

Bootstrapping Collinear Poisson Regression System: A Simulation Study

Mohamed A. Gharib *, Ahmed H. Yousef, Salah M. Mohamed

Department of Applied Statistics and Econometrics, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt.

* 12422019512772@pg.cu.edu.eg

Article History:

Received: 25-10-2024

Revised: 15-11-2024

Accepted: 28-12-2024

Abstract:

Count data modelling is used in many fields, such as predicting the football goals number a player will score. Poisson regression is the most popular model when modeling the effect of a dependent variable that takes non-negative integer values on one or more explanatory variable(s). Poisson regression model is classified as a generalized linear model which depends on the log-link function. The problem of multicollinearity causes inflation of standard errors estimates, which leads to wider confidence intervals. Increasing error of omission like failing to reject the false null hypothesis, which is known as type II error or false negative, generally leads to produce incorrect statistical decisions and consequently misleading decision making. This research offers a framework for handling the collinear system of multiple Poisson regression equations in all fields. This research addresses firstly the case of single equation and then the system of seemingly unrelated Poisson regression equations, focusing on two bivariate dependent Poisson distributed variables. the research will use bootstrap method to estimate parameters across some factors which are levels of collinearity, sample sizes, and correlation structures. Simulation study showed that bootstrap estimators have lower standard errors than traditional maximum likelihood (ML) estimators in all degrees of collinearity ($\rho = 0.75, 0.85, 0.95, 0.99$) and for all sample sizes ($n = 25, 50, 100, 150, 200$).

Keywords: Poisson Regression, Seemingly Unrelated Poisson Regression, Multicollinearity, Bootstrap, Simulation Study.

1. Introduction

The most famous statistical technique which applied in many fields is the linear regression analysis. Linear regression is useful for modeling the relationship between a continuous dependent variable and one or more explanatory variables. However, in many real applied fields, the data the dependent variable is not continuous. The dependent variable may consist of count data (0, 1, 2, ...), such as the number of correct answers in a test, the number of demanded/supplied units of a product, or the number of cases of a disease like COVID-19. These non-negative integer outcomes are not suitable to be modeled with linear regression. So, the need of count data regression models is emerged. The Poisson regression model is one of more appropriate models for the nature of count dependent variables.

Independence of explanatory variables is an assumption in multiple regression model. Multicollinearity problem is a common empirical phenomenon which exist when this assumption is not met. Multicollinearity concept was firstly used by Frisch (1934) to describe a linearly related explanatory variables in a multiple regression model. Multicollinearity leads to inflated standard

errors of the parameter estimates. This inflation makes confidence intervals wider, and consequently insignificant regression coefficients. It generally increases the probability of Type II error (Lavery *et al.*, 2019). So, violating this assumption leads to incorrect statistical inference and misleading decisions.

Many methods were proposed to combat the problem of multicollinearity such as ridge regression for the linear regression model presents for the first time by Hoerl and Kennard (1970) or using principal component analysis in context of the panel data model (Youssef *et al.*, 2023). In this research, we tried to handle this problem and overcome the bad effects of the multicollinearity problem, by presenting a proposed solution using a resampling technique which is the bootstrap method invented by Efron (1979) and developed by Efron and Tibshirani (1993). for estimating parameters of Poisson regression model under multicollinearity side by side with Maximum Likelihood (ML) estimation. The use of bootstrap method aims to enhance the precision of statistical inference in the existence of multicollinearity problem in Poisson regression model. The bootstrap method's performance will be evaluated through the simulation. Also, the simulation will compare the estimates performance of both bootstrap method and maximum likelihood alone.

The structure of this paper contains six sections, it is organized as follows. Research methodology will be presented in Section 2, it contains of research problem and question, objective of research, and the importance of research. In Section 3, the Poisson regression model will be presented as a popular count regression model. Bootstrap method as a suggested solution will be illustrated in Section 4. A Monte Carlo simulation will be presented in Section 5. At last, simulation results and concluded remarks will be presented in Section 6.

2. Research Methodology

Research methodology contains of the problem, objective, and Importance of research.

2.1 Research Problem

In practice, the researchers which use the Poisson regression models are faced by ill-conditioned data with multicollinear design matrix. Lack of independence assumption with varying degrees of multicollinearity in the explanatory variables leads to misleading results. The inaccurate statistical conclusions were due to the inflated standard errors of the regression coefficients and so destabilizing of maximum likelihood estimators (MLE). Generally, multicollinearity results inability to conduct accurate statistical inference for the parameters of Poisson regression model. The research problem can be formulated through the following research question: Is bootstrap an appropriate solution for handling the problem of multicollinearity, and in what cases is it preferable to use the bootstrap method?

2.2 Research Objective

Despite the many studies have tried to deal with multicollinearity in the single linear regression equation model such as Ridge regression invented by Hoerl and Kennard (1970), this research aims to address the problem of multicollinearity using the bootstrap method in the Poisson regression equation. The main concern of the research is to estimate the parameters of Poisson regression

model under varying degrees of multicollinearity and sample sizes. The simulation study uses criterion of the lowest standard error to determine which cases is it optimal to use of bootstrap method.

2.3 Research Importance

Many academic and applied fields concern with the issue of multicollinearity because ignoring it or not addressing it effectively leads to misleading results. Therefore, it is important to present a proposed solution to estimate model parameters more accurately to increase the predictive ability as well as to test hypotheses properly. The importance of this research is divided into two parts. The first is that the proposed solution which is bootstrap method, unlike ridge regression method, does not modify the data or the design matrix. The second is applying the bootstrap method to the most well-known count regression method, the Poisson regression.

