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Laplacian Minimum Dominating Quotient Energy of a Graph

Ramesha M S[1], Purushothama S[2] and Puttaswamy[3]

[1]Department of Mathematics, Government First Grade College, Kuvempunagara Mysore, Karanataka, India
E-mail: profmsr1978@gmail.com
[2]Department of Mathematics, Maharaja Institute of Technology Mysore, Mandya, Karanataka, India
Corresponding author: E-mail: psmandya@gmail.com
[3]Department of Mathematics, P.E.S. College of Engineering, Mandya, Karanataka, India.
E-mail: prof.puttaswamy@gmail.com

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Abstract:
In this paper, we present the idea of Laplacian minimum dominating quotient energy of graph, $LQ_DE(G)$ and compute the Laplacian minimum dominating quotient energy of $LQ_DE(G)$ of few families of graphs. Additionally, we derive bounds for the Laplacian minimum dominating quotient energy, providing a comprehensive understanding of its behavior and properties in different graph structures.

Objectives: Finding the Laplacian minimum dominating quotient energy of different graph

Methods: To establish the upper and lower bounds for the energy of graphs we employ the Standard methods of proofs namely direct methods and using Matlab to compute the minimum pendant dominating partition eigen values of a graph G.

Results: We obtain the Laplacian minimum dominating quotient energy of $LQ_DE(G)$ of well-known families of graphs. Additionally we obtain upper and lower bounds

Conclusions: Nowadays, the study of theory of domination and energy of graph is an important area in Graph theory and also remarkable research is going on in this area. In recent years many scholars are working in this area and also they are introducing new domination parameters. In this paper we have initiated the study of Laplacian minimum dominating quotient energy of graph. We have calculated the energies for some standard family graphs and we have established some bounds for this parameter. Further, we have studied some important properties of Laplacian minimum quotient dominating eigenvalues

Keywords: Laplacian Minimum Dominating set, Minimum Dominating Set, Quotient Energy, Laplacian Dominating Quotient Matrix.

1. Introduction

Let $G = (V, E)$ be a graph with n nodes and m edges. The degree of v_i written by $d(v_i)$ is the number of edges incident with v_i. The maximum node of degree is denoted by $\Delta(G)$ and minimum node of degree is denoted by $\delta(G)$. The adjacency matrix $A_D(G)$ of G is defined by its entries as $a_{ij} = 1$ if $v_iv_j \in E(G)\ or\ v_i \in D\ if\ (i = j)$ where D is a dominating set of G and 0 otherwise. The eigen values of graph G are the eigenvalues of its adjacency matrix$A_D(G)$, denoted by$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. A graph G is considered singular if it has at least one eigenvalue equal to zero. In the case of singular graphs, it is clear that $det(A) = 0$. A graph is said to be nonsingular if all of its eigenvalues are nonzero. A graph G is referred to be k-regular if every node in G has degree k.

The energy of a graph G is defined as $\mathrm{E(G)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} |\lambda_{\mathrm{i}}|$. This concept was introduced by I. Gutman in 1978 [3]. I.Gutman and B.Zhou [4] defined the Laplacian energy of a graph G in the year 2006. Let G be a graph with n nodes and m edges. The Laplacian matrix of the graph G, denoted by $L = L_{i,j}$, is a square matrix of order n. The elements of the Laplacian matrix are defined as

$$L_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if } v_i \text{ and } v_j \text{ are non adjacent,} \\ d_i & if\ i = j. \end{cases}$$

Where d_i is the vertex's v_i degree

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of Laplacian matrix G. Laplacian energy of G is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \lambda_i - \frac{2m}{n} \right|$$

The key characteristics of Laplacian energy, including various upper and lower bounds, have been established in [4, 5] It has been found that Laplacian graph energy has notable applications in areas such as chemical analysis, high-resolution satellite image classification and segmentation, as well as identifying semantic structures in image hierarchies.

