Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 3 (2025)

Compactness and Rough Isomorphism on Topological Simple Rough Groups

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract: In this paper, we study about the compactness on topological simple rough groups. In particular we discuss the open mapping theorems and rough isomorphism theorems in topological simple rough groups. Also, we define a rough double coset space and discuss their role in topological simple rough groups. Further, we examine the relationship between compactness and continuity of quotient maps.

Keywords: Rough groups, Rough subgroups, Topological simple rough groups, Compact, Topological rough group homeomorphism, Rough double coset spaces, Quotient spaces.

2020 Mathematics Subject Classification: 20E32, 22C05, 22D05, 54D45.

1.Introduction:

The rough set theory was introduced by Pawlak [14] in 1982 which is based on the equivalence relations. After more than 30 years of research, the theory of rough set has been continuously improved and widely expanded in applications. In 1994, Biswas and Nanda [3] introduced the notion of rough groups and rough subgroups, which depends on the upper approximation. Then, Bagirmaz et al. (2016) [13] introduced the concept of topological rough group and extended the notion of a topological group to include algebraic structures of rough groups.

In this paper, we discussed compactness and open mapping theorems in topological simple rough groups and we examine the relationship between compactness and continuity of quotient maps. Further, we analysed some results related to topological rough group homeomorphism and also rough isomorphism theorems are discussed. Finally, we defined a rough double coset space and discuss their role in topological simple rough groups. By utilizing techniques from topology and fundamental analysis, we establish criteria for compactness in rough double coset spaces.

2. Preliminaries:

Definition 2.1. [4] Let $K = (U, R)$ be an approximation space and $*$ be a binary operation defined on U. A subset G of universe U is called a rough group if the following properties are satisfied:

(i) $\forall$ x, y $\in$ G, $x*y \in \overline{G}$;

(ii) Association property holds in $\overline{G}$;
(iii) $\exists$ e $\in \overline{G}$ such that $\forall$ x $\in$ G, x∗e = e∗x = x; e is called the rough identity element of rough group G;
(iv) $\forall$ x $\in$ G, $\exists$ y $\in$ G such that x∗y = y∗x = e; y is called the rough inverse element of x in G;

Theorem 2.2. [4] A necessary and sufficient condition for a subset H of rough group G to be a rough subgroup is that:

(i) $\forall$ x, y $\in$ H, x∗y $\in \overline{G}$;
(ii) $\forall$ x $\in$ H, $x^{-1} \in$ H.

Definition 2.3. [13] A topological rough group is a rough group (G, ∗) together with a topology T on $\overline{G}$ satisfying the following two properties:

(i) the mapping f: G × G → $\overline{G}$ defined by f(x, y) = xy is continuous with respect to product topology on G × G and the topology T_G on G induced by T,
(ii) the inverse mapping g: G → G defined by g(x) = x^{-1} is continuous with respect to the topology T_G on G induced by T.

Definition 2.4. [4] Let (U_1, R_1), (U_2, R_2) be two approximation spaces, and $*$, $\overline{*}$ be binary operations over universes U_1, U_2 respectively. Let $G_1 \subset U_1$ and $G_2 \subset U_2$ be rough groups. G_1, G_2 are called rough homomorphism sets if there exists a surjection $\varphi: \overline{G_1} \to \overline{G_2}$ such that $\forall\, x, y \in G_1 \cup \{e\}$, we have $\varphi(x * y) = \varphi(x) \,\overline{*}\, \varphi(y)$.

If a rough homomorphism is a bijection, then we say that G_1 and G_2 are rough isomorphism.

Definition 2.5. [1] A mapping $f: \overline{G_1} \to \overline{G_2}$ is called a topological rough group homomorphism, if f is a rough homomorphism and continuous with respect to the topology τ_2 on $\overline{G_2}$ inducing τ_{G_2} on G_2 and a topology τ_1 on $\overline{G_1}$ inducing τ_{G_1} on G_1.

Definition 2.6. [1] Topological rough group homomorphism $f: \overline{G_1} \to \overline{G_2}$ is called a topological rough group homeomorphism, if there exists a topological rough homomorphism f^{-1} such that $f^{-1} \circ f = 1_{G_1}$.

Definition 2.7. [1] Let $\Phi: \overline{G_1} \to \overline{G_2}$ be a topological rough group homomorphism and let e_2 be the rough identity element in G_2. Then

$\ker(\Phi) = \{g \in G_1 : \Phi(g) = e_2\}$.

is called the rough kernel associated to the map Φ.

Definition 2.8. [12] Let G be a rough group such that $\overline{G}$ is a group and H is a rough subgroup of G. If H is a normal subgroup in $\overline{G}$, then $\overline{G}/H$ is a rough quotient group.

Definition 2.9. [16] A rough group $G_{\mathfrak{R}}$ is called a simple rough group if it contains no proper non-trivial rough normal subgroups.

That is, $G_{\mathfrak{R}}$ has only the rough normal subgroups {e} and $G_{\mathfrak{R}}$.

Definition 2.10. [16] A topological simple rough group is a simple rough group $(G_{\mathfrak{R}}, *)$ together with a topology $\bar{\tau}$ on $\overline{G_{\mathfrak{R}}}$ satisfying the following two properties:

(i) The mapping f: $G_{\mathfrak{R}} \times G_{\mathfrak{R}} \to \overline{G_{\mathfrak{R}}}$ defined by f(x, y) = xy, x, y $\in G_{\mathfrak{R}}$ is continuous with respect to the product topology on $G_{\mathfrak{R}} \times G_{\mathfrak{R}}$ and the topology τ on $G_{\mathfrak{R}}$ induced by $\bar{\tau}$
(ii) The inverse mapping g: $G_{\mathfrak{R}} \to G_{\mathfrak{R}}$ defined by g(x) = x^{-1}, x $\in G_{\mathfrak{R}}$ is continuous with respect to the topology τ on $G_{\mathfrak{R}}$ induced by $\bar{\tau}$.

Proposition 2.11. [16] Let $G_{\mathfrak{R}}$ be a topological simple rough group. If U $\subseteq \overline{G_{\mathfrak{R}}}$ is an open set with e $\in$ U, then there exists a symmetric open set V of e in $G_{\mathfrak{R}}$ such that VV $\subseteq$ U.

Lemma 2.12. [17] Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$ and W be a neighbourhood of e in $\overline{G_{\mathfrak{R}}}$. Then there is an open set U of e in $G_{\mathfrak{R}}$ such that $U \subseteq U^n \subseteq W$, for every $n \in \mathbb{N} - \{0\}$.

Lemma 2.13. [7] Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Theorem 2.14. [5] Every locally compact subspace M of a Hausdorff space X is an open subset of the closure $\bar{M}$ of the set M in the space X.

Remark 2.15. [16] The topological closure of $H_{\mathfrak{R}}$, $cl(H_{\mathfrak{R}})$, in $\overline{G_{\mathfrak{R}}}$ is a topological rough subgroup in $\overline{G_{\mathfrak{R}}}$.

