize the alike powers of differentials to 0. After this, we will be able to pertain the system of determining equations as under:

$(i)\ \tau_x = 0, \tau_u = 0, \tau_v = 0$
$(ii)\ \xi_u = 0, \xi_v = 0$

$(iii)\ -a\phi_v + a\eta_u + 2a\xi_x - a\tau_t = 0$

$(iv)\ \phi_{uu} = 0, \phi_{uv} = 0, \phi_{vv} = 0$

$(v)\ -3u^2\xi_x + 3u^2\tau_t b + 2uvb\phi_u - 2uvb\eta_v - v^2b\xi_x + v^2b\tau_t + 6bu\eta + 2bv\phi - 2a\phi_{xu} - \xi_t = 0$

$(vi)\ 2u^2b\eta_v + 2uvb\phi_v - 2uvb\eta_u - 2uvb\xi_x + 2uvb\tau_t - 2bv^2\eta_v + 2bu\phi + 2bv\eta + a\xi_{xx} - 2a\phi_{xv} = 0$

$(vii)\ \phi_u = -\eta_v$

$(viii)\ v^2b\eta_x + 2uvb\phi_x - a\phi_{xx} + \eta_t + 3u^2b\eta_x = 0$

$(ix)\ \eta_{vv} = 0, \eta_{uu} = 0, \eta_{uv} = 0$

$(x)\ -bu^2\xi_x + bu^2\tau_t - 2buv\phi_u + 2buv\eta_v - 3bv^2\xi_x + 3bv^2\tau_t + 2bu\eta + 6bv\phi + 2a\eta_{xv} - \xi_x = 0$

$(xi)\ -2bu^2\phi_u - 2buv\xi_x + 2buv\eta_u + 2buv\tau_t - 2buv\phi_v + 2bv^2\phi_u + 2bu\phi + 2bv\eta + 2a\eta_{xu} - a\xi_{xx} = 0$

$(xii)\ -2a\xi_x + a\eta_u + a\tau_t - a\phi_v = 0$

$(xiii)\ \phi_u + \eta_v = 0$

$(xiv)\ a\eta_{xx} + 2buv\eta_x + 3bv^2\phi_x + bu^2\phi_x + \phi_t = 0$

(2.7)

By working through these determining equations, the below-mentioned solutions will be obtained-

$$\begin{aligned}
&\eta = uC_2 - vC_1,\\
&\phi = uC_1 + vC_2,\\
&\xi = -2xC_2 + C_3,\\
&\tau = -4tC_2 + C_4,
\end{aligned} \tag{2.8}$$

where C_1, C_2, C_3, C_4 illustrates the arbitrary constants. From the above obtained Lie symmetries the vector fields are as follows:

$$\begin{aligned}
&V_1 = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u},\\
&V_2 = v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u} - 2x\frac{\partial}{\partial x} - 4t\frac{\partial}{\partial t},\\
&V_3 = \frac{\partial}{\partial x},\\
&V_4 = \frac{\partial}{\partial t}.
\end{aligned} \tag{2.9}$$

By utilizing the various linear combinations of the above mentioned vector fields i.e., V_1, V_2, V_3, V_4 we avail the similarity reductions of the Kaup-Newell equation.

3 Reductions and Exact Solutions of Kaup-Newell Equation

A prominent motive to calculate symmetries of differential equations aims to utilize those to generate the exact solutions and obtain symmetry reductions. In this subsection, we use the different combinations of vector fields and reduce the Kaup-Newell equation to ordinary differential equations in each case to further explore the exact solutions.

We will consider the following vector fields for the reduction of the system of equations:

$$\begin{aligned}
&(1)\ \ V_1 + \alpha V_3 + \beta V_4,\\
&(2)\ \ V_2,\\
&(3)\ \ V_3 + \omega V_4.
\end{aligned} \tag{3.1}$$

In each case, by using characteristics equation, as under, we will try to obtain the reductions:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta} = \frac{dv}{\phi}. \tag{3.2}$$

3.1 Vector field $V_1 + \alpha V_3 + \beta V_4$

On using the characteristic equations (3.2) we acquire the similarity variables as shown below-

$$q = F(\xi)e^{\iota(\frac{t}{\beta}+G(\xi))}, \xi = \beta x - \alpha t. \tag{3.3}$$

