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Singularly Perturbed Initial Value Problem for First Order Delay Differential Equation Magdm Problem using a Magdm Problem

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Abstract:

This paper presents a novel approach to Dynamic Intuitionistic Fuzzy Multiple Attribute Group Decision Making (DIF-MAGDM) problems, where decision attributes are represented by intuitionistic fuzzy numbers collected over multiple time periods. To determine the unknown decision-maker weights, a singularly perturbed initial value problem is formulated, incorporating a transition parameter τ for computing the weighting vectors. The proposed method employs the Dynamic Intuitionistic Fuzzy Weighted Averaging (DIFWA) operator to transform all dynamic intuitionistic fuzzy decision matrices into a column matrix. To rank and select the most suitable alternative, a proximity coefficient function is applied. The effectiveness and practicality of the approach are demonstrated through a numerical case study on supplier selection. The results validate the robustness and applicability of the proposed decision-making framework in dynamic and uncertain environments.

Keywords: Intuitionistic Fuzzy Sets (IFSs), MAGDM, Dynamic Intuitionistic Fuzzy Weighted Averaging (DIFWA) operator, Singular Perturbation Problem.

Introduction

Multi-Attribute Group Decision-Making (MAGDM) plays a crucial role in modern decision theory, providing a framework for evaluating multiple alternatives based on multiple criteria. When intuitionistic fuzzy sets (IFSs) are incorporated into MAGDM and extended across multiple time periods, the process is referred to as Dynamic Intuitionistic Fuzzy Multi-Attribute Group Decision-Making (DIF-MAGDM).

Since Atanassov [1] introduced the concept of IFSs, significant advancements have been made in their theoretical development and practical applications. Intuitionistic fuzzy sets have been widely utilized in various real-world decision-making scenarios [2]. Atanassov et al. [3] explored the intuitionistic fuzzy interpretations of multi-criteria and multi-measurement decision-making methods, while Bustine and Burillo [4] extended the concept to vague sets. Several operations on IFSs have been further refined by Biswas and Roy [5].

In parallel, singular perturbation theory in ordinary differential equations has been a subject of extensive research. Doolan et al. [6] introduced this theory, and Kadalbajoo [7] expanded its application to singularly perturbed differential equations, incorporating both positive and negative shifts. Liu et al. [8] proposed several improvements to MAGDM and multiple criteria decision-making models using IFSs. Furthermore, Miller et al. [9] developed fitted numerical techniques for singular perturbation problems in both one and two dimensions.

The study of distances between intuitionistic fuzzy sets for group decision-making was introduced by Szmidt and Kacprzyk [10, 11], while Xu and Yager [12–15] developed the Dynamic Intuitionistic Fuzzy Weighted Averaging (DIFWA) operator to enhance decision-making processes. The foundational concept of fuzzy sets, which provides a powerful approach to managing uncertainty, was first introduced by Zadeh [16].

Building upon these advancements, this paper presents an approach for solving DIF-MAGDM problems by incorporating intuitionistic fuzzy numbers across various time periods. The proposed method leverages the DIFWA operator and a proximity coefficient function to facilitate effective decision-making, as demonstrated through a numerical case study on supplier selection.

PRELIMINARIES

This section discusses key concepts of Dynamic Intuitionistic Fuzzy Multi-Attribute Group Decision-Making (DIF-MAGDM) and the different types of aggregation operators. These operators are essential for integrating intuitionistic fuzzy information from multiple decision-makers over various time periods, ensuring a systematic approach to decision-making. The section highlights various aggregation techniques, including the Dynamic Intuitionistic Fuzzy Weighted Averaging (DIFWA) operator, which plays a crucial role in managing uncertainty in group decision-making processes.

Definition

Let $\sigma(t_1), \sigma(t_2), \ldots\ldots\ldots, \sigma(t_p)$be a collection of IFNs collected at p different periods $t_k(k = 1,2,\ldots\ldots, p), and \lambda(t) = \left(\lambda(t_1), \lambda(t_2), \ldots\ldots, \lambda(t_p)\right)^T$ be the weight vector of the periods $t_k(k = 1,2,\ldots\ldots, p)$, then we call

$$DIFWA_{\lambda(t)}\left(\alpha(t_1), \alpha(t_1), \ldots\ldots, \alpha(t_p)\right) = \lambda(t_1)\alpha(t_1) \oplus, \lambda(t_2)\alpha(t_2) \oplus, \ldots\ldots, \lambda(t_p)\alpha(t_p)$$