In this article, we are defining a matrix, called the Laplacian minimum dominating quotient matrix denoted by $LQ_D(G)$ and we study its eigenvalues and the energy. Further, we study the mathematical aspects of the Laplacian minimum dominating quotient energy of a graph. It is possible that the Laplacian minimum dominating quotient energy discussed in this article could have uses in other fields of science, such as chemistry, and beyond. The graphs under consideration are assumed to be finite, simple, undirected, with no isolated nodes, and of order at least two.

2. Quotient Energy of Graphs

For a graph G, the quotient matrix $Q = Q(G) = q_{ij}$ is a $p \times p$ matrix defined as

$$q_{ij} = \begin{cases} \dfrac{d(v_i)}{d(v_j)}, & for \quad v_i v_j \in E, \\ 0, & otherwise. \end{cases}$$

The characteristic polynomial of $Q(G)$ is $f(G, \lambda) = det\,(Q - \lambda\, I)$. The quotient spectrum of the graph G is the eigenvalues of the matrix Q and is denoted as $Q - Spec(G)$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the spectrum of $Q(G)$. Then the quotient energy is defined as $\mathrm{QE(G)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} |\lambda_{\mathrm{i}}|$. For additional details about quotient energy of a graph refer [6]

3. Minimum Dominating Quotient Energy of Graph

Let G be simple graph of order n with node set $V = \{v_1, v_2, \ldots, v_n\}$ edge set E. Let D be the minimum dominating set of a graph G. The minimum dominating quotient matrix of G is the $n \times n$ matrix defined by $A_Q(G) = a_{ij}$ where

$$a_{ij} = \begin{cases} \dfrac{d(v_i)}{d(v_j)}, & if\ v_i v_j \in E, \\ 1, & if\ v_i = v_j\ and\ v_i \in D, \\ 0, & if\ otherwise\,. \end{cases}$$

The characteristic polynomial of $A_Q(G)$ is indicated by$f(G,\lambda) = det\ (\lambda\, I - A_Q(G))$. The minimum dominating quotient eigenvalues of the graph G are the eigenvalues of $A_Q(G)$. Since $A_Q(G)$ is real and symmetric, its eigenvalues are real numbers and are labelled in non-increasing order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. The minimum dominating quotient energy of G is defined as

$$\mathrm{Q_D E(G)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} |\lambda_{\mathrm{i}}|$$

4. The Laplacian Minimum Dominating Quotient Energy of a Graph

Let $D(G)$ represent the diagonal matrix of the node degrees of the graph G. Then the Laplacian minimum dominating quotient matrix of G is denoted by $LQ_D E(G)$ and is defined as follows $LQ_D E(G) = D(G) - A_D(G)$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigen values $LQ_D E(G)$ organized in non-increasing order. These eigen values are called Laplacian minimum dominating quotient eigen values of G. The Laplacian minimum dominating quotient energy of a graph G is defined as

$$\mathrm{LQ_D E(G)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \left| \lambda_{\mathrm{i}} - \frac{\mathrm{2m}}{\mathrm{n}} \right|$$

Where m is the number of edges of G and $\frac{2m}{n}$ is the average degree of G.

Example.4.1
Let G be a graph with 6 nodes, as illustrated in Figure 4.1. The possible γ −sets are $(i) D_1 = \{v_2, v_4\}$ $(ii) D_2 = \{v_1 v_4\}$

