

A Study on the Differential Value of Total Graph

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract:

Let $G = (V, E)$ be a graph and X be a subset of V . Let $B(X)$ be the set of vertices in $V - X$ that has a neighbour in a set X . The differential of a set X , is defined as $\partial(X)$ which is $|B(X)| - |X|$ and the differential of a graph is $\partial(G) = \max \{ \partial(X) / X \subset V \}$. The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are either adjacent or incident. In this paper, we study the differential value of total graph for some standard graphs and its bounds.

Keywords: total graph, domination number

Mathematics Subject Classification 05C38, 05C69

1. Introduction

Throughout this paper, $G = (V, E)$ is a simple finite graph of n vertices. For theoretical terminology about graph which is not given here, we refer to Harary [7]. For a vertex $v \in V$, the *open neighbourhood* of v is $N(v) = \{u \in V / uv \in E\}$ and the *closed neighbourhood* of the set $N[v] = N(v) \cup \{v\}$. For a set $X \subset V$, its open neighbourhood $N(X) = \bigcup_{v \in V} N(v)$ and $N[X] = N(X) \cup X$ is the closed neighbourhood. A set $D \subseteq V$ is a *dominating set* [6,10] of G if every vertex in $V - D$ is adjacent to some vertex in D . The *boundary* $B(X)$ of a set X is defined to be the set of vertices in $V - X$ dominated by vertices in X , that is $B(X) = (V - X) \cap N(X)$. The differential $\partial(X)$ of X is defined as $|B(X)| - |X|$. The *differential* of a graph G is $\partial(G) = \max\{\partial(X) / X \subset V\}$. If $S \subset V$ and $\partial(G) = \partial(S)$, then S is a ∂ -set. The differential of a set was first defined by Hedetniemi and later studied by Mashburn et al. and Goddard and Henning [3,4,9,13]. A graph G is *complete graph* if every distinct pair of vertices are adjacent. A *complete bipartite graph* is a special type of bipartite graph where every vertex of one set is connected to every other vertex of other set. A *complete binary tree* is a special type of binary tree where all the levels of the tree are filled completely except the lowest level nodes which are filled from as left as possible. A graph G is said to be *dominant differential* if it contains a ∂ - set which is also a dominating set. Some examples are complete graph and wheel graph.

The *total graph* $T(G)$ [1,2,5,8,11,12,14] of G is the graph whose vertex set is $V(G) \cup E(G)$ with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are either adjacent or incident and $|V(T(G))| = l$.

In this paper, we study the differential value of total graph of some standard graphs and its bounds.

2. Results

Theorem 1. For any graph G of n vertices, $1 \leq \partial(T(G)) \leq \frac{n(n-1)}{2}$

Theorem 2. If $T(G)$ is the total graph of a graph G , then $\partial(T(G)) \leq \frac{n(n-1)}{2}$ for $n \geq 2$.

Proof. We have to prove by induction method. We have to prove that the result is true for $n = 2$. When $n = 2$, then $\partial(T(G)) \leq \frac{2(2-1)}{2} = 1$. It is always true. We assume that the result is true for $n = k$. Then, $\partial(T(G)) \leq \frac{k(k-1)}{2}$. We have to prove that the result is true for $n = k + 1$. Consider G with $n = k + 1$ vertices. Removing a vertex $v \in T(G)$ and $|V(T(G)) - \{v\}| = k$, hence by induction hypothesis, $\partial(T(G) - \{v\}) + k \leq \frac{k(k-1)}{2} + k = \frac{k^2 - k + 2k}{2} = \frac{k^2 + k}{2} = \frac{k(k+1)}{2}$. So, $\partial(T(G)) \leq \frac{k(k+1)}{2}$. Therefore, the result is true for $n = k + 1$ and the result is true for all n . So, $\partial(T(G)) \leq \frac{n(n-1)}{2}$.

Theorem 3. If G is a complete graph, then $\partial(T(G)) = \frac{n(n-1)}{2}$, for $n \geq 2$.