Using equation (3.3) in equation (2.2) and (2.3), we get ordinary differential equation as:

$$-a\beta^2 F(\xi)G'(\xi)^2 + F(\xi)(\frac{1}{\beta} - \alpha G'(\xi)) + a\beta^2 F''(\xi) + b\beta F(\xi)^3 G'(\xi) = 0, \tag{3.4}$$

and

$$2\beta^2 F'(\xi)G'(\xi) + \beta^2 F(\xi)G''(\xi) + 3b\beta F'(\xi)F(\xi)^2 - \alpha F'(\xi) = 0. \tag{3.5}$$

In this context, the prime(') signifies the differentiation w.r.t variable ξ.
On Solving these equations we get

$$F(\xi) = K_1, \tag{3.6}$$

where K_1 is any constant and

$$G(\xi) = \left(\frac{bc^2\beta - \alpha + \sqrt{b^2c^4\beta^2 - 2bc^2\alpha\beta + 4a\beta + \alpha^2}}{2a\beta^2}\right)\xi + C_1. \tag{3.7}$$

Using these values we get the solution of Kaup-Newell equation as

$$q(x,t) = K_1 e^{\iota\left(\frac{t}{\beta} + \left(\frac{bc^2\beta - \alpha + \sqrt{b^2c^4\beta^2 - 2bc^2\alpha\beta + 4a\beta + \alpha^2}}{2a\beta^2}\right)(\beta x - \alpha t) + C_1\right)}. \tag{3.8}$$

3.2 Vector field V_2

By Solving the characteristic equation (3.2) aforementioned similarity variable is calculated as

$$u = F(\xi)t^{\frac{-1}{4}}, v = G(\xi)t^{\frac{-1}{4}}, \xi = \frac{x^2}{t}, \tag{3.9}$$

On using equation (3.9) in equation (2.2) and (2.3), we get ordinary differential equation as:

$$\frac{-1}{4}F(\xi) - \xi F'(\xi) - 4a\xi G''(\xi) - 2aG'(\xi) + 6b\sqrt{\xi}F'(\xi) + 2b\sqrt{\xi}G^2(\xi)F'(\xi) + 4b\sqrt{\xi}G(\xi)G'(\xi) = 0,$$

and

$$\frac{-1}{4}G(\xi) - \xi G'(\xi) + 2aF'(\xi) + 4a\xi F''(\xi) + 4bF(\xi)G(\xi)F'(\xi)\sqrt{\xi} + 6b\sqrt{\xi}G^2(\xi)G'(\xi) + 2b\sqrt{\xi}F^2(\xi)G'(\xi) = 0. \tag{3.10}$$

where prime(') denotes the differentiation w.r.t variable ξ. Due to complexity of the reduced equations, we just obtain a constant solution for the equation.

3.3 Vector field $V_3 + \omega V_4$

Similarity variables are acquired in the light of the characteristic equation as:

$$\begin{aligned} \xi &= x - \tfrac{t}{\omega}, \\ q &= F(\xi)e^{\iota G(\xi)}. \end{aligned} \tag{3.11}$$

On applying these values we acquire the aforementioned system of Ordinary differential equations as

$$-\frac{1}{\omega}F(\xi)G'(\xi) + aF''(\xi) - aF(\xi)(G'(\xi))^2 + bF(\xi)^3G'(\xi) = 0, \tag{3.12}$$

and

$$-\frac{1}{\omega}F'(\xi) - 2aF'(\xi)G'(\xi) - aF(\xi)G''(\xi) + 3bF(\xi)^2F'(\xi) = 0. \tag{3.13}$$

Multiply equation(3.13) with $F(\xi)$ and on integrating we obtain

$$-aG'(\xi)F(\xi)^2 - \frac{F(\xi)^2}{2\omega} + \frac{3}{4}bF(\xi)^4 = 0, \tag{3.14}$$

Above equation leads to:

$$G'(\xi) = -12a\omega + \frac{3b}{4a}F(\xi)^2 + B, \tag{3.15}$$

where B is referred to as the arbitrary constant. On substituting this value in (3.12) and again multiply with $F'(\xi)$ and integrating, we get

$$\frac{b^2}{32a}F(\xi)^6 - \frac{b}{8aw}F(\xi)^4 + \frac{1}{8a\omega^2}F(\xi)^2 + \frac{a}{2}(F'(\xi))^2 = 0, \tag{3.16}$$