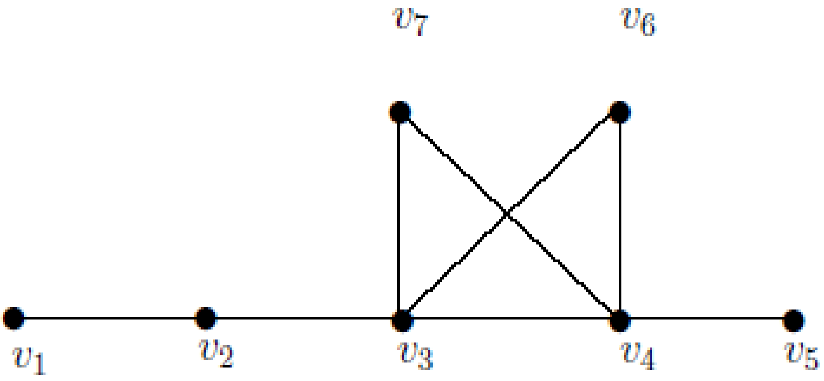

Figure. 4.1

(i) If the $\gamma -$ set is $D_1 = \{v_2, v_4\}$ then

$$A_{D_1}(G) = \begin{pmatrix} 1 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 4 & 2 & 2 \\ 0 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \end{pmatrix} \text{ and } D(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$LQ_D E_{D_1}(G) = D(G) - A_{D_1}(G) = \begin{pmatrix} 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 4 & -1 & 0 & -2 & -2 \\ 0 & 0 & -1 & 3 & -4 & --2 & -2 \\ 0 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & -1/2 & -1/2 & 0 & 2 & 0 \\ 0 & 0 & -1/2 & -1/2 & 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is expressed as
$f_{\{n\}}(G,\lambda) = \lambda^7 - 14\lambda^6 + 72\lambda^5 - 166\lambda^4 + 162\lambda^3 - 30\lambda^2 - 30\lambda + 4 = 0.$
The Laplacian minimum dominating quotient eigen values are $\lambda_1 = -0.5597, \lambda_2 = -0.1842, \lambda_3 = 0.9583, \lambda_4 = 2, \lambda_5 = 2.1960, \lambda_6 = 4.1923,$ λ_7 $= 4.3973$. The mean degree of the graph $\frac{2m}{n} = \frac{2\times 8}{7} = \frac{16}{7}$
Hence, Laplacian minimum dominating quotient energy, $LQ_{D_1}E(G) \approx 12.43638$
(ii) If the $\gamma -$ set is $D_2 = \{v_1, v_4\}$ then

$$A_{D_2}(G) = \begin{pmatrix} 1 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 4 & 2 & 2 \\ 0 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \end{pmatrix} \text{ and } D(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

$$LQ_D E_{D_2}(G) = D(G) - A_{D_2}(G) = \begin{pmatrix} 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 4 & -1 & 0 & -2 & -2 \\ 0 & 0 & -1 & 3 & -4 & -2 & -2 \\ 0 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & -1/2 & -1/2 & 0 & 2 & 0 \\ 0 & 0 & -1/2 & -1/2 & 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is expressed as
$f_{\{n\}}(G,\lambda) = \lambda^7 - 14\lambda^6 + 71\lambda^5 - 155\lambda^4 + 121\lambda^3 + 27\lambda^2 - 48\lambda - 4 = 0.$
The Laplacian minimum dominating quotient eigen values are$\lambda_1 = -0.4956, \lambda_2 = -0.0811, \lambda_3 = 1.0381, \lambda_4 = 2.2342, \lambda_5 = 5.09551, \lambda_6 = 4.2094, \lambda_7 = 2$. Average degree of the graph $\frac{2m}{n} = \frac{2\times 8}{7} = \frac{16}{7}$

Thus, the Laplacian minimum dominating quotient energy, $LQ_{D_2}E(G) \approx 12.8677$
Therefore, based on the above example, it is evident that the Laplacian minimum dominating quotient energy of a graph G is influenced by the minimum dominating set of G

5. Laplacian Minimum Dominating Quotient Energy of Some Standard Graphs

Theorem 5.1. If K_n is the complete graph with n nodes, then $LQ_DE(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}$.
Proof: Let K_n be the complete graph with node set $V = \{v_1, v_2, \dots, v_n\}$. The γ −set $D = \{v_1\}$.

$$A_D(K_n) = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 1 & 1 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 1 & 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