3. Poisson Regression

The Poisson regression model is the most popular count regression model when the dependent or response variable follows a discrete probability distribution with only one parameter, which is the Poisson distribution with a positive parameter λ , with the following probability mass function (PMF):

$$\Pr(Y = y) = \frac{e^{-\lambda} \lambda^y}{y!}; y = 0, 1, \dots; \lambda > 0 \quad (1)$$

Simplicity of Poisson distribution is one of the advantages which lead to prefer it for modeling count data. It's single parameter λ represents both the mean and variance of the distribution. Poisson distribution has a positive skewness $\frac{1}{\sqrt{\lambda}}$ and has a kurtosis of $3 + \frac{1}{\lambda}$ (Evans & Adenomon, 2014). Another advantage is that the non-negative values of the variable start from zero with no upper limit.

The natural logarithm properties can be used to ensure that the parameter is positive for any value of the explanatory variables. Also, the rule for raising the natural logarithm of a function to the power of the natural number base (e) produce the function itself $f(y) = e^{\log[f(y)]}$ can be used to rewrite the Poisson probability distribution function in exponential form with parameter $\theta = \log(\lambda)$ as follows:

$$\begin{aligned} \Pr(Y = y) &= e^{\log\left[\frac{e^{-\lambda} \lambda^y}{y!}\right]} = e^{\log(e^{-\lambda}) + \log(\lambda^y) - \log(y!)} \\ &= e^{-\lambda + y \cdot \log(\lambda) - \log(y!)} \\ &= e^{y \cdot \log(\lambda) - \lambda} \frac{1}{y!} \end{aligned} \quad (2)$$

The Poisson regression model is considered as one of the generalized linear models (GLM) which model data belonging to the exponential family. The generalized linear model contains three components: the random component which is Poisson-distributed response, the systematic component which is linear predictor, and the link function (Nelder & Wedderburn, 1972). Poisson regression is a log-linear model because the link function that links the mean of the dependent or

response variable to at least two explanatory variables in the case of multiple regression is a logarithmic function as follows:

$$\log(\mu_i) = \log(\lambda_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} \tag{3}$$

When the values of the dependent or response variable are non-negative discrete, the Poisson distribution is the default distribution in the count regression. maximum likelihood method can be used to estimate the parameters of Poisson regression, unlike ordinary least squares (OLS) method is used in classical linear regression. Ordinary least squares method allows estimating the mean of response variable to be negative (Chatterjee & Simonoff, 2020).

The likelihood function and the logarithm of the likelihood function can be derived as follows:

$$L = \frac{(e^{-\sum_{i=1}^n \lambda_i}) (\prod_{i=1}^n \lambda_i^{y_i})}{\prod_{i=1}^n y_i!}$$

$$\ln L = l = \ln(e^{-\sum_{i=1}^n \lambda_i}) + \ln(\prod_{i=1}^n \lambda_i^{y_i}) - \ln(\prod_{i=1}^n y_i!)$$

$$l = -\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \ln(\lambda_i^{y_i}) - \sum_{i=1}^n \ln(y_i!)$$

$$l = -\sum_{i=1}^n \lambda_i + \sum_{i=1}^n y_i \ln(\lambda_i) - \sum_{i=1}^n \ln(y_i!)$$

(4)

Single Poisson regression models can be viewed as a special case of a system of Poisson regression equations which is useful in many applied studies when count outcomes are interrelated across equations although seemingly unrelated. This issue is pronounced in Seemingly Unrelated Poisson Regression (SUPR) framework or Poisson SUR system.

To present the simplest form, systems of two Poisson regression equations, consider two dependent variables Y_1 and Y_2 , each Poisson distributed with parameters λ_1 and λ_2 and

$$\log(\lambda_1) = \beta_{10} + \beta_{11} x_{11} + \beta_{12} x_{12} ,$$

$$\log(\lambda_2) = \beta_{20} + \beta_{21} x_{21} + \beta_{22} x_{22} \tag{5}$$

Where x_{11}, x_{12} are explanatory variables in the first regression equation of the system, and x_{21}, x_{22} are explanatory variables in the second regression equation of the system.

4. Bootstrap Method

This section presents the main idea of bootstrap method. Bootstrap method was introduced by Efron (1979) for the first time, and refined by Efron and Tibshirani (1993) as a method of resampling technique. Bootstrap method is used for estimating statistical properties without strong distributional assumptions. So, the bootstrap method can be used in both linear and non-linear regression models

(Freedman, 1981). This research uses the nonparametric bootstrap, which requires no assumptions about the population distribution. The idea of bootstrapping is to generate a large number of samples with the same size from the original sample using random sampling with replacement. For each bootstrapped sample, statistic of interest which is regression coefficient was computed. All the estimated coefficients construct an empirical sampling distribution which is the bootstrap distribution. The Parameter can be estimated by the expected value of the bootstrap distribution, and its standard deviation approximates the standard error of the estimate.

The nonparametric bootstrap algorithm is as follows:

- 1- Draw a random sample with replacement of the same size of the original sample, say (n).
- 2- Use the new sample to compute the statistic of interest, which is $\hat{\beta}$ in this research.
- 3- repeat steps (1) and (2) for (B) iterations, say B=1000
- 4- From the (B) bootstrapped estimates, we have a random variable. calculate the expected value to estimate the true parameter, and standard deviation to estimate its standard error.