Proof. Given that G is a complete graph. Consider S is a ∂ -set of $T(G)$. When n is even, choose any arbitrary vertex u_1 in S and choose next vertex u_2 which is not adjacent with u_1 . Choose next vertex u_3 which is not adjacent with u_1 and u_2 . Continuing this process until $|S| = \frac{n}{2}$. Clearly, S is a dominant differential of $T(G)$. Therefore, $\partial(T(G)) = |B(S)| - |S| = \left[\frac{n(n+1)}{2} - \frac{n}{2} \right] - \frac{n}{2} = \frac{n(n-1)}{2}$. If n is odd, we choose the vertices in S as we discussed in the above case and S dominates all the vertices except one in the case. Therefore, $\partial(T(G)) = \left[\frac{n(n+1)}{2} - \frac{n-1}{2} - 1 \right] - \frac{n-1}{2} = \frac{n(n-1)}{2}$. In both the cases, $\partial(T(G)) = \frac{n(n-1)}{2}$. Hence the proof.

Theorem 4. Given a positive integer k , there exist a graph on n vertices whose total graph on l vertices with $\partial(T(G)) = k$

Proof. If n is odd, we consider the graph with $n = \frac{k+5}{2}$ vertices whose total graph has $\frac{k+3}{2}$ copies of K_3 with exactly one vertex as common, say v . Clearly, ∂ -set of $T(G)$ contains v only. Hence $\partial(T(G)) = k$. If n is even, when $n = 4$, we consider the circulant graph $C_{4,2}$. For other cases, we consider the circulant graph $G = C_{4,2}$ in which $\frac{k-6}{2}$ copies of P_2 which is attached with a vertex, say u of $C_{4,2}$. Let S be the ∂ -set of above graphs. Consider $S = \{u, v\}$ where $v \notin N(u)$. So, $\partial(T(G)) = k$.

Observation 5. For any graph G with n vertices, $\partial(T(G)) \neq 2$.

Theorem 6. For any graph G with n vertices, $1 \leq \partial(T(G)) - \partial(G) \leq \frac{n^2 - 3n + 4}{2}$

Proof. Given that G is a graph of n vertices. The maximum value of the differential of any total graph is less than or equal to $\frac{n(n-1)}{2}$ and clearly $\partial(G) \leq n - 2$. Therefore, $\partial(T(G)) - \partial(G) \leq \frac{n(n-1)}{2} - (n - 2) = \frac{n^2 - 3n + 4}{2}$.

Observation 7. For any graph G with n vertices, $\partial(T(G)) - \partial(G) = \frac{n^2 - 3n + 4}{2}$ if and only if G is a complete graph.

Theorem 8. If $G = K_{1,n-1}$ is a star graph, then $\partial(T(K_{1,n-1})) = 2n - 3$

Proof. Let $V(K_{1,n-1}) = \{v, v_1, v_2 \dots v_{n-1}\}$ and $E(K_{1,n-1}) = \{e_1, e_2 \dots e_{n-1}\}$ where $e_i = vv_i$, $i = 1, 2, \dots, n - 1$ and v is a head vertex of the star graph. By the definition of total graph, $V(T(K_{1,n-1})) = \{v_1, v_2 \dots v_{n-1}, e_1, e_2 \dots e_{n-1}\}$ and $|V(T(K_{1,n-1}))| = 2n + 1$. Since $deg(v) = 2n$ and $S = \{v\}$ is the differential set, then $\partial(T(K_{1,n-1})) = 2n - 3$

Theorem 9. For any graph $G = C_n$, then $\partial(T(P_n)) = \begin{cases} 3 \left\lfloor \frac{2n}{5} \right\rfloor, & n \equiv 0, 1, 3 \pmod{5} \\ 3 \left\lfloor \frac{2n}{5} \right\rfloor + 1, & n \equiv 4 \pmod{5} \\ 3 \left\lfloor \frac{2n}{5} \right\rfloor + 2, & n \equiv 2 \pmod{5} \end{cases}$