Definition 2.16. [16] A continuous mapping $f: X \to Y$ is perfect if X is a Hausdorff space, f is a closed mapping and all fibers $f^{-1}(y)$ are compact subsets of X.

Result 2.17. [5] A T_1- space X is a regular space if and only if for every $x \in X$ and every neighbourhood V of x there exists a neighbourhood U of x such that $\overline{V} \subseteq U$.

Theorem 2.18. [5] A continuous mapping $f: X \to Y$ is closed if and only if for every point $y \in Y$ and every open set $U \subset X$ which contains $f^{-1}(y)$, there exists a neighbourhood V of the point y in Y such that $f^{-1}(V) \subset U$.

Throughout this paper, we consider X be the universal set, $G_{\mathfrak{R}}$ be a simple rough group with identity e and $\overline{G_{\mathfrak{R}}}$ be the upper rough approximation of $G_{\mathfrak{R}}$. Also, the corresponding topologies are denoted by $\bar{\tau}$ for $\overline{G_{\mathfrak{R}}}$ and τ for $G_{\mathfrak{R}}$ induced from $\bar{\tau}$.

3. Compactness:

Theorem 3.1. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$ and A be a compact subset of $\overline{G_{\mathfrak{R}}}$. If M is a closed subset of $\overline{G_{\mathfrak{R}}}$ with $A \cap M = \emptyset$, then there is an open neighbourhood V of e in $G_{\mathfrak{R}}$ such that $AV \cap M = \emptyset$ and $VA \cap M = \emptyset$.

Proof: Since the map $L_g: G_{\mathfrak{R}} \to \overline{G_{\mathfrak{R}}}$ is continuous and M is closed, there exists an open neighbourhood U_a of e in $\overline{G_{\mathfrak{R}}}$ such that $aU_a \cap M = \emptyset$, for all $a \in A$. By proposition 2.9, there is a symmetric open set V_a of e in $G_{\mathfrak{R}}$ such that $V_aV_a \subseteq U_a$. Since A is compact, there exists an open cover $\bigcup_{a\in A} a\, V_a$ such that $A \subseteq \bigcup_{a\in A} a\, V_a$. Let $V = \bigcap_{a\in A} V_a$. Suppose there exists an arbitrary element $b \in A$, then $b \in aV_a$. Now, $bV \subseteq bV_a \subseteq aV_aV_a \subseteq aU_a$ which implies $bV \cap M = \emptyset$. Therefore, $AV \cap M = \emptyset$. Similarly, we prove $VA \cap M = \emptyset$.

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 3 (2025)

Theorem 3.2. (Second Closure Lemma) Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. Suppose A is a compact subset of $\overline{G_{\mathfrak{R}}}$ and M is a closed subset of $\overline{G_{\mathfrak{R}}}$. Then AM and MA are closed sets in $\overline{G_{\mathfrak{R}}}$.

Proof: Let $a \notin AM$. Then $A^{-1}a \cap M = \emptyset$. Since A is a compact subset, $A^{-1}a$ is compact. Therefore, by theorem 3.1, there is an open neighbourhood V of e in $G_{\mathfrak{R}}$ such that $A^{-1}aV \cap M = \emptyset$ which implies $aV \cap AM = \emptyset$ and aV is an open neighbourhood of a in the complement of AM. Hence AM is a closed subset in $\overline{G_{\mathfrak{R}}}$. Similarly, MA is a closed subset in $\overline{G_{\mathfrak{R}}}$.

Theorem 3.3. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. Suppose A is a compact subset of $\overline{G_{\mathfrak{R}}}$. Then there exists an identity neighbourhood $V \subseteq \overline{G_{\mathfrak{R}}}$ such that $aVa^{-1} \subseteq W$, for every open neighbourhood W of e in $\overline{G_{\mathfrak{R}}}$ and $a \in A$.

Proof: Let W be a neighbourhood of e in $\overline{G_{\mathfrak{R}}}$. From lemma 2.10, there is an open set U of e in $G_{\mathfrak{R}}$ such that $U \subseteq U^n \subseteq W$, for every $n \in \mathbb{N} - \{0\}$. Since A is compact, there exists an open cover $A \subseteq UM$ such that M is a finite subset of A. Consider $V = \bigcap_{x \in M} x^{-1}Ux$. Then e ∈ V is open in $G_{\mathfrak{R}}$. Also, by theorem 2.11, V is open in $\overline{G_{\mathfrak{R}}}$. Now we choose an element $a \in A$ such that $a = ux$, for some $u \in U$ and $x \in M$. Hence, $aVa^{-1} = uxVx^{-1}u^{-1} \subseteq uUu^{-1} \subseteq U^3 \subseteq W$, for any $a \in A$.

Theorem 3.4. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $\overline{G_{\mathfrak{R}}}$ is a group and $H_{\mathfrak{R}}$ is a subgroup of $\overline{G_{\mathfrak{R}}}$. If $H_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$, then $H_{\mathfrak{R}}$ is closed in $\overline{G_{\mathfrak{R}}}$.

Proof: The rough quotient space $\overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}} = \{aH_{\mathfrak{R}} : a \in \overline{G_{\mathfrak{R}}}\}$. It is a disjoint open cover of $\overline{G_{\mathfrak{R}}}$. Since $H_{\mathfrak{R}}$ is open, $aH_{\mathfrak{R}}$ is also open. Therefore, the complement of $H_{\mathfrak{R}} = \bigcup_{a \notin H_{\mathfrak{R}}} aH_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. Hence $H_{\mathfrak{R}}$ is closed in $\overline{G_{\mathfrak{R}}}$.

Theorem 3.5. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $\overline{G_{\mathfrak{R}}}$ is a group and $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. If A is a compact open neighbourhood of e in $\overline{G_{\mathfrak{R}}}$, there is a compact subgroup $H_{\mathfrak{R}}$ of $\overline{G_{\mathfrak{R}}}$ such that $H_{\mathfrak{R}} \subseteq A$.

Proof: Since $A \subseteq \overline{G_{\mathfrak{R}}}$ is an open neighbourhood of e, there exists a symmetric open neighbourhood V of e in $G_{\mathfrak{R}}$ such that $VV \subseteq A$ and by lemma 2.10, $V \subseteq V^n \subseteq A$, for every $n \in \mathbb{N} - \{0\}$. Now consider $H_{\mathfrak{R}} = \bigcup_{n \in \mathbb{N} - \{0\}} V^n$. Then $H_{\mathfrak{R}}$ is open in $G_{\mathfrak{R}}$ and $H_{\mathfrak{R}} \subseteq A$. Since $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$, $H_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. Let us prove $H_{\mathfrak{R}}$ is a subgroup of $\overline{G_{\mathfrak{R}}}$. Let $a, b \in H_{\mathfrak{R}}$. Then $a \in V^n$ and $b \in V^m$, for some $n, m \in \mathbb{N} - \{0\}$ which implies $ab \in V^{n+m} \in H_{\mathfrak{R}}$. Also, $a \in H_{\mathfrak{R}}$ implies $a \in V^n$ and $a^{-1} \in (V^n)^{-1} = (V^{-1})^n = V^n \in H_{\mathfrak{R}}$. Therefore, $H_{\mathfrak{R}}$ is a subgroup of $\overline{G_{\mathfrak{R}}}$. Applying theorem 3.4, $H_{\mathfrak{R}}$ is closed in $\overline{G_{\mathfrak{R}}}$. Hence $H_{\mathfrak{R}}$ is a compact subgroup of $\overline{G_{\mathfrak{R}}}$.