And

$$D(K_n) = \begin{pmatrix} n-1 & 0 & 0 & \dots & 0 & 0 \\ 0 & n-1 & 0 & \dots & 0 & 0 \\ 0 & 0 & n-1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & n-1 & 0 \\ 0 & 0 & 0 & \dots & 0 & n-1 \end{pmatrix}.$$

$$LQ_DE(K_n) = D(K_n) - A_D(G) = \begin{pmatrix} n-2 & -1 & -1 & \dots & -1 & -1 \\ -1 & n-1 & 1 & \dots & -1 & -1 \\ -1 & -1 & n-1 & \dots & -1 & -1 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ -1 & -1 & -1 & \dots & n-1 & -1 \\ -1 & -1 & -1 & \dots & -1 & n-1 \end{pmatrix}.$$

Its characteristic polynomial is

$$[\lambda - n]^{(n-2)}[\lambda^2 - (n-1)\lambda - 1]$$

The Laplacian minimum dominating quotient eigen values are:

$$\lambda = n[(n-2)time], \qquad \lambda = \frac{(n-1) \pm \sqrt{n^2 - 2n + 5}}{2} \ [one\ time\ each]$$

Average degree of $K_n = \frac{2m}{n} = \frac{2^{\frac{n(n-1)}{2}}}{n} = n - 1$
Hence, the Laplacian minimum dominating quotient energy of K_n is

$$\begin{aligned} LQ_DE(K_n) &= |n-(n-1)|\,(n-2) + \left|\frac{(n-1)+\sqrt{n^2-2n+5}}{2} - (n-1)\right| \\ &\quad + \left|\frac{(n-1)-\sqrt{n^2-2n+5}}{2} - (n-1)\right| \end{aligned}$$

$$LQ_DE(K_n) = (n-2) + \left|\frac{-n+1+\sqrt{n^2-2n+5}}{2}\right| + \left|\frac{-n+1-\sqrt{n^2-2n+5}}{2}\right|$$

Therefore, $LQ_DE(K_n) = (n-2) + \sqrt{n^2-2n+5}$

Theorem. 5.2. If $K_{1,n-1}$ is the star graph with n node, then $LQ_DE(G) = \frac{(n-2)^2}{n} + \sqrt{n^2-2n+5}$

Proof: Let $K_{1,n-1}$ be the star graph with node set $V = \{v_1, v_2 \ldots., v_{n-1}\}$. the minimum dominating set $D = \{v_1\}$.

$$A_D(K_{1,n-1}) = \begin{pmatrix} 1 & 3 & 3 & \ldots & 3 & 3 \\ 1/3 & 0 & 0 & \ldots & 0 & 0 \\ 1/3 & 0 & 0 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 1/3 & 0 & 0 & \ldots & 0 & 0 \\ 1/3 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

And

$$D(K_{1,n-1}) = \begin{pmatrix} n-1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

$$LQ_DE(K_n) = D(K_{1,n-1}) - A_D(K_{1,n-1}) =$$
$$\begin{pmatrix} n-2 & n-1 & n-1 & \ldots & n-1 & n-1 \\ -1/n-1 & 1 & 0 & \ldots & 0 & 0 \\ -1/n-1 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ -1/n-1 & 0 & 0 & \ldots & 1 & 0 \\ -1/n-1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Its characteristic polynomial is

$$[\lambda-1]^{(n-2)}[\lambda^2-(n-1)\lambda-1]$$

The Laplacian minimum dominating quotient eigen values are:

$\lambda = 1[(n-2)time]$, $\quad \lambda = \dfrac{(n-1) \pm \sqrt{n^2-2n+5}}{2}$ $[one\ time\ each]$

Average degree of $K_{1,n-1} = \frac{2m}{n} = \frac{2(n-1)}{n}$

Hence, the Laplacian minimum dominating quotient energy of $K_{1,n-1}$ is

$$LQ_DE(K_{1,n-1}) = \left|n - \frac{2(n-1)}{n}\right| (n-2) + \left|\frac{(n-1)+\sqrt{n^2-2n+5}}{2} - \frac{2(n-1)}{n}\right| + \left|\frac{(n-1)-\sqrt{n^2-2n+5}}{2} - \frac{2(n-1)}{n}\right|$$

$$LQ_DE(K_{1,n-1}) = \frac{(n-2)^2}{n} + \left|\frac{-n+1+\sqrt{n^2-2n+5}}{2}\right| + \left|\frac{-n+1-\sqrt{n^2-2n+5}}{2}\right|$$