5. Simulation Study

Monte Carlo Simulation Study was designed in order to compare the performance of estimates of maximum likelihood and bootstrap of the multiple Poisson regression parameters under multicollinearity. The simulation study uses different degrees of multicollinearity and different sample sizes (Khan *et al.*, 2021). The R language is used to conduct the simulation study, based on two factors, which are:

1- The correlation between explanatory variables (r): the degree of correlation represents the degree of multicollinearity, where it takes the levels of 0.75, 0.85, 0.95, and 0.99, representing increasing of multicollinearity.

2- Sample size (n): It's determined by levels of 25, 50, 100, 150, and 200.

The algorithm of Halawa and Azzam (1995) study was used to simulate explanatory variables with different degrees of predefined correlations. In order to building a multiple regression model with two explanatory variables ($p = 2$), true parameters were set as $\beta_0 = 1.0$, $\beta_1 = 0.35$, $\beta_2 = -0.95$.

The dependent variable was simulated from a Poisson distribution with the parameter:

$$\mu_i = \lambda_i = \exp(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}) \quad (6)$$

The R function used to generate the dependent variable as a random Poisson distribution is “*rpois*”. It uses two arguments, Sample size (n), and Poisson parameter shown in equation (6). The R code used is shown as follows:

```
# Set betas  
b0 = 1.00  
b1 = 0.35
```

$b_2 = -0.95$

```
# Compute lambda
```

```
lambda = exp(b0 + b1 * X1 + b2 * X2)
```

```
# Generate the dependent variable
```

```
y <- rpois(n, lambda)
```

Appendix (1) shows the generated data in case of $n = 25$ and all levels of multicollinearity as an example.

Model was estimated using MLE, and standard errors were calculated using the following R code:

```
p.model = glm(y ~ X1 + X2, family = poisson(link = log))
```

```
p.se = c((summary(p.model))$coefficients[,2])
```

Then, model was estimated using bootstrap, and standard errors were calculated to compare it with the MLE standard errors, using the following R code:

```
boot.coef <- matrix(c(0), nrow = 1000, ncol = p+1)
```

```
boot.coef[1, ] = p.model$coefficients
```

```
for (i in 1: 999) {
```

```
boot.data <- data[sample(nrow(data), n, replace = TRUE),]
```

```
boot.model = glm(boot.data$y ~ boot.data$x1 + boot.data$x2,
```

```
family = poisson(link = log))
```

```
boot.coef[(i+1), ] = boot.model$coefficients
```

```
}
```

```
boot.b0 = mean(boot.coef[,1])
```

```
boot.se.b0 = sqrt(((var(boot.coef[,1]))*(n-1)/n))
```

```
boot.b1 = mean(boot.coef[,2])
```

```
boot.se.b1 = sqrt(((var(boot.coef[,2]))*(n-1)/n))
```

```
boot.b2 = mean(boot.coef[,3])
```

```
boot.se.b2 = sqrt(((var(boot.coef[,3]))*(n-1)/n))
```

```
boot.results <- matrix(c(boot.b0, boot.b1, boot.b2, boot.se.b0,
```

```
boot.se.b1, boot.se.b2), nrow = p+1, ncol = 2)
```

In context of Poisson regression systems, equation (5) can be applied with the true parameters for the first regression equation $\beta_{10} = 1.0$, $\beta_{11} = 0.35$, $\beta_{12} = -0.95$, and for the second regression equation $\beta_{20} = 0.8$, $\beta_{21} = 0.50$, $\beta_{22} = -0.70$.

6. Simulation Results

The results of the simulation study (standard errors for both methods) are presented in Appendix (2). Simulation results reveal that bootstrap standard errors are consistently lower than maximum likelihood standard errors across all sample sizes and under all levels of multicollinearity.

Excel was used to plot the standard errors for the three estimates (b_0 , b_1 , and b_2) of both maximum likelihood and bootstrap methods.

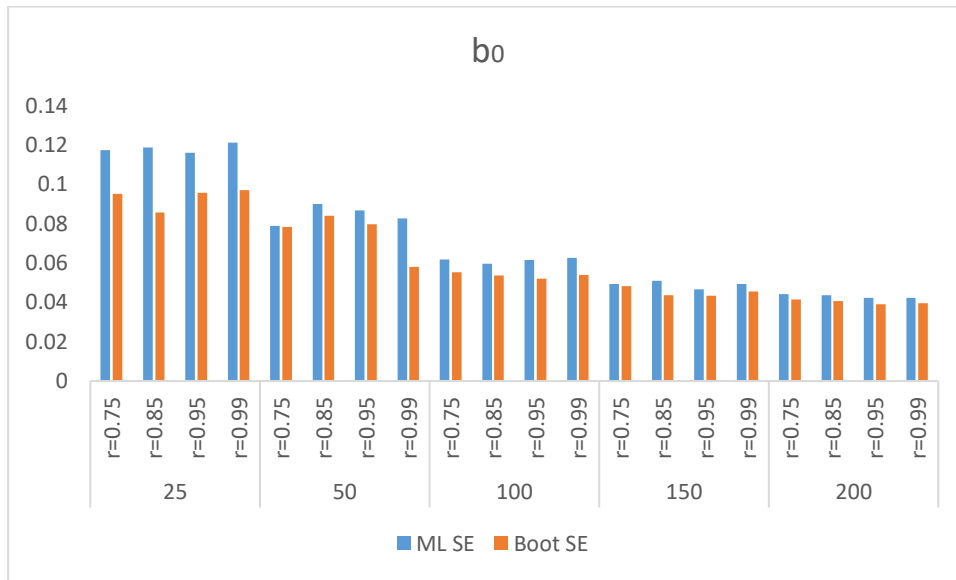


Figure (1) Standard errors of b_0 using the maximum likelihood and bootstrap methods

Figure (1) shows that standard errors of the Intercept (b_0) decrease with increasing sample size for both methods, with bootstrap outperforming MLE for all cases.