Proposition 3.6. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $\overline{G_{\mathfrak{R}}}$ is a group and $H_{\mathfrak{R}}$ be a locally compact subgroup of a Hausdorff topological group $\overline{G_{\mathfrak{R}}}$. Then $H_{\mathfrak{R}}$ is closed in $\overline{G_{\mathfrak{R}}}$.

Proof: Since $H_{\mathfrak{R}}$ be a locally compact subgroup of a Hausdorff topological group $\overline{G_{\mathfrak{R}}}$, $H_{\mathfrak{R}}$ is open in $cl(H_{\mathfrak{R}})$, closure of $H_{\mathfrak{R}}$ and $cl(H_{\mathfrak{R}})$ is a topological rough subgroup in $\overline{G_{\mathfrak{R}}}$. Therefore, by theorem 3.4, $H_{\mathfrak{R}}$ is closed in $\overline{G_{\mathfrak{R}}}$.

Theorem 3.7. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $\overline{G_{\mathfrak{R}}}$ is a Hausdorff topological group and $H_{\mathfrak{R}}$ is a compact subgroup $\overline{G_{\mathfrak{R}}}$. Then the rough quotient mapping $\varphi: \overline{G_{\mathfrak{R}}} \to \overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}}$ is perfect.

Proof: Let M be a closed subset of $\overline{G_{\mathfrak{R}}}$. Then by the second closure lemma, $MH_{\mathfrak{R}}$ is closed in $\overline{G_{\mathfrak{R}}}$. That is, $\varphi(M)$ is closed in rough quotient space $\overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}}$. Therefore, the rough quotient mapping φ is closed. Let $b \in \overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}}$ and $a \in \overline{G_{\mathfrak{R}}}$ such that $\varphi(a) = b$. Then $\varphi^{-1}(b) = aH_{\mathfrak{R}}$, is a compact subset of $\overline{G_{\mathfrak{R}}}$. Hence, by the definition of 2.14, the rough quotient mapping φ is perfect.

Theorem 3.8. Suppose M is a compact subset of a topological simple rough group $G_{\mathfrak{R}}$ such that $\overline{G_{\mathfrak{R}}}$ is a group. Then there exists a smallest rough subgroup $H_{\mathfrak{R}}$ in $\overline{G_{\mathfrak{R}}}$ containing M such that $H_{\mathfrak{R}}$ is σ-compact.

Proof: Let $A = M \cup \{e\} \cup M^{-1}$. Since M is a compact subset of a topological simple rough group $G_{\mathfrak{R}}$, A is compact in $G_{\mathfrak{R}}$. Now define the multiplication mapping $f_i: G_{\mathfrak{R}}^i \to \overline{G_{\mathfrak{R}}}$ by $f_i(x_1, x_2, \dots, x_i) = x_1 x_2 \dots x_i$, for $x_1, x_2, \dots, x_i \in G_{\mathfrak{R}}$ and for every $i \in \mathbb{N}$. Since $G_{\mathfrak{R}}$ is a topological simple rough group and continuous image of compact set is compact, the mappings f_i are continuous which implies $f_i(A^i)$ is compact in $\overline{G_{\mathfrak{R}}}$, for every $i \in \mathbb{N}$. Therefore, the rough subgroup $H_{\mathfrak{R}} = \bigcup_{i=1}^{n} f_i(A^i)$ is generated by M. Hence $H_{\mathfrak{R}}$ is σ-compact.

Theorem 3.9. (Open Mapping Theorem I) Let $G_{\mathfrak{R}}$ and $H_{\mathfrak{R}}$ be topological simple rough groups such that $G_{\mathfrak{R}}$ and $H_{\mathfrak{R}}$ are open in $\overline{G_{\mathfrak{R}}}$ and $\overline{H_{\mathfrak{R}}}$. Let $\pi: \overline{G_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}$ be a surjective mapping topological rough group homomorphism. If $\overline{G_{\mathfrak{R}}}$ is a compact space and $\overline{H_{\mathfrak{R}}}$ is a Hausdorff space, then the mapping π is open.

Proof: Since $\overline{G_{\mathfrak{R}}}$ is a compact space and $\overline{H_{\mathfrak{R}}}$ is a Hausdorff space, π is closed. Also, by the continuity of π, the mapping π is a quotient map that means a subset $U \subseteq \overline{H_{\mathfrak{R}}}$ is open if and only if $\pi^{-1}(U)$ is open in $\overline{G_{\mathfrak{R}}}$. Let us prove π is an open mapping. Let V be an open set in $\overline{G_{\mathfrak{R}}}$. Then $\pi^{-1}\left(\pi(V)\right) = \mathcal{K}_\pi V$ is open in $\overline{G_{\mathfrak{R}}}$, where $\mathcal{K}_\pi$ is the rough kernel of π. Now consider $U = \pi(V)$ that implies $\pi^{-1}(U)$ is open in $\overline{G_{\mathfrak{R}}}$. Since π is a quotient map, $U = \pi(V)$ is open. Hence the mapping π is open.

Proposition 3.10. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $G_{\mathfrak{R}}$ be an open set in $\overline{G_{\mathfrak{R}}}$. Suppose the subset $A \subseteq G_{\mathfrak{R}}$ and $int(xA \cap G_{\mathfrak{R}}) \neq \emptyset$. Then $int(A) \neq \emptyset$, where int means interior of the set.

Proof: Consider an arbitrary element $b \in int(xA \cap G_{\mathfrak{R}})$. Then there is a point $a \in A$ such that $b = xa$. Since $G_{\mathfrak{R}}$ is a topological simple rough group and $G_{\mathfrak{R}}$ is an open set in $\overline{G_{\mathfrak{R}}}$, there is a neighbourhood U of a in $G_{\mathfrak{R}}$ such that $xU \subseteq int(xA \cap G_{\mathfrak{R}})$ which implies $xU \subseteq xA$ that is, $U \subseteq A$. Hence, $int(A) \neq \emptyset$.