A dynamic intuitionistic fuzzy weighted a averaging (DIFWA) operator.

$$DIFWA_{\lambda(t)}\left(\alpha(t_1), \alpha(t_1), \ldots\ldots, \alpha(t_p)\right)$$
$$= \left(1 - \prod_{k=1}^{p}\left(1 - \mu_{\sigma(t_k)}\right)^{\lambda(t_k)}, \prod_{k=1}^{p}\gamma_{\sigma(t_k)}{}^{\lambda(t_k)}, \prod_{k=1}^{p}\left(1 - \mu_{\sigma(t_k)}\right)^{\lambda(t_k)} - \prod_{k=1}^{p}\gamma_{\sigma(t_k)}{}^{\lambda(t_k)}\right)$$

Where $\lambda(t_k) \geq 0, k = 1,2,\ldots\ldots, p, \sum_{k=1}^{p}\lambda(t_k) = 1$

A METHOD OF USING INTUITIONISTIC FUZZY INFORMATION IN GROUP DECISION MAKING:

Step 1: To create a collective intuitionistic fuzzy decision matrix $=(r_{ij})_{mxn}$.

, combine all of the individual intuitionistic fuzzy decision matrices using the DIFWA operator.

Step 2: Define $\boldsymbol{a}^{+} = (\boldsymbol{a}_{1}^{+}, \boldsymbol{a}_{2}^{+}, \ldots\ldots, \boldsymbol{a}_{m}^{+})^{\boldsymbol{T}}$ and $\boldsymbol{a}^{-} = (\boldsymbol{a}_{1}^{-}, \boldsymbol{a}_{2}^{-}, \ldots\ldots, \boldsymbol{a}_{m}^{-})^{\boldsymbol{T}}$ as the intuitionistic fuzzy negative ideal solution (IFNIS) and the intuitionistic fuzzy ideal solution (IFIS), respectively, where $a_i^{+} = (1,0,0)(i = 1,2, \ldots, m)$ are the m smallest and $a_i^{-} = (0,1,0)(i = 1,2, \ldots, m)$ m biggest IFNs. Additionally, for ease of illustration, we indicate the options $x_i (i = 1,2, \ldots, n) by x_i = (r_{i1}, r_{i2}, \ldots, r_{im})^{T}, i = 1,2, \ldots\ldots, n$.

Step 3: Determine the proximity coefficient, where, $\tilde{r}_i = (1,0)$between the positive ideal value $\tilde{r}_i$ and the collective overall preference values.

Step 4: Sort each option based on the proximity coefficient $C(x_i)(i = 1,2, \ldots., n)$,; the better option is indicated by a higher number x_i.

SINGULAR PERTURBATION PROBLEMS:

To differentiate between regular and singular perturbation problems, first take a look at a family of boundary value problems $p_{\in}$ that, depending on a small parameter ε. can have a solution $y_{\varepsilon}(x) of P_{\varepsilon}$constructed by a well-known "method of perturbation"; that is, as a power series ε in which the problem's solution p_0 (obtained by putting $\in = 0$ in $p_{\in}$) is the first term y_0. A power series expansion as $\in \rightarrow 0$ is said to be a regular perturbation issue when it converges evenly in x. is referred to as a single perturbation problem when $y_{\in}(x)$ it lacks a uniform limit in x as $\in \rightarrow 0$, as this regular perturbation procedure fails.

A singly perturbed delay differential equation is an ordinary differential equation that includes at least one delay term and has its highest derivative multiplied by a small parameter. These equations frequently arise in various applications, such as control theory variation problems, depolarization in Stein's model, human pupil-light reflex modeling, and the first exit time problem in neuronal variability activation.

This study focuses on boundary value problems for singularly perturbed linear second-order differential-difference equations, where a small parameter multiplies the highest-order derivative. The delay term u(x−1) is expressed using a Taylor series expansion at x=1. To address the numerical challenges posed by these equations, a numerical method based on a piecewise uniform **Shishkin-type mesh** is implemented within a classical finite difference scheme.

A carefully constructed piecewise uniform Shishkin mesh is derived to ensure numerical stability and accuracy. It is proven that the proposed approach achieves **first-order convergence in the**

maximum norm, regardless of the small parameter. The Taylor series expansion at x is employed to effectively represent the delay term u(x−1), ensuring an accurate numerical solution.