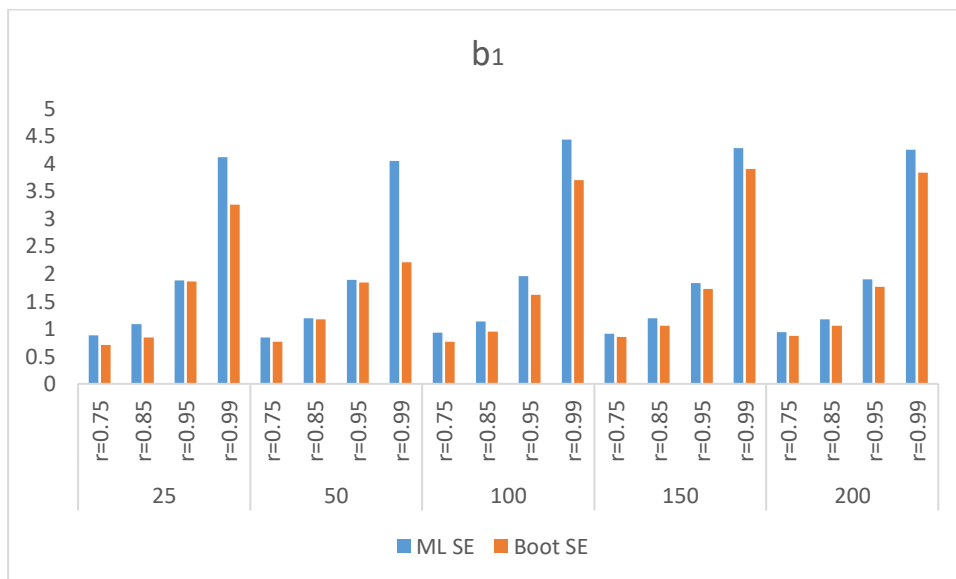


Figure (2) Standard errors of b_1 using the maximum likelihood and bootstrap methods

Figures (2), and (3) show that standard errors of slopes (b_1 , b_2) generally increase with multicollinearity, but for all cases, the standard errors of the bootstrap method were less than the standard errors of the maximum likelihood method, which means superiority of bootstrap method.

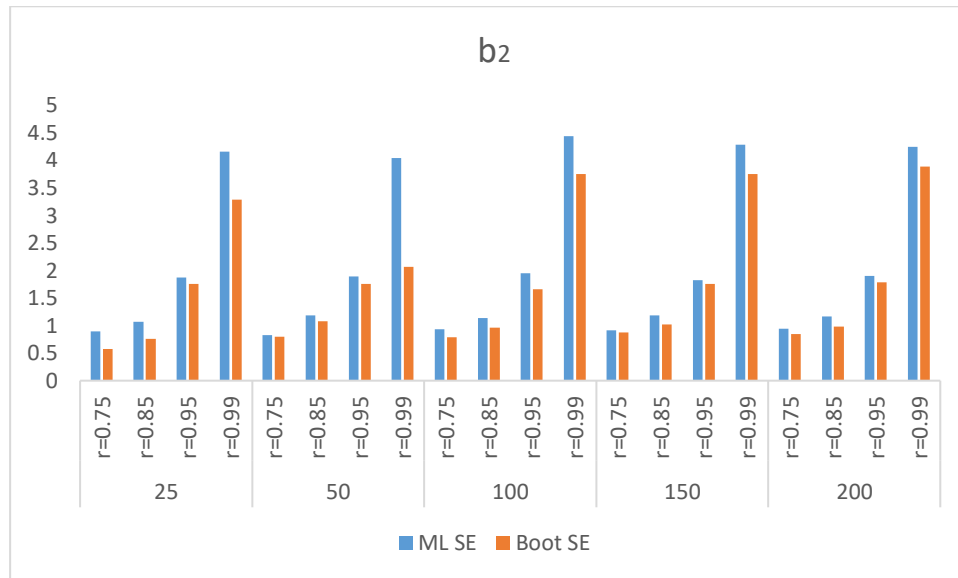
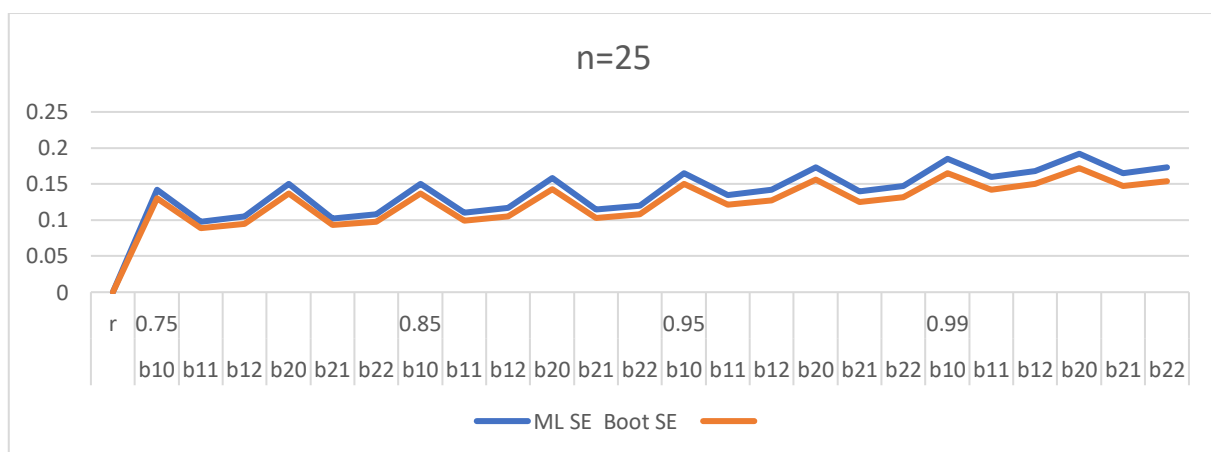