Therefore, $LQ_DE(K_{1,n-1}) = \frac{(n-2)^2}{n} + \sqrt{n^2-2n+5}$

6. Bounds on Laplacian Minimum Dominating Quotient Energy of Graphs

Theorem 6.1. Let D be a minimum dominating set of a graph G and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of $LQ_D(G)$ then

$$(i) \quad \sum_{i=1}^{n} \lambda_i = 2|E| - |D|$$

$$(ii) \quad \sum_{i=1}^{n} {\lambda_i}^2 = 2|E| + \sum_{i=1}^{n} (d_i - h_i)^2 \ where\ h_i = \begin{cases} 1, & if\ v_i \in D \\ 0, & if\ v_i \notin D \end{cases}$$

Proof: (i) By definition, the sum of the principal diagonal elements of $LQ_D(G)$ is equal to

$$\sum_{i=1}^{n} d_i - |D| = 2|E| - |D|$$

Also, the sum of the eigenvalues of the matrix $LQ_D(G)$ is equal to the trace$LQ_D(G)$.

(ii) The result that the sum of the squares of the eigenvalues of $LQ_D(G)$ is equal to the trace of $LQ_D(G)^{\,2}$ is a direct application of a general property of matrices.

Therefore,

$$\sum_{i=1}^{n} {\lambda_i}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}l_{ji} = \sum_{i=1}^{n} (l_{ij})^2 + \sum_{j=1}^{n} (l_{ji})^2$$

$$= 2\sum_{i<j}^{n} (l_{ij})^2 + \sum_{j=1}^{n} (l_{ji})^2$$

$$\sum_{i=1}^{n} {\lambda_i}^2 = 2|E| + \sum_{i=1}^{n} (d_i - h_i)^2 \ \text{where } h_i = \begin{cases} 1, \text{if } v_i \in D \\ 0, \text{if } v_i \notin D \end{cases}$$

$$\sum_{i=1}^{n} \lambda_i^{\,2} = 2N \text{ where } N = |E| + \frac{1}{2}\sum_{i=1}^{n}(d_i - h_i)^2$$

Theorem 6.2. Given the graph G with n vertices and m edges, and D is a minimum dominating set of G then $LQ_D(G) \le \sqrt{2Nn} + 2m$

Proof: Given a graph G with n vertices and m edges, and the eigenvalues $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ of the Laplacian matrix $LQ_D(G)$. By using Cauchy's - Schwarz inequality we have,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^{\,2}\right)\left(\sum_{i=1}^{n} b_i^{\,2}\right)$$

Put $a_i = 1, b_i = \lambda_i$ then,

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \le \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \le (n)(2N)$$

$$\therefore \left(\sum_{i=1}^{n} |\lambda_i|\right) \le \sqrt{2Nn}$$

By Triangle inequality,

$$\left|\lambda_i - \frac{2m}{n}\right| \le |\lambda_i| + \left|\frac{2m}{n}\right| \forall\, i = 1,2,\ldots,n$$

$$i.e., \left|\lambda_i - \frac{2m}{n}\right| \le |\lambda_i| + \frac{2m}{n} \;\forall\, i$$

$$\left(\sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|\right) \le \left(\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{n} \frac{2m}{n}\right)$$

$$\le \sqrt{2Nn} + 2m$$

$$LQ_D(G) \le \sqrt{2Nn} + 2m$$

Theorem 6.3. Given the graph G with n vertices and m edges, and D is a minimum dominating set of G and if $D = \left|\det LE_{pe}(G)\right|$ then $LQ_D(G) \ge \sqrt{2N + n(n-1)D^{\frac{2}{n}}} - 2m$.