$\boldsymbol{Lu(x) = -\varepsilon u' + a(x)u(x) + b(x)u(x-1) = f(x) on [0,2],}$

with $\boldsymbol{u = \phi on [-1,0] and u(2) = l.}$

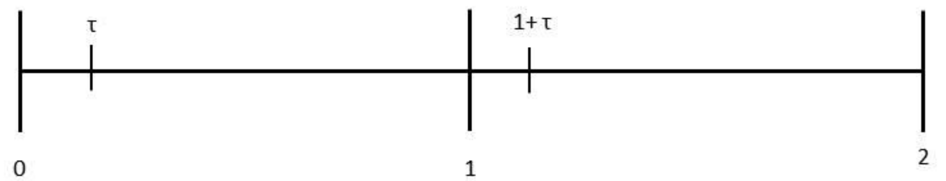
Where ϕ is sufficiently smooth on [-1, 0]. For all $x \in [0,2]$, it is assumed that a(x) and b(x) satisfy,

$\boldsymbol{a(x) + b(x) > \alpha,}$ and $\boldsymbol{b(x) < 0, for some real number \alpha > 0}$

Shishkin mesh:

To construct an effective piecewise uniform mesh within the domain of definition, it is essential to account for the boundary layer behavior near x=0. The mesh should be refined in this region while remaining coarser elsewhere.

If the total number of mesh points is N, they are distributed such that **N/2 points are allocated to the inner region**, where the solution exhibits rapid variation, and **the remaining N/2 points are placed in the outer region**, where the solution changes more gradually. The resulting mesh structure is outlined as follows.



Where the parameter τ is defined as

$$\tau = min\left\{\frac{1}{2}, \frac{\varepsilon}{\alpha} ln\, N\right\}$$

Discrete problem:

The finite difference method is now applied to formulate the discrete two-point boundary value problem on an arbitrary grid.

$$L^N U(x_j) = \varepsilon D^- U(x_j) + a(x_j)U(x_j) + b(x_j)U(x_j - 1) = f(x_j) for all (x_j), j = 1, N$$

$L^N U = f, U(0) = u(0), U(1) = u(1).$

Where

$$U(x_j - 1) = U(x_j) - D^- U(x_j)$$
$$D^- U(x_j) = \frac{U(x_j) - U(x_{j-1})}{h_j}$$

MAGDM PROBLEM DETERMINATION EXPERTS WEIGHTS USING A SINGULARLY PERTURBED INITIAL VALUE PROBLEM:

Problem proposed by decision maker.

The unknown decision-maker weights are determined by solving a singularly perturbed initial value problem.

$$-\varepsilon u' + 1.5u - 0.1u(x-1) = 1 for x \in (0,1) \cup (1,2)$$
$$u(0) = 1 for x \in [-1,0]$$

The numerical solution presented above can be obtained using the classical finite difference scheme. The normalization of weighting vectors is shown in Tables 1 and 2, and the weighting vectors can be calculated for different values of τ.

Table 1: Numerical solution of $-\boldsymbol{\varepsilon u' + a(x)u(x) + b(x)u(x-1)}$

X	$u(x)$	Normalize
0.1406080	0.9994382	0.33333
0.1407457	0.9994377	0.33333
0.1408834	0.9994371	0.33333

Table 2: Numerical solution of $-\boldsymbol{\varepsilon u' + a(x)u(x) + b(x)u(x-1)}$

X	$u(x)$	Normalize
0.1410212	0.9994366	0.20152
0.1117446	0.9955917	0.20075
0.2093871	0.9917987	0.19998
0.3070296	0.9880568	0.19923
0.4046720	0.9843654	0.19849

Hence, the weighting vectors given by the decision maker is calculated as $\omega = (0.33333,0.33333,0.33333)^T, w = (0.20152,0.20075,0.19998,0.19923,0.19849)^T$

The two mesh algorithm is used to calculate the order of convergence and the maximum pointwise error.

Table 1: Values of $\boldsymbol{D_\varepsilon^N, D^N, P^N, P^* and C_P^N for \varepsilon_1 = \frac{\eta}{8} and \alpha = 0.9}$