Theorem 3.11. (Open Mapping Theorem II) Let $G_{\mathfrak{R}}$ and $H_{\mathfrak{R}}$ be two locally compact Hausdorff topological simple rough groups such that $G_{\mathfrak{R}}$ and $H_{\mathfrak{R}}$ are open in $\overline{G_{\mathfrak{R}}}$ and $\overline{H_{\mathfrak{R}}}$. Suppose a surjective mapping $\pi: G_{\mathfrak{R}} \to H_{\mathfrak{R}}$ is a continuous rough homomorphism and $G_{\mathfrak{R}}$ is a σ-compact space. Then π is an open mapping.

Proof: Let $U \subseteq G_{\mathfrak{R}}$ be a symmetric identity neighbourhood. Since $G_{\mathfrak{R}}$ is locally compact, there exists a symmetric open neighbourhood N of e in $G_{\mathfrak{R}}$ such that $cl(N)$ is compact and $cl(N)cl(N) \subseteq U$, where $cl(N)$ is the closure of N. Since N is open, xN is open, for every $x \in G_{\mathfrak{R}}$. So, $\bigcup_{x \in G_{\mathfrak{R}}} xN$ covers $G_{\mathfrak{R}}$. Therefore, $G_{\mathfrak{R}} = G_{\mathfrak{R}} \cap \bigcup_{x \in G_{\mathfrak{R}}} xN$. Since $G_{\mathfrak{R}}$ is a σ-compact space, there exists a countable set $\{x_i\}_{i \in \mathbb{N}}$, such that $G_{\mathfrak{R}} = G_{\mathfrak{R}} \cap \bigcup_{i \in \mathbb{N}} x_i N$. Since $cl(N)$ is compact and π is continuous, $\pi(G_{\mathfrak{R}}) = \pi(G_{\mathfrak{R}}) \cap \bigcup_{i \in \mathbb{N}} \pi(x_i cl(N))$, for every $x_i \in G_{\mathfrak{R}}$ implies $H_{\mathfrak{R}} = H_{\mathfrak{R}} \cap \bigcup_{i \in \mathbb{N}} \pi(x_i)\, \pi(cl(N)) = H_{\mathfrak{R}} \cap$

$\bigcup_{i\in\mathbb{N}} y_i\,\pi(cl(N))$, where $y_i = \pi(x_i) \in H_{\mathfrak{R}}$. Thus, $y_i\,\pi(cl(N))$ is closed in $\overline{H_{\mathfrak{R}}}$, for every $i \in \mathbb{N}$. Therefore, $H_{\mathfrak{R}} \cap y_i\,\pi(cl(N))$ is closed in $H_{\mathfrak{R}}$. Since $H_{\mathfrak{R}}$ is locally compact, $int(H_{\mathfrak{R}} \cap y_i\,\pi(cl(N))) \neq \emptyset$, for every $i \in \mathbb{N}$. By proposition 3.10, we get $int(\pi(cl(N))) \neq \emptyset$. Then there exists an open set $V \subseteq H_{\mathfrak{R}}$ such that $V \subseteq \pi(cl(N))$. Let $v \in V$. Then there exists a point $n \in cl(N)$ such that $\pi(n) = v$. Therefore, $e' \in v^{-1}V \subseteq v^{-1}\pi(cl(N)) = \pi(n^{-1})\pi(cl(N)) \subseteq \pi(n^{-1}cl(N)) \subseteq \pi(cl(N)cl(N)) \subseteq \pi(U)$. Hence π is an open mapping.

4. Topological rough group homomorphism:

Proposition 4.1. Let $G_{\mathfrak{R}}$ and $H_{\mathfrak{R}}$ be topological simple rough groups such that $G_{\mathfrak{R}}$ and $H_{\mathfrak{R}}$ are open in $\overline{G_{\mathfrak{R}}}$ and $\overline{H_{\mathfrak{R}}}$. Let the map $f: \overline{G_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}$ be a continuous topological rough group homomorphism. Suppose for every open neighbourhood N of e in $\overline{G_{\mathfrak{R}}}$, $f(N)$ has a non-empty open set in $\overline{H_{\mathfrak{R}}}$. Then f is an open mapping.

Proof: Let U be an open neighbourhood of e in $\overline{G_{\mathfrak{R}}}$ such that $U^{-1}U \subseteq N$. But by the hypothesis, $f(U)$ has a non-empty open set in $\overline{H_{\mathfrak{R}}}$. Consider that open set V in $\overline{H_{\mathfrak{R}}}$. So, $V^{-1}V$ is an identity neighbourhood. Therefore, $V^{-1}V \subseteq f(U)^{-1}f(U) = f(U^{-1}U) \subseteq f(N)$ which implies $f(N)$ has an identity in $\overline{H_{\mathfrak{R}}}$. Let $b \in f(N)$. Since f is a continuous topological rough group homomorphism, there exists an arbitrary element $a \in$ N such that $f(a) = b$ and $aU \subseteq N$. Also, $V \subseteq f(U)$ and bV is an open neighbourhood in $\overline{H_{\mathfrak{R}}}$. Then $bV \subseteq f(aU) \subseteq f(N)$. Hence the map f is open.

Proposition 4.2.

(i) Let $G_{\mathfrak{R}}$, $H_{\mathfrak{R}}$ and $K_{\mathfrak{R}}$ be simple rough groups. Let $f: \overline{G_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}$ and $g: \overline{G_{\mathfrak{R}}} \to \overline{K_{\mathfrak{R}}}$ be rough group homomorphisms, where $g(\overline{G_{\mathfrak{R}}}) = \overline{K_{\mathfrak{R}}}$ and $\mathcal{K}_f \subseteq \mathcal{K}_g$, where $\mathcal{K}_f$ and $\mathcal{K}_g$ represent the rough kernel of f and g. Then $h: \overline{K_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}$ is a rough group homomorphism such that $f = h \circ g$.
(ii) Let $G_{\mathfrak{R}}$, $H_{\mathfrak{R}}$ and $K_{\mathfrak{R}}$ be topological simple rough groups such that $G_{\mathfrak{R}}$, $H_{\mathfrak{R}}$ and $K_{\mathfrak{R}}$ are open in $\overline{G_{\mathfrak{R}}}, \overline{H_{\mathfrak{R}}}$ and $\overline{K_{\mathfrak{R}}}$. Suppose $g^{-1}(U) \subseteq f^{-1}(V)$, for every identity neighbourhood U in $\overline{K_{\mathfrak{R}}}$, there exists an identity neighbourhood V in $\overline{H_{\mathfrak{R}}}$, then $h: \overline{K_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}$ is continuous.

Proof:

(i) Since f and g are rough group homomorphism and $f = h \circ g$, h is a rough group homomorphism.
(ii) Let U be an identity neighbourhood in $\overline{H_{\mathfrak{R}}}$. By the hypothesis, there exists an identity neighbourhood V in $\overline{H_{\mathfrak{R}}}$ such that $g^{-1}(V) \subseteq f^{-1}(U)$. Consider $N = g^{-1}(V)$. Therefore, $f(N) \subseteq U$ and $h(V) = f(N)$ which implies $h(V) \subseteq U$. Hence h is continuous at e in $\overline{K_{\mathfrak{R}}}$, which implies h is continuous.