Substitute $(F(\xi))^2 = P(\xi)$ into (3.16) equation, we get

$$b^2\omega^2P(\xi)^4 - 4b\omega P(\xi)^3 + 4P(\xi)^2 + 4a^2\omega^2(P'(\xi))^2 = 0. \tag{3.17}$$

On Solving the equation (3.17), we acquire the solutions which are given underneath:

$$\begin{aligned}
&(i)q(x,t) = \sqrt{\frac{2}{b\omega}}e^{\iota(-12a\omega+\frac{3\omega}{2a}+B)(x-\frac{t}{\omega})+C_2},\\
&(ii)q(x,t) = \sqrt{\frac{1}{b\omega} - \frac{tanh\left(C_1-\frac{\frac{1}{2}\iota(x-\frac{t}{\omega})}{\omega a}\right)}{b\omega}}\times\\
&e^{\iota\left(-12a\omega(x-\frac{t}{\omega})+\frac{3(x-\frac{t}{\omega})}{4a\omega}+\frac{3\iota\log(\tanh(C_1-\frac{\iota}{2}(x-\frac{t}{\omega}))-1)}{4a\omega}+\frac{3\iota\log(\tanh(C_1-\frac{\iota}{2}(x-\frac{t}{\omega}))+1)}{4a\omega}+B(x-\frac{t}{\omega})\right)}
\end{aligned} \tag{3.18}$$

3.4 Exact traveling wave solutions using $(\frac{G'}{G})$-Expansion Method

In this section, we aim to determine wave solutions by using $(\frac{G'}{G})$-Expansion Method for equation (3.17). The progression of wave solutions originates from these procedures:
1. Taking into Consideration, the structure of Ordinary Differential Equation (3.17) is formulated as $\left(\frac{G'}{G}\right)$:

$$\widetilde{F}(\chi) = a_q\left(\frac{G'}{G}\right)^q + a_{q-1}\left(\frac{G'}{G}\right)^{q-1} + ... \tag{3.19}$$

Here a_q are constants to be determine, where q ranges from 0 to infinity. As $G = G(\chi)$ persuades the LDE of second order which is of the form written below:

$$G'' + \widetilde{\lambda}G' + \widetilde{\mu}G = 0. \tag{3.20}$$

where $a_q(a_q \neq 0)$; where q is called the balance number, a_{q-1},.....a_0, $\widetilde{\lambda}$ and $\widetilde{\mu}$ are constants to be determine later.

2. Determining the positive integer q in (3.19), by equalizing the higher order nonlinear terms to the higher order derivatives in (3.17)

3. Replacing (3.19) into (3.17) and pertain the ODE (3.20), then gather all the entire terms of $\left(\frac{G'}{G}\right)$ containing homogeneous power, and equalize each coefficient to zero, resulting in a variety of algebraic equations for anaysis a_q, a_{q-1},..... a_0 ,c, $\widetilde{\lambda}$ and $\widetilde{\mu}$.

4. The general solution of (3.20) is noted, then substituting the values of a_q in (3.19) we will get the exact solutions of (1.1)

The general solutions of Equation(3.20) can be written as

$$\left(\frac{G'}{G}\right)=\begin{cases} \frac{\sqrt{\widetilde{\lambda}^2-4\widetilde{\mu}}}{2}\left(\frac{C_1 sinh\left(\frac{1}{2}\sqrt{\widetilde{\lambda}^2-4\widetilde{\mu}}\right)\chi+C_2 cosh\left(\frac{1}{2}\sqrt{\widetilde{\lambda}^2-4\widetilde{\mu}}\right)\chi}{C_1 cosh\left(\frac{1}{2}\sqrt{\widetilde{\lambda}^2-4\widetilde{\mu}}\right)\chi+C_2 sinh\left(\frac{1}{2}\sqrt{\widetilde{\lambda}^2-4\widetilde{\mu}}\right)\chi}\right)-\frac{\widetilde{\lambda}}{2},\ \widetilde{\lambda}^2-4\widetilde{\mu}>0 \\ \frac{\sqrt{4\widetilde{\mu}-\widetilde{\lambda}^2}}{2}\left(\frac{-C_1 sin\left(\frac{1}{2}\sqrt{4\widetilde{\mu}-\widetilde{\lambda}^2}\right)\chi+C_2 cos\left(\frac{1}{2}\sqrt{4\widetilde{\mu}-\widetilde{\lambda}^2}\right)\chi}{C_1 cos\left(\frac{1}{2}\sqrt{4\widetilde{\mu}-\widetilde{\lambda}^2}\right)\chi+C_2 sin\left(\frac{1}{2}\sqrt{4\widetilde{\mu}-\widetilde{\lambda}^2}\right)\chi}\right)-\frac{\widetilde{\lambda}}{2},\ \widetilde{\lambda}^2-4\widetilde{\mu}<0 \\ \frac{c_2}{c_1+c_2\chi}-\frac{\widetilde{\lambda}}{2}\ ,\widetilde{\lambda}^2-4\widetilde{\mu}=0. \end{cases} \tag{3.21}$$