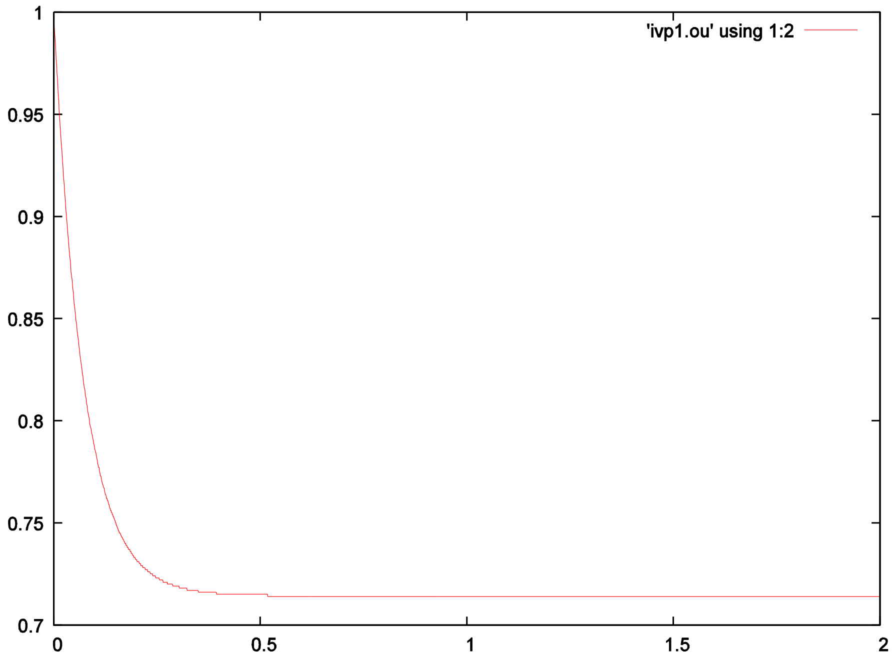
$\boldsymbol{\eta}$	**128**	**256**	**512**	**1024**	**2048**
$\mathbf{2^{-3}}$	0.625E-01	0.147E-02	0.862E-03	0.491E-03	0.275E-03

2^{-4}	0.391E-02	0.147E-02	0.862E-03	0.491E-03	0.275E-03
2^{-5}	0.244E-03	0.147E-02	0.862E-03	0.491E-03	0.275E-03
2^{-6}	0.153E-04	0`147E-02	0.862E-03	0.491E-03	0.275E-03
2^{-7}	0.954E-06	0.147E-02	0.862E-03	0.491E-03	0.275E-03
D^N	0.147E-02	0.862E-03	0.491E-03	0.275E-03	
P^N	0.771E+00	0.813E+00	0.838E+00		
C_P^N	0.256E+00	0.256E+00	0.249E+00	0.237E+00	

The order of convergence=0.7713155E+00

The error constant=0.2559836E+00

Figure 1: NUMERICAL SOLUTION OF$-\varepsilon u' + a(x)u(x) + b(x)u(x-1) = f(x)$



NUMERICAL ILLUSTRATION:

A company is looking to invest in a renewable energy source and has identified four potential alternatives: **geothermal, solar, wind, and biomass**. To evaluate these options, the company has established five key criteria: **risk factor, industry growth rate, payback reliability, social benefits, and demand change**.

Given the increasing energy demand, growing environmental awareness, and strong government support for renewable energy projects in recent years, the company recognizes the need for a

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dynamic assessment approach. As a result, renewable energy sources have been evaluated over the past six years in **two-year intervals** to account for changing conditions.

To ensure a well-informed decision, a **three-member decision-making group** has been assembled to analyze the data and determine the most suitable investment option.

$$R^1 = \begin{pmatrix} (0.6,0.3,0.1) & (0.7,0.2,0.1) & (0.3,0.6,0.1) & (0.6,0.2,0.2) & (0.2,0.6,0.1) \\ (0.4,0.5,0.1) & (0.9,0.1,0.0) & (0.1,0.8,0.1) & (0.4,0.5,0.1) & (0.6,0.3,0.1) \\ (0.7,0.1,0.2) & (0.4,0.5,0.1) & (0.5,0.4,0.1) & (0.8,0.2,0.0) & (0.6,0.2,0.1) \\ (0.5,0.4,0.1) & (0.5,0.3,0.2) & (0.5,0.2,0.3) & (0.4,0.6,0.0) & (0.8,0.1,0.1) \end{pmatrix}$$

$$R^2 = \begin{pmatrix} (0.9,0.1,0.0) & (0.7,0.2,0.1) & (0.8,0.2,0.0) & (0.7,0.2,0.1) & (0.3,0.7,0.0) \\ (0.2,0.6,0.2) & (0.8,0.1,0.1) & (0.1,0.8,0.1) & (0.3,0.5,0.2) & (0.9,0.1,0.0) \\ (0.7,0.3,0.0) & (0.3,0.4,0.3) & (0.5,0.4,0.1) & (0.9,0.1,0.0) & (0.7,0.1,0.2) \\ (0.6,0.3,0.1) & (0.3,0.5,0.2) & (0.5,0.3,0.2) & (0.5,0.4,0.1) & (0.4,0.6,0.0) \end{pmatrix}$$