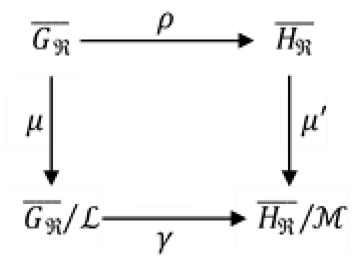
Corollary 4.3. Let $G_{\mathfrak{R}}$, $H_{\mathfrak{R}}$ and $K_{\mathfrak{R}}$ be topological simple rough groups, where $G_{\mathfrak{R}}$, $H_{\mathfrak{R}}$ and $K_{\mathfrak{R}}$ are open in $\overline{G_{\mathfrak{R}}}, \overline{H_{\mathfrak{R}}}$ and $\overline{K_{\mathfrak{R}}}$. Let $f: \overline{G_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}$ and $g: \overline{G_{\mathfrak{R}}} \to \overline{K_{\mathfrak{R}}}$ be continuous rough group homomorphisms such that $g(\overline{G_{\mathfrak{R}}}) = \overline{K_{\mathfrak{R}}}$ and $\mathcal{K}_f \subseteq \mathcal{K}_g$, where $\mathcal{K}_f$ and $\mathcal{K}_g$ represent the rough kernel of f and g. Suppose g is open, then there exists a continuous rough group homomorphism $h: \overline{K_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}$ such that $f = h \circ g$.

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 3 (2025)

Proof: By proposition 4.2 (i), there exists a rough group homomorphism $h: \overline{K_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}$ such that $f = h \circ g$. Let U be an open set in $\overline{H_{\mathfrak{R}}}$. Then $h^{-1}(U) = g(f^{-1}(U))$. Since f is continuous and g is open, $h^{-1}(U)$ is open. Therefore, h is continuous and hence h is continuous rough group homomorphism.

Proposition 4.4. Let $G_{\mathfrak{R}}$ and $H_{\mathfrak{R}}$ be topological simple rough groups such that $\overline{G_{\mathfrak{R}}}$ and $\overline{K_{\mathfrak{R}}}$ are groups. Let $\mathcal{L}$ be a subgroup of $G_{\mathfrak{R}}$ and normal subgroup of $\overline{G_{\mathfrak{R}}}$. Let $\mathcal{M} = \rho(\mathcal{L})$ be a subgroup of $H_{\mathfrak{R}}$ and normal subgroup of $\overline{H_{\mathfrak{R}}}$. Suppose $\rho: \overline{G_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}$ is a rough homeomorphism. Then the quotient map $\gamma: \overline{G_{\mathfrak{R}}}/\mathcal{L} \to \overline{H_{\mathfrak{R}}}/\mathcal{M}$ is a topological rough group homeomorphism which is defined by $\gamma(a\mathcal{L}) = b\mathcal{M}$, for $a \in \overline{G_{\mathfrak{R}}}$ and $b = \rho(a)$.

Proof: Consider the quotient maps $\mu: \overline{G_{\mathfrak{R}}} \to \overline{G_{\mathfrak{R}}}/\mathcal{L}$ and $\mu': \overline{H_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}/\mathcal{M}$. Then $\mu' \circ \rho = \gamma \circ \mu$. Since the maps μ', ρ and μ are open continuous rough homomorphisms, the map γ is an open



continuous homomorphism. Let $a\mathcal{L} \in \overline{G_{\mathfrak{R}}}/\mathcal{L}$ and $\gamma(a\mathcal{L}) = \mathcal{M}$. Then $\mu'(b) = \mathcal{M}$ and $b\mathcal{M} = \mathcal{M}$ which implies $b = \rho(a) \in \mathcal{M}$. Since ρ is a rough homeomorphism and $\mathcal{M} = \rho(\mathcal{L})$, $\rho(a) = \rho(c)$, for some $c \in \mathcal{L}$ which implies $a = c$. Therefore, $a \in \mathcal{L}$ implies the kernel of γ is $\mathcal{L}$ that mean the quotient map γ is injective. Hence the quotient map γ is a topological rough group homeomorphism.

5. Rough isomorphism:

Theorem 5.1. (Rough Isomorphism Theorem - I) Let $G_{\mathfrak{R}}$ and $H_{\mathfrak{R}}$ be topological simple rough groups such that $\overline{G_{\mathfrak{R}}}$ is a group. Let $\rho: \overline{G_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}$ be a topological rough group homomorphism and $\mathcal{K}_\rho$ be the rough kernel of ρ. Then the map $\varphi: \overline{G_{\mathfrak{R}}}/\mathcal{K}_\rho \to \overline{H_{\mathfrak{R}}}$ is a continuous rough isomorphism which is defined by $\varphi(a\mathcal{K}_\rho) = \rho(a)$, for every $a \in \overline{G_{\mathfrak{R}}}$. Also, if ρ is open, the map φ is a rough homeomorphism.

Proof: Let $\mu: \overline{G_{\mathfrak{R}}} \to \overline{G_{\mathfrak{R}}}/\mathcal{K}_\rho$ be a quotient map. Then $\rho = \varphi \circ \mu$.

(i) φ is injective:

Since ρ is a topological rough group homomorphism and $\mathcal{K}_\rho$ be the rough kernel of ρ, for some a, $b \in \overline{G_{\mathfrak{R}}}$, $a\mathcal{K}_\rho = b\mathcal{K}_\rho$ which implies $ab^{-1} \in \mathcal{K}_\rho$. Then $\rho(a)\rho(b)^{-1} = \rho(ab^{-1}) = e'$, where e' is the identity in $\overline{H_{\mathfrak{R}}}$. Therefore, $\rho(a) = \rho(b)$ implies that φ is one-one.

(ii) φ is surjective:

Let $x \in \overline{H_{\mathfrak{R}}}$. Then there exists an element $a \in \overline{G_{\mathfrak{R}}}$ such that $\rho(a) = x$. Since $\varphi(a\mathcal{K}_\rho) = \rho(a)$, for every element of $\overline{H_{\mathfrak{R}}}$ is in the image of φ. Hence φ is onto.

(iii) *φ is rough homomorphism:*

Let a, $b \in \overline{G_{\mathfrak{R}}}$. Then $a\mathcal{K}_\rho, b\mathcal{K}_\rho \in \overline{G_{\mathfrak{R}}}/\mathcal{K}_\rho$. Since ρ is a topological rough group homomorphism, $\varphi(ab\mathcal{K}_\rho) = \rho(ab) = \rho(a)\rho(b) = \varphi(a\mathcal{K}_\rho)\varphi(b\mathcal{K}_\rho)$. Therefore, φ is a rough homomorphism.