In this context, C_1 and C_2 both illustrates the arbitrary constants.

Considering equation (3.17), balancing with $P'(\xi)^2$ and $P(\xi)^4$ bestows $q=1$, thus, the result is of the below-stated form

$$P(\xi)=a_0+a_1(\frac{G'}{G}), \tag{3.22}$$

We substitute the equation (3.22) in the equation(3.17) and after this, we equalize the coefficients of the algebraic equation to zero, we will get the following algebraic system of equations:-

$$\begin{aligned} &(\tfrac{G'}{G})^4: b^2w^2a_1^4+4w^2a^2a_1^2=0,\\ &(\tfrac{G'}{G})^3: -4ba_1^3+8w^2a^2a_1^2\lambda+4b^2w^2a_0a_1^3=0,\\ &(\tfrac{G'}{G})^2: -12ba_0a_1^2+4a_1^2+6b^2w^2a_0^2a_1^2+4w^2a^2a_1^2\lambda^2+8w^2a^2a_1^2\mu=0,\\ &(\tfrac{G'}{G})^1: -12ba_0^2a_1+8a_0a_1+4b^2w^2a^2a_1^2\mu\lambda=0,\\ &(\tfrac{G'}{G})^0: b^2w^2a_0^4-4ba_0^3+4w^2a^2a_1^2\mu^2+4a_0^2=0. \end{aligned} \tag{3.23}$$

Resolving the overhead algebraic equations; yields the following cases:

Case-I

$$V=V, a=\frac{1}{2}\iota a_1 b, -\frac{1}{2}\iota a_1 b, b=b, \mu=\frac{a_0(-2+ba_0)}{ba_1^2}, w=1, \lambda=\frac{2(-1+ba_0)}{ba_1}, a_0=a_0, a_1=a_1. \tag{3.24}$$

Case-II

$$V=V, a=\frac{1}{2}\iota a_1 b, -\frac{1}{2}\iota a_1 b, b=b, \mu=\frac{a_0(-2+ba_0)}{ba_1^2}, w=-1, \lambda=\frac{2(-1+ba_0)}{ba_1}, a_0=a_0, a_1=a_1. \tag{3.25}$$

Corresponding to the above cases we will obtain the traveling wave solutions.

The general outcomes of the equation (3.24) can be expressed as:

Subcase-I when $\lambda^2-4\mu>0$, then solution is:

$$P(\xi)=a_0+a_1\left(-\frac{(-1+ba_0)}{a_1b}+\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}\left(\frac{C_1 sinh\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})+C_2 cosh\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})}{C_1 cosh\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})+C_2 sinh\sqrt{\frac{(-1+ba_0^2)}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})}\right)\right). \tag{3.26}$$

$$F(\xi)=\sqrt{a_0+a_1\left(-\frac{(-1+ba_0)}{a_1b}+\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}\left(\frac{C_1 sinh\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})+C_2 cosh\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})}{C_1 cosh\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})+C_2 sinh\sqrt{\frac{(-1+ba_0^2)}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})}\right)\right)}, \tag{3.27}$$

$$G(\xi)=6\iota a_1b\xi-\frac{\frac{3}{2}\iota\sqrt{\frac{(-1+ba_0)^2}{b^2a_1^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}\log\left(C_1\cosh(\sqrt{\frac{1}{b^2a_1^2}}\xi)+C_2\sinh(\sqrt{\frac{1}{b^2a_1^2}}\xi)\right)}{\sqrt{\frac{1}{b^2a_1^2}}}-\frac{3\iota\xi}{2a_1b}+B\xi, \tag{3.28}$$