Proof. let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ be the vertices of cycle of length n ($n \geq 3$) and $E(C_n) = \{e_1, e_2, \dots, e_n\}$ be the edges of the corresponding vertices $\{v_1, v_2, \dots, v_n\}$. Then, $V(T(C_n)) = \{v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_n\}$ and $E(T(G)) = \{v_i v_{i+1} / 1 \leq i \leq n - 1\} \cup \{e_i e_{i+1} / 1 \leq i \leq n - 1\} \cup \{v_i e_i / 1 \leq i \leq n\} \cup \{e_i v_{i+1} / 1 \leq i \leq n - 1\} \cup \{e_n v_1, e_n e_1, v_n v_1\}$ and hence $|V(T(C_n))| = 2n$. Let S be the differential set of $T(C_n)$. When $n \equiv 0 \pmod{5}$. Consider $S = \{v_1, e_3, v_6, \dots, v_{n-4}, e_{n-2}\}$ be a ∂ -set and hence. $|S| = \frac{2n}{5}$ and $|B(S)| = \frac{8n}{5}$. Then, $\partial(T(G)) = |B(S)| - |S| = 3 \left\lfloor \frac{2n}{5} \right\rfloor$. When $n \equiv 1 \pmod{5}$. Consider the differential set $S = \{v_1, e_3, v_6, \dots, e_{n-3}, v_{n-5}\}$ and $C(S) = \{e_{n-1}, v_{n-1}\}$. Clearly, $|S| = \left\lfloor \frac{2n}{5} \right\rfloor$ and $|B(S)| = \left\lfloor \frac{8n}{5} \right\rfloor$. So, $\partial(T(G)) = 3 \left\lfloor \frac{2n}{5} \right\rfloor$. When $n \equiv 3 \pmod{5}$. Consider the differential set $S = \{v_1, e_3, v_6, \dots, e_{n-5}, v_{n-2}\}$ and $C(S) = \{e_{n-1}\}$. Clearly, $|S| = \left\lfloor \frac{2n}{5} \right\rfloor$ and $|B(S)| = \left\lfloor \frac{8n}{5} \right\rfloor$. So, $\partial(T(G)) = |B(S)| - |S| = 3 \left\lfloor \frac{2n}{5} \right\rfloor$. When $n \equiv 4 \pmod{5}$ Consider the set $S = S_1 \cup S_2$ where $S_1 = \{v_1, e_3, v_6, \dots, e_{n-6}, v_{n-3}\}$ and $S_2 = \{v_{n-1}\}$. Then $\partial(S_2) = 1$. So, $\partial(T(G)) = 3 \left\lfloor \frac{2n}{5} \right\rfloor + 1$. When $n \equiv 2 \pmod{5}$. Consider the set $S = S_1 \cup S_2$ where $S_1 = \{v_1, e_3, v_6, \dots, e_{n-4}, v_{n-6}\}$ and $S_2 = \{v_{n-1}\}$. Then $\partial(S_2) = 2$. So, $\partial(T(G)) = 3 \left\lfloor \frac{2n}{5} \right\rfloor + 2$.

Theorem 10. For any path $G = P_n$ ($n \geq 5$), then

$$\partial(T(P_n)) = \begin{cases} 3 \left\lfloor \frac{2n-1}{5} \right\rfloor, & n \equiv 1,3,4 \pmod{5} \\ 3 \left\lfloor \frac{2n-1}{5} \right\rfloor + 1, & n \equiv 2 \pmod{5} \\ 3 \left\lfloor \frac{2n-1}{5} \right\rfloor + 2, & n \equiv 0 \pmod{5} \end{cases}$$