(iv) φ *is continuous:*

Since ρ is continuous and μ is a quotient map, $\varphi = \rho \circ \mu^{-1}$ is continuous. Hence the map $\varphi: \overline{G_{\mathfrak{R}}}/\mathcal{K}_{\rho} \to \overline{H_{\mathfrak{R}}}$ is a continuous rough isomorphism.

(v) φ *is a rough homeomorphism if* ρ *is open:*

Let U be an open neighbourhood in $\overline{G_{\mathfrak{R}}}$. Since ρ is open, $\rho(U)$ is open. Then $\varphi(U) = \rho(\mu^{-1}(U))$ is open. Therefore, φ^{-1} is continuous. Hence φ is a rough homeomorphism.

Theorem 5.2. (Rough Isomorphism Theorem - II) Let $G_{\mathfrak{R}}$ and $H_{\mathfrak{R}}$ be topological simple rough groups. Let $\rho: \overline{G_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}$ be a topological rough group homomorphism such that $\overline{G_{\mathfrak{R}}}$ and $\overline{H_{\mathfrak{R}}}$ are groups and $\mathcal{L}$ be a normal subgroup of $\overline{H_{\mathfrak{R}}}$. Define $\mathcal{M} = \rho^{-1}(\mathcal{L})$ and $\mathcal{K}_{\rho}$ be the rough kernel of ρ. Then the map $\varphi: (\overline{G_{\mathfrak{R}}}/\mathcal{K}_{\rho})/(\mathcal{M}/\mathcal{K}_{\rho}) \to \overline{H_{\mathfrak{R}}}/\mathcal{L}$ is a topological rough group homeomorphism.

Proof: Consider the rough quotient map $\mu: \overline{H_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}/\mathcal{L}$ which is an open continuous rough homomorphism. Then $\mu \circ \rho : \overline{G_{\mathfrak{R}}} \to \overline{H_{\mathfrak{R}}}/\mathcal{L}$ is also a continuous open rough homomorphism. Let $\rho' = \mu \circ \rho$. Then the rough kernel of ρ', $\mathcal{K}_{\rho'} = \{a \in \overline{G_{\mathfrak{R}}} : \rho'(a) = \mathcal{L}\}$. Since $\rho'(a) = \mu(\rho(a)) = \rho(a)\mathcal{L}$, $\mathcal{K}_{\rho'} = \{a \in \overline{G_{\mathfrak{R}}} : \rho(a) \in \mathcal{L}\}$. But $\mathcal{M} = \rho^{-1}(\mathcal{L}) = \{a \in \overline{G_{\mathfrak{R}}} : \rho(a) \in \mathcal{L}\}$. Therefore, $\mathcal{K}_{\rho'} = \mathcal{M}$. By theorem 5.1, $\overline{G_{\mathfrak{R}}}/\mathcal{M}$ is topological rough group homeomorphism to $\overline{H_{\mathfrak{R}}}/\mathcal{L}$. In the similar way, define the map $\varphi: \overline{G_{\mathfrak{R}}}/\mathcal{K}_{\rho} \to \overline{H_{\mathfrak{R}}}/\mathcal{L}$ by $\varphi(a\mathcal{K}_{\rho}) = \rho(a)\mathcal{L}$. Then the rough kernel of φ, $\mathcal{K}_{\varphi} = \{a\mathcal{K}_{\rho} \in \overline{G_{\mathfrak{R}}}/\mathcal{K}_{\rho} : \varphi(a\mathcal{K}_{\rho}) = \mathcal{L}\}$ which implies $\mathcal{K}_{\varphi} = \{a\mathcal{K}_{\rho} \in \overline{G_{\mathfrak{R}}}/\mathcal{K}_{\rho} : \rho(a) \in \mathcal{L}\}$. But $\mathcal{M} = \rho^{-1}(\mathcal{L}) = \{a \in \overline{G_{\mathfrak{R}}} : \rho(a) \in \mathcal{L}\}$. That is, the element $a \in \mathcal{M}$ mapped to the identity $\mathcal{L}$ in $\overline{H_{\mathfrak{R}}}/\mathcal{L}$. Therefore, the element $a\mathcal{K}_{\rho} \in \mathcal{M}/\mathcal{K}_{\rho}$ mapped to the identity $\mathcal{L}$ in $\overline{H_{\mathfrak{R}}}/\mathcal{L}$ which implies the rough kernel of φ, $\mathcal{K}_{\varphi} = \mathcal{M}/\mathcal{K}_{\rho}$. By theorem 5.1, $(\overline{G_{\mathfrak{R}}}/\mathcal{K}_{\rho})/(\mathcal{M}/\mathcal{K}_{\rho})$ is topological rough group homeomorphism to $\overline{H_{\mathfrak{R}}}/\mathcal{L}$.

Theorem 5.3. (Rough Isomorphism Theorem - III) Let $G_{\mathfrak{R}}$ be a topological simple rough group and let $\mathcal{L}$ be a normal rough subgroup of $\overline{G_{\mathfrak{R}}}$. For any topological rough subgroup $H_{\mathfrak{R}}$ of $G_{\mathfrak{R}}$, if $\overline{G_{\mathfrak{R}}}$, $\overline{H_{\mathfrak{R}}}$ and $\bar{\mathcal{L}}$ are groups, where $\bar{\mathcal{L}}$ is the upper approximation of $\mathcal{L}$, then the rough quotient map $\gamma: \overline{H_{\mathfrak{R}}}\mathcal{L}/\mathcal{L} \to \varphi(\overline{H_{\mathfrak{R}}})$ is a topological rough group homeomorphism, where $\varphi: \overline{G_{\mathfrak{R}}} \to \overline{G_{\mathfrak{R}}}/\mathcal{L}$ is a rough quotient map and $\varphi(\overline{H_{\mathfrak{R}}})$ is a subgroup of $\overline{G_{\mathfrak{R}}}/\mathcal{L}$.

Proof: By our assumption, $\overline{H_{\mathfrak{R}}}\mathcal{L} = \varphi^{-1}(\varphi(\overline{H_{\mathfrak{R}}}))$. Let $\mu: \overline{H_{\mathfrak{R}}}\mathcal{L} \to \varphi(\overline{H_{\mathfrak{R}}})$ defined by $\mu(a\mathcal{L}) = \varphi(a)$. Since φ is homomorphism, the rough quotient map μ is homomorphism. Then the rough kernel of μ, $\mathcal{K}_{\mu} = \{a \in \overline{H_{\mathfrak{R}}}\mathcal{L} : \mu(a\mathcal{L}) = e,$ is the identity of $\varphi(\overline{H_{\mathfrak{R}}})\}$. But the identity of $\overline{G_{\mathfrak{R}}}/\mathcal{L}$ is $\mathcal{L}$ and $\varphi(\overline{H_{\mathfrak{R}}})$ is a subgroup of $\overline{G_{\mathfrak{R}}}/\mathcal{L}$. So, $\mathcal{K}_{\mu} = \{a\mathcal{L} \in \overline{H_{\mathfrak{R}}}\mathcal{L} : \varphi(a) = \mathcal{L}\} = \mathcal{L}$. Therefore, by the theorem 5.1, the rough quotient map $\gamma: \overline{H_{\mathfrak{R}}}\mathcal{L}/\mathcal{L} \to \varphi(\overline{H_{\mathfrak{R}}})$ is a topological rough group homeomorphism.