Figure (3) Standard errors of b_2 using the maximum likelihood and bootstrap methods

It is also clear from Figures (1), (2), and (3) that the plot pattern differs in the case of the intercept b_0 from the case of the two slopes b_1 and b_2 which represent estimators of the effect of the explanatory variables while the bootstrap method remains superior.

Figures (2), and (3) reflect the advantage of using the bootstrap method in the case of multicollinearity in multiple Poisson regression for all sample sizes under study, as evidenced by lower standard errors in the simulation study. We recommend its adoption when explanatory variables are correlated especially in the presence of severe multicollinearity.

Appendix (3) shows the bootstrap's performance under multicollinearity in Poisson Seemingly Unrelated Regression (SUR) system through presenting the standard errors for Poisson SUR System ($b_{10}, b_{11}, b_{12}, b_{20}, b_{21}, b_{22}$).



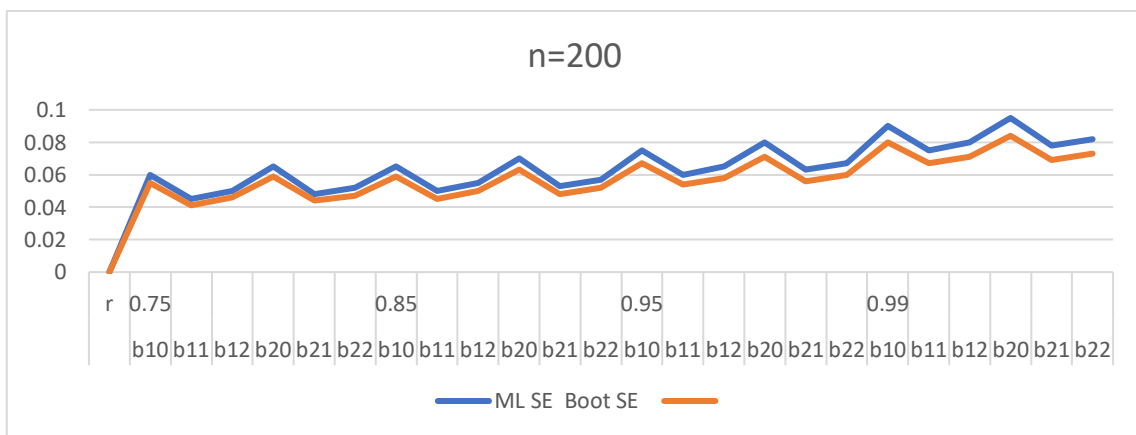
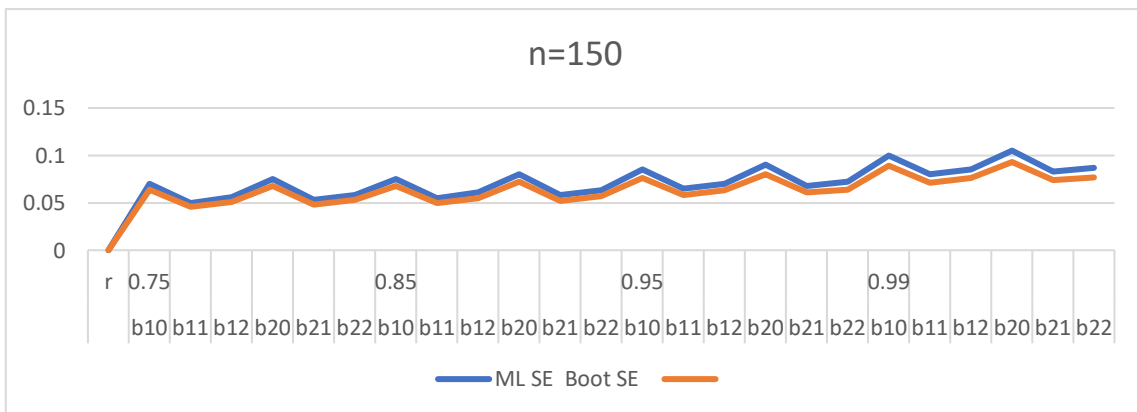
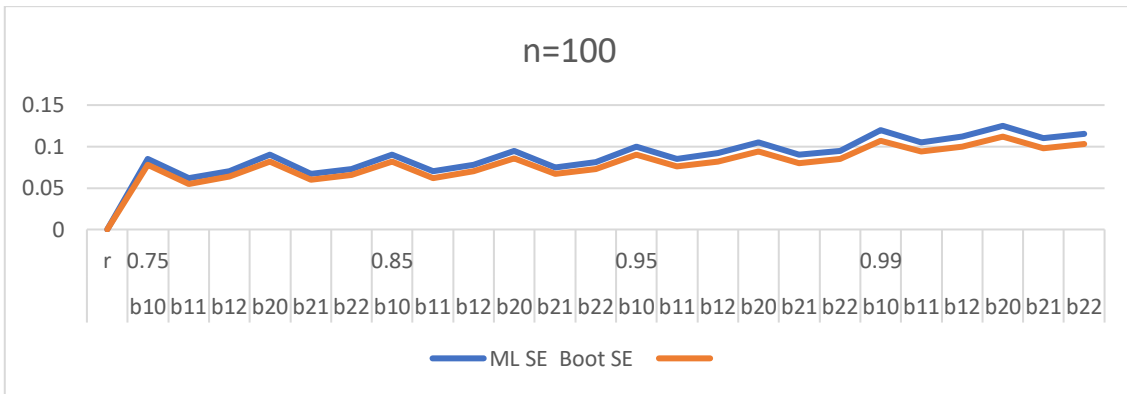
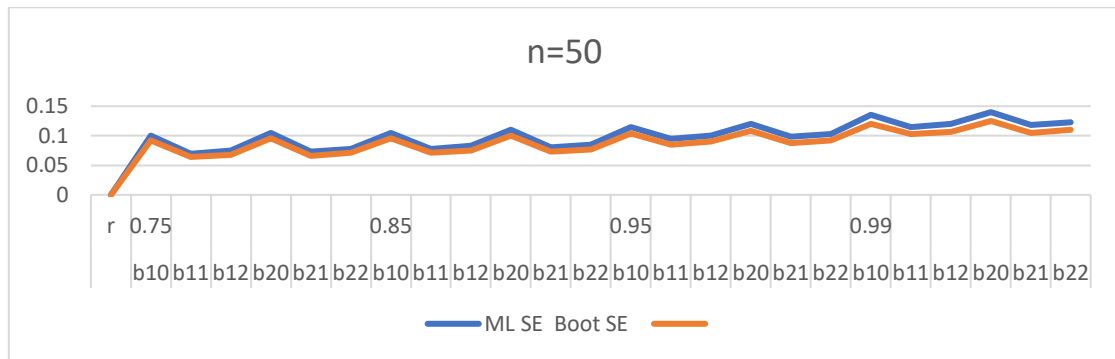