Proof. let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ be the vertices of path of length n ($n \geq 3$) and $E(P_n) = \{e_1, e_2, \dots, e_{n-1}\}$. Then, $V(T(P_n)) = \{v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_{n-1}\}$ and $|V(T(P_n))| = 2n - 1$. When $n \equiv 1 \pmod{5}$. Consider the differential set of a total graph $S = \{v_2, e_4, \dots, e_{n-2}, v_{n-4}\}$ and $C(S) = \{v_n\}$. Then, $|B(S)| = \left\lfloor \frac{8n-4}{5} \right\rfloor$ and $|S| = \left\lfloor \frac{2n-1}{5} \right\rfloor$. So, $\partial(T(G)) = 3 \left\lfloor \frac{2n-1}{5} \right\rfloor$. When $n \equiv 3 \pmod{5}$, Consider the differential set of a total graph $S = \{v_2, e_4, \dots, e_{n-4}, v_{n-1}\}$. Then $\partial(T(G)) = 3 \left\lfloor \frac{2n-1}{5} \right\rfloor$. When $n \equiv 4 \pmod{5}$. Consider the differential set $S = \{v_2, e_4, \dots, e_{n-5}, v_{n-2}\}$ and $C(S) = \{v_n, e_{n-1}\}$. Then $\partial(T(G)) = 3 \left\lfloor \frac{2n-1}{5} \right\rfloor$. When $n \equiv 2 \pmod{5}$, there are three possible differential sets $S = S_1 \cup \{v_{n-1}\}$ or $S = S_1 \cup \{v_n\}$ or $S = S_1 \cup \{e_{n-1}\}$ where $S_1 = \{v_2, e_4, \dots, e_{n-3}, v_{n-5}\}$. Then $\partial(T(G)) = 3 \left\lfloor \frac{2n-1}{5} \right\rfloor + 1$. When $n \equiv 0 \pmod{5}$, the possible differential set $S = S_1 \cup S_2$ where $S_1 = \{v_2, e_4, \dots, e_{n-6}, v_{n-3}\}$ and $S_2 = \{e_{n-2}\}$. Then, $\partial(T(G)) = 3 \left\lfloor \frac{2n-1}{5} \right\rfloor + 2$.

Theorem 11. If $G = K_{m_1 \times n_1}$ is a complete bipartite graph, then $\partial(T(K_{m_1 \times n_1})) = (m_1 + 1)n - m_1$.

Proof. Since $G = K_{m_1 \times n_1}$ is a complete bipartite graph, the vertex set can be partitioned into two disjoint non empty sets $V_1(G) = \{u_1, u_2 \dots u_{m_1}\}$ and $V_2(G) = \{v_1, v_2, \dots, v_{n_1}\}$. Here, $V(T(K_{m_1 \times n_1})) = \{u_i/1 \leq i \leq m_1\} \cup \{v_j/1 \leq j \leq n_1\} \cup \{e_{ij}/1 \leq i \leq m_1, 1 \leq j \leq n_1\}$ and $|V(T(K_{m_1 \times n_1}))| = m_1 + n_1 + m_1 n_1$. Clearly, the differential set $S = \{u_1, u_2 \dots u_{m_1}\}$. Since u_1 is adjacent with $2n_1$ vertices and u_2 is adjacent with n_1 vertices and so on. Therefore $\partial(T(K_{m_1 \times n_1})) = (2n_1 - 1) + (n_1 - 1) + \dots + (n_1 - 1) = (m_1 + 1)n_1 - m_1$.

Theorem 12. If G is a complete binary tree, then

$$\partial(T(G)) = \begin{cases} \frac{2^{k+3}-7}{3}, & k \text{ is an odd integer} \\ \frac{2^{k+3}-5}{3}, & k \text{ is an even integer} \end{cases}$$

Proof. Let S_k be the set of all vertices in level k and $|S_k| = 2^k$ where $0 \leq k \leq m$ and m is a positive integer.

case(i) k is an odd integer. Clearly, $S_{k-1} \cup S_{k-3} \cup \dots \cup S_2 \cup S_0$ is a ∂ -set.