6. Rough double coset spaces:

Definition 6.1. Let $G_{\mathfrak{R}}$ be a rough group such that $\overline{G_{\mathfrak{R}}}$ is a group and $H_{\mathfrak{R}}$, $K_{\mathfrak{R}}$ be rough subgroups in $G_{\mathfrak{R}}$. If $H_{\mathfrak{R}}$, $K_{\mathfrak{R}}$ be subgroups in $\overline{G_{\mathfrak{R}}}$, then

$$K_{\mathfrak{R}} \backslash \overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}} = \{K_{\mathfrak{R}} x H_{\mathfrak{R}} : x \in \overline{G_{\mathfrak{R}}}\}$$

is a *rough double coset space* ($\mathcal{D}_{\mathfrak{C}}$). Also, for any $x, y \in \overline{G_{\mathfrak{R}}}$, either $K_{\mathfrak{R}} x H_{\mathfrak{R}} = K_{\mathfrak{R}} y H_{\mathfrak{R}}$ or $K_{\mathfrak{R}} x H_{\mathfrak{R}} \cap K_{\mathfrak{R}} y H_{\mathfrak{R}} = \emptyset$.

Lemma 6.2. Let $G_{\mathfrak{R}}$ be a rough group such that $\overline{G_{\mathfrak{R}}}$ is a group and $H_{\mathfrak{R}}$, $K_{\mathfrak{R}}$ be rough subgroups in $G_{\mathfrak{R}}$. If $H_{\mathfrak{R}}$, $K_{\mathfrak{R}}$ be subgroups in $\overline{G_{\mathfrak{R}}}$, then all the double cosets form a partition of $\overline{G_{\mathfrak{R}}}$.

Proof: Let $x, y \in \overline{G_{\mathfrak{R}}}$. Then the corresponding cosets are $K_{\mathfrak{R}}xH_{\mathfrak{R}}$, $K_{\mathfrak{R}}yH_{\mathfrak{R}}$. These are either disjoint or coincide. Consider an arbitrary element $z \in K_{\mathfrak{R}}xH_{\mathfrak{R}} \cap K_{\mathfrak{R}}yH_{\mathfrak{R}}$, that is, $z = k_1xh_1 = k_2yh_2$, for some $k_1, k_2 \in K_{\mathfrak{R}}$ and $h_1, h_2 \in H_{\mathfrak{R}}$. Therefore, $x \in K_{\mathfrak{R}}yH_{\mathfrak{R}}$ which implies $K_{\mathfrak{R}}xH_{\mathfrak{R}} \subseteq K_{\mathfrak{R}}yH_{\mathfrak{R}}$. Similarly, we can prove $K_{\mathfrak{R}}yH_{\mathfrak{R}} \subseteq K_{\mathfrak{R}}xH_{\mathfrak{R}}$. Hence, $K_{\mathfrak{R}}xH_{\mathfrak{R}} = K_{\mathfrak{R}}yH_{\mathfrak{R}}$.

Lemma 6.3. Let $H_{\mathfrak{R}}$ and $K_{\mathfrak{R}}$ be rough subgroups of a topological simple rough group $G_{\mathfrak{R}}$ such that $\overline{G_{\mathfrak{R}}}$ is a group and $H_{\mathfrak{R}}$, $K_{\mathfrak{R}}$ are subgroups of $\overline{G_{\mathfrak{R}}}$. If A is a compact subset in $\mathcal{D}_{\mathfrak{C}}$ and the rough quotient map $\varphi: \overline{G_{\mathfrak{R}}} \to \mathcal{D}_{\mathfrak{C}}$ is defined by $\varphi(x) = K_{\mathfrak{R}}xH_{\mathfrak{R}}$, for $x \in \overline{G_{\mathfrak{R}}}$, then there exists a compact subset B in $\overline{G_{\mathfrak{R}}}$ such that $\varphi(B) = A$.

Proof: Let U be an identity neighbourhood in $\overline{G_{\mathfrak{R}}}$ which has a compact closure. Since A is compact, there exists a cover $\bigcup_{i=1}^{n} \varphi(x_iU)$, where $x_1, x_2, \ldots x_n \in \overline{G_{\mathfrak{R}}}$. That is, $A \subseteq \bigcup_{i=1}^{n} \varphi(x_iU)$. Now consider $B = \varphi^{-1}(A) \cap \bigcup_{i=1}^{n} x_i cl(U)$, $cl(U)$ means closure of U. Since $cl(U)$ is compact, $\varphi^{-1}(A)$ lies in the compact set $\bigcup_{i=1}^{n} x_i cl(U)$. Therefore, B is compact and $\varphi(B) = \varphi(\varphi^{-1}(A) \cap \bigcup_{i=1}^{n} x_i cl(U)) = A \cap \bigcup_{i=1}^{n} \varphi(x_i cl(U)) = A$.

Lemma 6.4. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $\overline{G_{\mathfrak{R}}}$ is a group and $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$. Suppose $K_{\mathfrak{R}}$ and $H_{\mathfrak{R}}$ are closed in $\overline{G_{\mathfrak{R}}}$ and $K_{\mathfrak{R}}$ is compact in $\overline{G_{\mathfrak{R}}}$. Then $\mathcal{D}_{\mathfrak{C}}$ is a closed set in $\overline{G_{\mathfrak{R}}}$.

Proof: Since $H_{\mathfrak{R}}$ is closed in $\overline{G_{\mathfrak{R}}}$, $xH_{\mathfrak{R}}$ is closed in $\overline{G_{\mathfrak{R}}}$. Then using theorem 3.2, $K_{\mathfrak{R}}xH_{\mathfrak{R}}$ is closed in $\overline{G_{\mathfrak{R}}}$ that is, $\mathcal{D}_{\mathfrak{C}}$ is a closed set in $\overline{G_{\mathfrak{R}}}$.

Proposition 6.5. Let $H_{\mathfrak{R}}$ and $K_{\mathfrak{R}}$ be rough subgroups of a topological simple rough group $G_{\mathfrak{R}}$ such that $\overline{G_{\mathfrak{R}}}$ is a group and $H_{\mathfrak{R}}$, $K_{\mathfrak{R}}$ are subgroups of $\overline{G_{\mathfrak{R}}}$. If $K_{\mathfrak{R}}$ is a compact subset in $\overline{G_{\mathfrak{R}}}$ and $H_{\mathfrak{R}}$ is a closed subset in $\overline{G_{\mathfrak{R}}}$, then the rough quotient map $\varphi: \overline{G_{\mathfrak{R}}} \to \mathcal{D}_{\mathfrak{C}}$ is open.