Figure (4) Standard errors of coefficients of Poisson SUR using the ML and bootstrap methods

The simulation results in Appendix (3) and Figure (4) confirm that the bootstrap method outperforms MLE in Poisson SUR systems under multicollinearity. Across all sample sizes and correlation levels, bootstrap standard errors (SEs) were consistently lower than MLE SEs, which proves the bootstrap's Superiority and demonstrates bootstrap's ability to mitigate variance inflation caused by multicollinearity particularly in context of Poisson SUR system.

This study demonstrates that the bootstrap method significantly enhances parameter estimation in Poisson SUR systems under multicollinearity, outperforming MLE by reducing standard errors by 7–12% across all tested conditions. The results highlight bootstrap's robustness in small and large samples ($n = 25$ to 200), and varying multicollinearity levels ($r = 0.75$ to 0.99), making it a valuable tool for count data modeling in the presence of collinearity.

References

1. Adane, M. D., Wale, T., & Meried, E. W. (2021). Determinants of automated teller machine deployment in commercial banks of Ethiopia. *Heliyon*, 7(8), e07712. <https://doi.org/10.1016/j.heliyon.2021.e07712>. Available at: <https://www.sciencedirect.com/science/article/pii/S2405844021018156>
2. Chatterjee, S., & Simonoff, J. S. (2020). *Handbook of regression analysis with applications in R*. John Wiley & Sons. P.P 187-207.
3. Efron, B. (1979), "Bootstrap Methods: Another Look at The Jackknife", *Annals of Statistics*, 7, 1-26.
4. Efron, B. and Tibshirani, R.J. (1993), *An Introduction to The Bootstrap*. Champman and Hall, NewYork.
5. Evans, O. P., & Adenomom, M. O. (2014). Modeling the prevalence of malaria in Niger State: An application of Poisson regression and negative binomial regression models. *Int. J. Phys. Sci*, 2, 61-68.
6. Freedman, D.A., (1981), "Bootstrapping Regression Models", *Annals of Statistics*, 9, 1218-1228.
7. Halawa, A.M., and Azzam, A.H. (1995), "A New Method of Generating the Design Matrix of Linear Regression Models", *The Egyptian Statistical Journal*, ISSR, Cairo University, 39, 106-119.
8. Khan, A., Amanullah, M., Aljohani, H. M., & Mubarak, S. A. (2021). Influence diagnostics for the Poisson regression model using two-parameter estimator. *Alexandria Engineering Journal*, 60(5), 4745-4759.
9. Lavery, M. R., Acharya, P., Sivo, S. A., & Xu, L. (2019). Number of predictors and multicollinearity: What are their effects on error and bias in regression?. *Communications in Statistics-Simulation and Computation*, 48(1), 27-38.