using these values we get solution

$$q(x,t)=\sqrt{a_0+a_1\left(-\frac{(-1+ba_0)}{a_1b}+\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}\left(\frac{C_1 sinh\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})+C_2 cosh\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})}{C_1 cosh\sqrt{\frac{(-1+ba_0)^2}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})+C_2 sinh\sqrt{\frac{(-1+ba_0^2)}{a_1^2b^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}(x-\frac{t}{\omega})}\right)\right)}\; e^{\iota\left(6\iota a_1b(x-\frac{t}{\omega})-\frac{\frac{3}{2}\iota\sqrt{\frac{(-1+ba_0)^2}{b^2a_1^2}-\frac{a_0(-2+ba_0)}{ba_1^2}}\log\left(C_1\cosh(\sqrt{\frac{1}{b^2a_1^2}}(x-\frac{t}{\omega}))+C_2\sinh(\sqrt{\frac{1}{b^2a_1^2}}(x-\frac{t}{\omega}))\right)}{\sqrt{\frac{1}{b^2a_1^2}}}-\frac{3\iota(x-\frac{t}{\omega})}{2a_1b}+B(x-\frac{t}{\omega})\right)}. \tag{3.29}$$

The Graphical representation of this case is represented in Figure 1 and in Figure 2.

Subcase-II when $\lambda^2-4\mu<0$, then solution is:

$$P(x,t)=a_0+a_1\left(-\frac{(-1+ba_0)}{a_1b}+\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}\left(\frac{-C_1 sin\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})+C_2 cos\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})}{C_1 cos\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})+C_2\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})}\right)\right). \tag{3.30}$$

$$F(\xi)=\sqrt{a_0+a_1\left(-\frac{(-1+ba_0)}{a_1b}+\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}\left(\frac{-C_1 sin\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})+C_2 cos\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})}{C_1 cos\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})+C_2\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})}\right)\right)}, \tag{3.31}$$

$$G(\xi)=-12a\omega\xi+B\xi\left(\frac{3b}{8a}\sqrt{\frac{-2(-2a_0C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2})-2a_0C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2})+\frac{\sqrt{4\mu-\lambda^2}}{2}a_1C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2})-\sqrt{4\mu-\lambda^2}a_1C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2})+\lambda C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2})+\lambda C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2})}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2})+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2})}}\right)\xi. \tag{3.32}$$

using these values we get the solution

$$q(x,t)=\sqrt{a_0+a_1\left(-\frac{(-1+ba_0)}{a_1b}+\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}\left(\frac{-C_1 sin\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})+C_2 cos\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})}{C_1 cos\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})+C_2\sqrt{\frac{a_0(-2+ba_0)}{ba_1^2}-\frac{(-1+ba_0)^2}{a_1^2b^2}}(x-\frac{t}{\omega})}\right)\right)}\; e^{\iota(-12a\omega\xi+B\xi)}\times e^{\iota\left(\frac{3b}{8a}\sqrt{\frac{-2(-2a_0C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2})-2a_0C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2})+\frac{\sqrt{4\mu-\lambda^2}}{2}a_1C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2})-\sqrt{4\mu-\lambda^2}a_1C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2})+\lambda C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2})+\lambda C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2})}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2})+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2})}}\right)\xi} \tag{3.33}$$

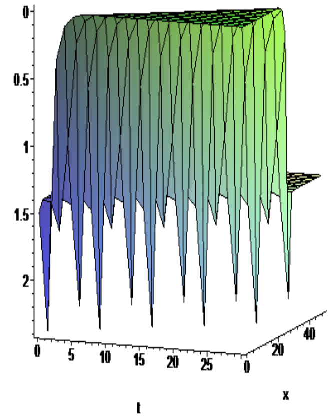

Figure 1: Traveling wave solution of $|q(x,t)|$ of equation (3.29) when $\lambda^2-4\mu > 0$ for case-I $a_0 = 2$, $a_1 = 1$, $C_1 = 4$, $C_2 = 5$, $w = 1$, $b = 1$, $\xi = x - \frac{t}{w}$.