Proof: Consider

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right) \cdot \left(\sum_{j=1}^{n} |\lambda_j|\right)$$

$$= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i\neq j} |\lambda_i|\,\left|\lambda_j\right|$$

$$\therefore \sum_{i\neq j} |\lambda_i|\,\left|\lambda_j\right| = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 - \sum_{i=1}^{n} |\lambda_i|^2 \quad (\text{Theorem } 6.1.)$$

using AM-GM inequality for $n(n-1)$ non-negative terms shows that the arithmetic mean of these terms is atleast as large as their geometric mean and thus it follows that:

$$\frac{\sum_{i\neq j} |\lambda_i|\,\left|\lambda_j\right|}{n(n-1)} \geq \left[\prod_{i\neq j} |\lambda_i|\,\left|\lambda_j\right|\right]^{\frac{1}{n(n-1)}}$$

$$i.e., \sum_{i\neq j} |\lambda_i|\,\left|\lambda_j\right| \geq n(n-1)\left[\prod_{i\neq j} |\lambda_i|\,\left|\lambda_j\right|\right]^{\frac{1}{n(n-1)}}$$

Using Theorem.6.1 we get,

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 - \sum_{i=1}^{n} |\lambda_i|^2 \geq n(n-1)\left[\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}}$$

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 - 2N \geq n(n-1)\left[\prod_{i=1}^{n} |\lambda_i|\right]^{\frac{2}{n}}$$

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \geq 2N + n(n-1)\left[\prod_{i=1}^{n} |\lambda_i|\right]^{\frac{2}{n}}$$

$$\therefore \sum_{i=1}^{n} |\lambda_i| \geq \sqrt{2N + n(n-1)D^{\frac{2}{n}}}$$

W.K.T

$$|\lambda_i| - \left|\frac{2m}{n}\right| \leq \left|\lambda_i - \frac{2m}{n}\right| \forall\, i$$

$$\sum_{i=1}^{n} |\lambda_i| - \sum_{i=1}^{n} \frac{2m}{n} \leq \sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|$$

$$\sum_{i=1}^{n} |\lambda_i| - 2m \leq LQ_D(G)$$

$$LQ_D(G) \geq \sum_{i=1}^{n} |\lambda_i| - 2m$$

$$\geq \sqrt{2\mathrm{N} + \mathrm{n}(\mathrm{n}-1)\mathrm{D}^{\frac{2}{n}}} - 2\mathrm{m}$$

$$\therefore \mathrm{LQ}_D(\mathrm{G}) \geq \sqrt{2\mathrm{N} + \mathrm{n}(\mathrm{n}-1)\mathrm{D}^{\frac{2}{n}}} - 2\mathrm{m}$$

Theorem 6.4. Given the graph G with n vertices and m edges, and D is a minimum dominating set of G then $\mathrm{LQ}_D(\mathrm{G}) \leq \sqrt{2\mathrm{Nn} + 4\mathrm{m}(|\mathrm{D}| - \mathrm{m})}$

Proof: By using Cauchy’s - Schwarz inequality we have,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right)$$

Put $a_i = 1, b_i = \left|\lambda_i - \frac{2m}{n}\right|$ then,

$$\left(\sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|^2\right)$$

$$[\mathrm{LQ}_D(\mathrm{G})]^2 = n\left[\sum_{i=1}^{n} {\lambda_i}^2 + \sum_{i=1}^{n} \frac{4m^2}{n^2} - \frac{4m}{n}\sum_{i=1}^{n} \lambda_i\right]$$

$$= n\left[2N + \frac{4m^2}{n^2} \cdot n - \frac{4m}{n}(2m - |D|)\right]$$

$$= n\left[2N + \frac{4m^2}{n} - \frac{8m^2}{n} + \frac{4m|D|}{n}\right]$$

$$= 2Nn + 4m(|D| - m)$$

$$\therefore \mathrm{LQ}_D(\mathrm{G}) \leq \sqrt{2Nn + 4m(|D| - m)}.$$

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