10. Nelder, J. A., & Wedderburn, R. W. (1972). Generalized linear models. *Journal of the Royal Statistical Society: Series A (General)*, 135(3), 370-384.
11. Youssef, A. H., Mohamed, E. S., & Latif, S. H. A. (2023). Handling multi-collinearity using principal component analysis with the panel data model. *EUREKA: Physics and Engineering*, (1), 177-188.

Appendices

Appendix 1: Simulated data for $n=25$

r = 0.75		r = 0.85		r = 0.95		r = 0.99					
y	x1	x2	y	x1	x2	y	x1	x2			
-	-	-	-	-	-	-	-	-			
3	0.10653	0.15733	6	0.05317	0.19849	3	0.09032	0.13814	1	0.34803	0.35618
-	-	-	-	-	-	-	-	-	-	-	-
3	0.02634	0.25128	3	0.12017	0.04111	7	0.18197	0.25845	3	0.04678	0.05462
-	-	-	-	-	-	-	-	-	-	-	-
3	0.05143	0.08582	2	0.09045	0.23153	2	0.05828	0.00791	3	0.09873	0.08080
-	-	-	-	-	-	-	-	-	-	-	-
4	0.10479	0.19103	1	0.33867	0.41615	5	0.07062	0.05373	5	0.18068	0.18657
-	-	-	-	-	-	-	-	-	-	-	-
2	0.17517	0.37533	5	0.08962	0.01117	4	0.10940	0.16856	2	0.17006	0.20796
-	-	-	-	-	-	-	-	-	-	-	-
3	0.25914	0.49927	3	0.20605	0.26936	3	0.28748	0.22129	1	0.39666	0.35248
-	-	-	-	-	-	-	-	-	-	-	-
1	0.02175	0.10858	4	0.14276	0.17007	2	0.29270	0.30187	1	0.12512	0.15048
-	-	-	-	-	-	-	-	-	-	-	-
2	0.15983	0.05384	3	0.02671	0.01639	5	0.11617	0.12905	5	0.13133	0.13381
-	-	-	-	-	-	-	-	-	-	-	-
2	0.19043	0.04342	4	0.19157	0.28835	4	0.12602	0.04645	2	0.00245	0.01251
-	-	-	-	-	-	-	-	-	-	-	-
5	0.05202	0.07405	4	0.10759	0.09800	3	0.14730	0.14465	2	0.16247	0.22316
-	-	-	-	-	-	-	-	-	-	-	-
6	0.03230	0.05364	4	0.12461	0.01298	1	0.37780	0.28374	3	0.15678	0.17130
-	-	-	-	-	-	-	-	-	-	-	-
4	0.10456	0.02594	1	0.09624	0.23662	1	0.22280	0.25375	4	0.06645	0.06001
-	-	-	-	-	-	-	-	-	-	-	-
4	0.7613	0.8552	1	0.2928	0.9675	1	0.4658	0.8515	4	0.5789	0.9184

	-	-	-	-	-	-	-	-	-
	0.13222	0.02869	0.22439	0.36848	0.25694	0.18198	0.07719	0.06471	
4	1036	625	6	8506	3039	2	7008	8757	1
	-	-	-	-	-	-	-	-	-
	0.32744	0.12795	0.21052	0.03271	0.26759	0.32207	0.33671	0.34740	
0	5794	0861	2	3943	1911	3	9133	3169	3
	-	-	-	-	-	-	-	-	-
	0.25497	0.15936	0.11856	0.07039	0.26709	0.30809	0.20182	0.21249	
5	9985	7525	4	8984	2259	2	831	5182	3
	-	-	-	-	-	-	-	-	-
	0.30349	0.26057	0.19101	0.18077	0.16047	0.14007	0.27032	0.23156	
2	7602	8867	4	6723	812	3	6262	151	3
	0.40506	0.27442	0.29852	0.31761	0.04452	0.02119	0.38022	0.40236	
3	7429	8003	1	1789	3561	3	7098	5712	1
	-	-	-	-	-	-	-	-	-
	0.18632	0.21783	0.38112	0.33703	0.34993	0.27648	0.21368	0.17331	
1	2583	7393	2	4876	232	1	9628	7885	6
	-	-	-	-	-	-	-	-	-
	0.02656	0.12903	0.29619	0.06982	0.16792	0.04445	0.03708	0.00663	
2	1579	7165	4	4486	8249	3	3491	925	2
	-	-	-	-	-	-	-	-	-
	0.11007	0.18369	0.22820	0.05199	0.04034	0.00214	0.17934	0.18800	
2	4416	5072	5	5026	1259	5	3781	5524	4
	-	-	-	-	-	-	-	-	-
	0.19013	0.08106	0.19500	0.15776	0.29556	0.38759	0.02207	0.03776	
3	5323	8841	1	651	022	1	9989	5726	5
	-	-	-	-	-	-	-	-	-
	0.15935	0.04624	0.10262	0.02863	0.12380	0.18195	0.05300	0.05792	
3	0795	9722	3	6261	125	2	2383	634	5
	-	-	-	-	-	-	-	-	-
	0.31624	0.18813	0.19841	0.05686	0.13074	0.03047	0.13671	0.13492	
3	9965	1401	2	1261	5638	7	0885	2142	3
	-	-	-	-	-	-	-	-	-
	0.35500	0.24401	0.32152	0.22139	0.01452	0.05526	0.28411	0.24364	
3	7231	8306	4	6472	9144	4	1983	7062	4
	-	-	-	-	-	-	-	-	-
	0.13594	0.25530	0.06697	0.08264	0.13272	0.20103	0.00889	0.06733	
5	8122	7109	1	3956	9072	2	5564	6274	1