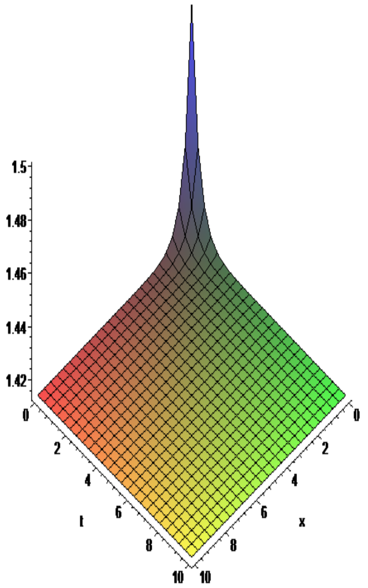

Figure 2: Traveling wave solution of $|q(x,t)|$ of equation (3.29) when $\lambda^2-4\mu > 0$ for case-II $a_0 = 2$, $a_1 = 1$, $C_1 = 4$, $C_2 = 5$, $w = -1$, $b = 1$, $\xi = x - \frac{t}{w}$.

where $\xi = x - \frac{t}{w}$.

Subcase-III when $\lambda^2 - 4\mu = 0$, then solution is:

$$P(x,t) = \left(-\frac{(-1+ba_0)}{a_1 b} + \frac{C_2}{C_1 + C_2\xi}\right). \tag{3.34}$$

In this instance, both C_1 and C_2 are treated the arbitrary constants.

$$F(x,t) = \sqrt{-K_2 + \left(\frac{C_2}{C_1 + C_2\xi}\right)}, \tag{3.35}$$

Where $K_2 = \frac{-1+ba_0}{a_1 b}$. is any constant.

$$G(x,t) = \\ -12a\xi + \frac{3}{8}\left(\frac{b\sqrt{\frac{-(K_2C_1+K_2C_2\xi-C_2)}{(C_1+C_2\xi)}}(C_1+C_2\xi)(2\sqrt{-(K_2C_1+K_2C_2\xi-C_2)(C_1+C_2\xi)}\sqrt{K_2C_2^2}+C_2^2 arctan(\frac{\sqrt{K_2C_2^2}(2K_2C_2\xi+2K_2C_1-C_2))}{2K_2C_2\sqrt{-(K_2C_1+K_2C_2\xi-C_2)(C_1+C_2\xi)}})}{aC_2\sqrt{-(K_2C_1+K_2C_2\xi-C_2)(C_1+C_2\xi)}\sqrt{K_2C_2^2}}\right) + B\xi. \tag{3.36}$$

using these values we get solution

$$q(x,t) = \sqrt{-K_2 + \left(\frac{C_2}{C_1+C_2\xi}\right)}\times \\ e^{\iota\left(-12a\xi+\frac{3}{8}\left(\frac{b\sqrt{\frac{-(K_2C_1+K_2C_2\xi-C_2)}{(C_1+C_2\xi)}}(C_1+C_2\xi)(2\sqrt{-(K_2C_1+K_2C_2\xi-C_2)(C_1+C_2\xi)}\sqrt{K_2C_2^2}+C_2^2 arctan(\frac{\sqrt{K_2C_2^2}(2K_2C_2\xi+2K_2C_1-C_2))}{2K_2C_2\sqrt{-(K_2C_1+K_2C_2\xi-C_2)(C_1+C_2\xi)}})}{aC_2\sqrt{-(K_2C_1+K_2C_2\xi-C_2)(C_1+C_2\xi)}\sqrt{K_2C_2^2}}\right)+B\xi\right)}. \tag{3.37}$$

where $\xi = x - \frac{t}{w}$.

4 Conclusion

The goal of this study was to explore the various Lie symmetries and the invariant solutions of Kaup-Newell equations. The first step is to determine the infinitesmal generators of the equation using the Lie group method. We then used the $\frac{G'}{G}$-expansion procedure to derive the traveling wave solutions. These traveling wave solutions are expressed using hyperbolic, trigonometric, and rational functions that incorporate arbitrary parameters. We derived exact soliton solutions for the Kaup-Newell equation. These solutions may hold importance across various scientific disciplines. The availability of advanced mathematical software such as Maple simplifies the complex algebraic computations.

5 Author Contribution

Anupma Bansal; Editing, original graph plotting and Investigation. Arshdeep Kaur; Writting, Editing and original graph plotting. Rajeev Budhiraja; Editing and Investigation.

6 Competing Interest

The authors declare that they have no known competing interest.

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7 Statements and declarations

The authors declares that this manuscript presents original research and confirm that they have no financial interests or personal relationships that could have impacted the work described in this paper.