$$\begin{aligned} \partial(T(G)) &= 2^{k+1} + 2^{k-1} + 2^{k-3} + \dots + 2^4 + 2^2 - 1 \\ &= 2^{k+1} \left[1 + \frac{2^{k-1}}{2^{k+1}} + \frac{2^{k-3}}{2^{k+1}} + \dots + \frac{2^4}{2^{k+1}} + \frac{2^2}{2^{k+1}} \right] - 1 \end{aligned}$$

$$\begin{aligned}
 &= 2^{k+1}[1 + 2^{-2} + 2^{-4} + \dots + 2^{-k+3} + 2^{-k+1}] - 1 \\
 &= 2^{k+1} \left[1 + \left(\frac{1}{2^2}\right)^1 + \left(\frac{1}{2^2}\right)^2 + \dots + \left(\frac{1}{2^2}\right)^{\frac{k-3}{2}} + \left(\frac{1}{2^2}\right)^{\frac{k-1}{2}} \right] - 1 \\
 &= 2^{k+1} \left[\frac{1 - \left(\frac{1}{2^2}\right)^{\frac{k+1}{2}}}{1 - \left(\frac{1}{2^2}\right)} \right] - 1 \\
 &= 2^{k+1} \left[\frac{1 - \left(\frac{1}{2^2}\right)^{\frac{k+1}{2}}}{\left(\frac{3}{2^2}\right)} \right] - 1 \\
 &= 2^{k+1} \times \frac{2^2}{3} \times \left[1 - \frac{1}{2^{k+1}} \right] - 1 \\
 &= \left[\frac{2^{k+3}-4}{3} \right] - 1 = \left[\frac{2^{k+3}-7}{3} \right]
 \end{aligned}$$

case(ii) k is an even integer. Clearly, $S_{k-1} \cup S_{k-3} \cup \dots \cup S_3 \cup S_1$ is a ∂ -set.

$$\begin{aligned}
 \partial(T(G)) &= 2^{k+1} + 2^{k-1} + 2^{k-3} + \dots + 2^5 + 2^3 + 1 \\
 &= 2^{k+1} \left[1 + \frac{2^{k-1}}{2^{k+1}} + \frac{2^{k-3}}{2^{k+1}} + \dots + \frac{2^5}{2^{k+1}} + \frac{2^3}{2^{k+1}} \right] + 1 \\
 &= 2^{k+1} [1 + 2^{-2} + 2^{-4} + \dots + 2^{-k+4} + 2^{-k+2}] + 1 \\
 &= 2^{k+1} \left[1 + \left(\frac{1}{2^2}\right)^1 + \left(\frac{1}{2^2}\right)^2 + \dots + \left(\frac{1}{2^2}\right)^{\frac{k-4}{2}} + \left(\frac{1}{2^2}\right)^{\frac{k-2}{2}} \right] + 1 \\
 &= 2^{k+1} \left[\frac{1 - \left(\frac{1}{2^2}\right)^{\frac{k}{2}}}{1 - \left(\frac{1}{2^2}\right)} \right] + 1 \\
 &= 2^{k+1} \times \frac{2^2}{3} \times \left[1 - \frac{1}{2^k} \right] + 1 \\
 &= \left[\frac{2^{k+3}-8}{3} \right] + 1 = \left[\frac{2^{k+3}-5}{3} \right]
 \end{aligned}$$

Theorem 13. If $G = P_2 \times P_{n_1}$ is a grid graph with then $\partial(T(P_2 \times P_n)) = l - 2n_1$ when $n_1 \geq 2$.

Proof. Let $T(P_2 \times P_{n_1})$ be the total graph of the grid graph and the vertex set of the graph is

$$V(T(P_2 \times P_{n_1})) = \{v_{11}, \dots, v_{1n_1}, v_{21}, \dots, v_{2n_1}, u_{11}, \dots, u_{1n_1}, e_{11}, \dots, e_{1(n_1-1)}, e_{21}, \dots, e_{2(n_1-1)}\}$$

where $u_{ij} = v_{ij}v_{(i+1)j}$, $i, j = 1, 2, \dots, n_1$ and $|V(T(P_2 \times P_{n_1}))| = 5n_1 - 2 = l$. Consider the

differential set of the given grid graph G is $S = \{v_{11}, v_{14}, u_{12}, u_{15}, v_{23}, \dots, v_{1(n_1-1)}, u_{1n_1}, v_{2(n_1-2)}\}$

which is also an γ -set. Therefore, $\partial(T(P_2 \times P_{n_1})) = l - 2\gamma(G) = 1 - 2n_1$.

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