Appendix 2: Simulation results (standard errors for single Poisson regression equation: b_0 , b_1 , b_2)

		25		50		100	
		ML SE	Boot SE	ML SE	Boot SE	ML SE	Boot SE
b_0	0.75	0.117189	0.095108	0.078695	0.078162	0.061773	0.055127
	0.85	0.118614	0.085532	0.089895	0.083876	0.059685	0.053549

	0.95	0.11602	0.095709	0.086605	0.079743	0.06135	0.052055
	0.99	0.12123	0.096972	0.082744	0.057873	0.062571	0.053888
b1	0.75	0.882124	0.705526	0.843708	0.762228	0.929538	0.767394
	0.85	1.081748	0.841829	1.194372	1.172992	1.135699	0.950911
	0.95	1.881286	1.857075	1.890855	1.838357	1.953252	1.616719
	0.99	4.113244	3.249813	4.046614	2.202222	4.437075	3.700107
b2	0.75	0.895983	0.575088	0.828156	0.802495	0.931182	0.79153
	0.85	1.071833	0.754677	1.183832	1.081668	1.133997	0.963876
	0.95	1.874871	1.751994	1.892222	1.757994	1.954476	1.661746
	0.99	4.156519	3.283358	4.038606	2.0637	4.434121	3.753883

		150		200		
		Boot SE	ML SE	Boot SE	ML SE	Boot SE
b0	0.75	0.055127	0.04934	0.048282	0.04401	0.041545
	0.85	0.053549	0.050905	0.04349	0.04365	0.04063
	0.95	0.052055	0.046659	0.043409	0.042185	0.039026
	0.99	0.053888	0.049218	0.045421	0.042335	0.039614
b1	0.75	0.767394	0.91357	0.848553	0.939454	0.866355
	0.85	0.950911	1.186948	1.056583	1.17008	1.049974
	0.95	1.616719	1.825074	1.717767	1.899774	1.760119
	0.99	3.700107	4.279265	3.897103	4.245321	3.83236
b2	0.75	0.79153	0.91268	0.874206	0.938709	0.84627
	0.85	0.963876	1.18424	1.022127	1.168602	0.9794
	0.95	1.661746	1.823047	1.754595	1.900257	1.785977
	0.99	3.753883	4.284385	3.748556	4.245246	3.884735

Appendix 3: Simulation results (standard errors for Poisson SUR System: b_{10} , b_{11} , b_{12} , b_{20} , b_{21} , b_{22})

		n=25		n=50		n=100		n=150		n=200	
R		ML SE	Boot SE	ML SE	Boot SE	ML SE	Boot SE	ML SE	Boot SE	ML SE	Boot SE
	b10	0.75	0.142	0.13	0.1	0.092	0.085	0.078	0.07	0.064	0.06

b11		0.098	0.089	0.07	0.064	0.062	0.055	0.05	0.046	0.045	0.041
b12		0.105	0.095	0.075	0.068	0.07	0.064	0.056	0.051	0.05	0.046
b20		0.15	0.137	0.105	0.096	0.09	0.082	0.075	0.068	0.065	0.059
b21		0.102	0.093	0.073	0.066	0.067	0.06	0.053	0.048	0.048	0.044
b22		0.108	0.098	0.078	0.071	0.073	0.066	0.058	0.053	0.052	0.047
b10	0.85	0.15	0.137	0.105	0.096	0.09	0.082	0.075	0.068	0.065	0.059
b11		0.11	0.099	0.078	0.071	0.07	0.062	0.055	0.05	0.05	0.045
b12		0.117	0.105	0.083	0.075	0.078	0.07	0.061	0.055	0.055	0.05
b20		0.158	0.143	0.11	0.1	0.095	0.086	0.08	0.072	0.07	0.063
b21		0.115	0.103	0.08	0.073	0.075	0.067	0.058	0.052	0.053	0.048
b22		0.12	0.108	0.085	0.077	0.081	0.073	0.063	0.057	0.057	0.052
b10	0.95	0.165	0.15	0.115	0.104	0.1	0.09	0.085	0.076	0.075	0.067
b11		0.135	0.121	0.095	0.085	0.085	0.076	0.065	0.058	0.06	0.054
b12		0.142	0.127	0.1	0.09	0.092	0.082	0.07	0.063	0.065	0.058
b20		0.173	0.156	0.12	0.108	0.105	0.094	0.09	0.08	0.08	0.071
b21		0.14	0.125	0.098	0.088	0.09	0.08	0.068	0.061	0.063	0.056
b22		0.147	0.132	0.103	0.092	0.095	0.085	0.072	0.064	0.067	0.06
b10	0.99	0.185	0.165	0.135	0.12	0.12	0.107	0.1	0.089	0.09	0.08
b11		0.16	0.142	0.115	0.103	0.105	0.094	0.08	0.071	0.075	0.067
b12		0.168	0.15	0.12	0.107	0.112	0.1	0.085	0.076	0.08	0.071
b20		0.192	0.172	0.14	0.125	0.125	0.112	0.105	0.093	0.095	0.084
b21		0.165	0.147	0.118	0.105	0.11	0.098	0.083	0.074	0.078	0.069
b22		0.173	0.154	0.123	0.11	0.115	0.103	0.087	0.077	0.082	0.073