Proof: Let U be an open set in $\overline{G_{\mathfrak{R}}}$. Then $\varphi^{-1}\varphi(U) = K_{\mathfrak{R}}UH_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$ which implies $\varphi(U)$ is open. Therefore, the rough quotient map φ is open.

Proposition 6.6. Let $H_{\mathfrak{R}}$ and $K_{\mathfrak{R}}$ be rough subgroups of a topological simple rough group $G_{\mathfrak{R}}$ such that $\overline{G_{\mathfrak{R}}}$ is a group and $H_{\mathfrak{R}}$, $K_{\mathfrak{R}}$ are subgroups of $\overline{G_{\mathfrak{R}}}$. Then the rough double coset space $\mathcal{D}_{\mathfrak{C}}$ is regular.

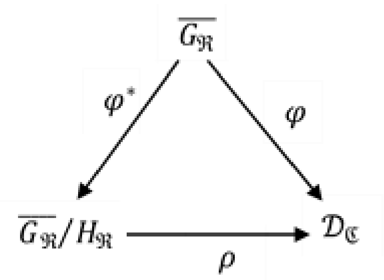
Proof: Consider an arbitrary point $c \in \mathcal{D}_{\mathfrak{C}}$ and $\varphi(x) = c$, for some $x \in \overline{G_{\mathfrak{R}}}$. Then $\varphi^{-1}(c) = K_{\mathfrak{R}}xH_{\mathfrak{R}}$. From theorem 4.2, $\varphi^{-1}(c)$ is closed which implies $\{c\}$ is closed in $\mathcal{D}_{\mathfrak{C}}$. Therefore, for any $c \in \mathcal{D}_{\mathfrak{C}}$, the rough double coset $\mathcal{D}_{\mathfrak{C}}$ is a T_1- space. Let U be an open neighbourhood of c in $\mathcal{D}_{\mathfrak{C}}$. Then there exist identity neighbourhoods V and W in $\overline{G_{\mathfrak{R}}}$ such that $\varphi(Vx) \subseteq U$ and WW $\subseteq$ V. Also, by theorem 3.3, there exists a symmetric identity neighbourhood $N \subseteq W$ in $\overline{G_{\mathfrak{R}}}$ such that $xNx^{-1} \subseteq W$, for every $x \in K_{\mathfrak{R}}$ which implies $NK_{\mathfrak{R}} \subseteq K_{\mathfrak{R}}W$. Since φ is an open mapping, $\varphi(Nx)$ is an open neighbourhood of $\varphi(x)$ in $\mathcal{D}_{\mathfrak{C}}$ and $\varphi(Nx) \subseteq U$. Now let us prove the closure of $\varphi(Nx)$ is contained in U. Let $\varphi(Ny)$ be an open neighbourhood of y in $\overline{G_{\mathfrak{R}}}$ and y be an accumulation point of $\varphi(Nx)$.That is, y in closure of $\varphi(Nx)$. Then $\varphi(Ny) \cap \varphi(Nx) \neq \emptyset$ implies $Ny \cap K_{\mathfrak{R}}NxH_{\mathfrak{R}} \neq \emptyset$. Therefore,

$$y \in NK_{\mathfrak{R}}NxH_{\mathfrak{R}} \subseteq K_{\mathfrak{R}}WNxH_{\mathfrak{R}} \subseteq K_{\mathfrak{R}}WWxH_{\mathfrak{R}} \subseteq K_{\mathfrak{R}}VxH_{\mathfrak{R}} = \varphi(Vx) \subseteq U.$$

So, closure of $\varphi(Nx)$ is contained in U. Hence $\mathcal{D}_{\mathfrak{C}}$ is a regular space.

Proposition 6.7. Let $H_{\mathfrak{R}}$ and $K_{\mathfrak{R}}$ be rough subgroups of a topological simple rough group $G_{\mathfrak{R}}$ such that $\overline{G_{\mathfrak{R}}}$ is a Hausdorff topological group and $H_{\mathfrak{R}}$, $K_{\mathfrak{R}}$ are subgroups of $\overline{G_{\mathfrak{R}}}$. Then the mapping $\rho: \overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}} \to \mathcal{D}_{\mathfrak{C}}$ defined by $\rho(xH_{\mathfrak{R}}) = K_{\mathfrak{R}}xH_{\mathfrak{R}}$, for every $x \in \overline{G_{\mathfrak{R}}}$ is open and perfect.

Proof: Let the mappings $\varphi: \overline{G_{\mathfrak{R}}} \to \mathcal{D}_{\mathfrak{C}}$ and $\varphi^*: \overline{G_{\mathfrak{R}}} \to \overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}}$ defined by $\varphi(x) = K_{\mathfrak{R}}xH_{\mathfrak{R}}$ and $\varphi^*(x) = xH_{\mathfrak{R}}$, for $x \in \overline{G_{\mathfrak{R}}}$. Then $\varphi = \rho \circ \varphi^*$. Since φ and φ^* are continuous and open, ρ is an open mapping.



Let $c \in \mathcal{D}_{\mathfrak{C}}$ such that $\varphi(x) = c$, for some $x \in \overline{G_{\mathfrak{R}}}$. Then $\varphi^{-1}(c) = K_{\mathfrak{R}}xH_{\mathfrak{R}}$ and the preimage $\rho^{-1}(c) = \varphi^*(K_{\mathfrak{R}}xH_{\mathfrak{R}}) = \varphi^*(K_{\mathfrak{R}}x)$, $K_{\mathfrak{R}}x \subseteq \overline{G_{\mathfrak{R}}}$. Since continuous image of a compact set is compact, the set of all preimages, $\rho^{-1}(c)$ is compact in $\overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}}$, for every $c \in \mathcal{D}_{\mathfrak{C}}$. Now let us prove ρ is closed, using the theorem 2.16. Let N be an open neighbourhood of $\varphi^*(K_{\mathfrak{R}}x)$ in $\overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}}$. Then, $K_{\mathfrak{R}}x \subseteq {\varphi^*}^{-1}(N)$ which implies there exists an open neighbourhood $U \subseteq \overline{G_{\mathfrak{R}}}$ such that $K_{\mathfrak{R}}xU \subseteq {\varphi^*}^{-1}(N)$ that is, $\varphi^*(K_{\mathfrak{R}}xU) \subseteq N$. Let $V = \varphi(xU)$ be an open neighbourhood of $c \in \mathcal{D}_{\mathfrak{C}}$. Therefore,

$$\rho^{-1}(V) = \rho^{-1}\big(\varphi(xU)\big) = \varphi^*(\varphi^{-1}(\varphi(xU))) = \varphi^*(K_{\mathfrak{R}}xUH_{\mathfrak{R}}) = \varphi^*(K_{\mathfrak{R}}xU) \subseteq N.$$

Hence, ρ is closed which implies ρ is a perfect mapping.

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