$$R^3 = \begin{pmatrix} (0.5,0.5,0.0) & (0.6,0.2,0.2) & (0.7,0.3,0.0) & (0.6,0.3,0.1) & (0.3,0.7,0.0) \\ (0.6,0.3,0.1) & (0.8,0.1,0.1) & (0.8,0.2,0.0) & (0.4,0.5,0.1) & (0.6,0.3,0.1) \\ (0.3,0.4,0.3) & (0.5,0.4,0.1) & (0.5,0.4,0.1) & (0.7,0.2,0.1) & (0.6,0.2,0.2) \\ (0.5,0.4,0.1) & (0.5,0.2,0.3) & (0.6,0.2,0.2) & (0.4,0.6,0.0) & (0.5,0.4,0.1) \end{pmatrix}$$

Step: 1 Use the DIFWA operator to aggregate all individual intuitionistic fuzzy decision matrices into a collective intuitionistic fuzzy decision matrix $R=(r_{ij})_{mxn}$.

$$R = \begin{bmatrix} [(0.72855,0.24662,0.02482)], [(0.66980,0.20000,0.13019)], [(0.65239,0.33019,0.01741)], \\ [(0.63657,0.22895,0.13448)], [(0.26813,0.66494,0.06692)] \\ [(0.42310,0.44814,0.12876)], [(0.84125,0.10000,0.05874)], [(0.45486,0.50397,0.04116)], \\ [(0.36836,0.46416,0.20075)], [(0.74801,0.20801,0.04397)] \\ [(0.60209,0.22895,0.16896)], [(0.40560,0.43089,0.16350)], [(0.49999,0.40000,0.10000)], \\ [(0.81828,0.15874,0.02297)], [(0.63657,0.15874,0.02297)] \\ [(0.53584,0.36343,0.10073)], [(0.44065,0.31073,0.24862)], [(0.53584,0.22895,0.23522)], \\ [(0.43537,0.52415,0.04047)], [(0.60851,0.28845,0.10304)] \end{bmatrix}$$

Using step2, we obtain

$$r_1 = \begin{bmatrix} (0.72855,0.24662,0.02482), (0.66980,0.20000,0.13019), (0.65239,0.33019,0.01741), \\ (0.63657,0.22895,0.13448), (0.26813,0.66494,0.06692) \end{bmatrix}^T$$

$$r_2 = \begin{bmatrix} (0.42310,0.44814,0.12876), (0.84125,0.10000,0.05874), (0.45486,0.50397,0.04116), \\ (0.36836,0.46416,0.20075), (0.74801,0.20801,0.04397) \end{bmatrix}^T$$

$$r_3 = \begin{bmatrix} (0.60209,0.22895,0.16896), (0.40560,0.43089,0.16350), (0.49999,0.40000,0.10000), \\ (0.81828,0.15874,0.02297), (0.63657,0.15874,0.02297) \end{bmatrix}^T$$

$$r_4 = \begin{bmatrix} (0.53584,0.36343,0.10073), (0.44065,0.31073,0.24862), (0.53584,0.22895,0.23522), \\ (0.43537,0.52415,0.04047), (0.60851,0.28845,0.10304) \end{bmatrix}^T$$

STEP: 3 calculate the closeness coefficient of each alternatives

$$c(x_i) = \frac{\sum_{j=1}^{n} \omega_j \left(1 - \gamma_{ij}\right)}{\sum_{j=1}^{n} \omega_j \left(1 + \pi_{ij}\right)}$$

$C(\tilde{r}_1, \tilde{r}^+)$ = 0.62019

$C(\tilde{r}_2, \tilde{r}^+)$ = 0.59839

$C(\tilde{r}_3, \tilde{r}^+)$ = 0.66079

$C(\tilde{r}_4, \tilde{r}^+)$ = 0.57333

Rank all the alternatives $x_i (i = 1,2,3,4)$ according to the closeness coefficient of $c(x_i)(i = 1,2,3,4)$.

$$x_3 > x_1 > x_2 > x_4.$$

Conclusion:

In this study, the numerical solution of the singularly perturbed initial value problem for the decision-making process is determined using the **unknown weights of the decision-makers**. To select the optimal alternative, the **dynamic intuitionistic multi-attribute decision-making approach** is applied, computing the **proximity coefficient** for each option. This method ensures a systematic and dynamic evaluation of alternatives, leading to a well-informed investment decision.

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