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Anisotropic Charged Spheres in Higher Dimensional Rosen's Bimetric Relativity Theory

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Abstract:

In the present study, we presented the field equations for anisotropic charged spheres in higher dimensional spherically symmetric space-time in the framework of Rosen's bimetric relativity theory (Rosen 1979, 1980). To obtain the determinate solutions of the higher dimensional field equations, the strategy we adopted is to assume that the space-time admits one parameter group of conformal motions generated by the vector field ξ^{μ} if $L\xi g_{\mu\nu} = \psi\ g_{\mu\nu}$.

Keywords: Bimetric theory, spherically symmetric model

1. Introduction

Rosen [1, 2, 3] proposed a new theory of gravitation called the bimetric general relativity theory. In this theory, in addition to the metric tensor g_{ij} (the gravitational or physical metric), a second metric tensor γ_{ij} (the background metric) has been introduced describing the geometry of space-time, which one would have if there were no matter in the universe. In this respect the theory is similar to an earlier version of bimetric theory formulated by Rosen [4, 5, 6]. The field equations of the present theory of gravitation are taken to be the same as in general relativity, except for the fact that ordinary derivatives of the physical metric are replaced by covariant derivatives with respect to the background metric. The background metric γ_{ij} is taken to describe a space-time of constant curvature. It was found by Rosen [2] that these field equations could be written in the form of Einstein field equations, but with an additional term on the right-hand side,

$$G_{ij} = S_{ij} - 8\pi\ T_{ij}, \tag{1}$$

where the Einstein tensor $G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$ and T_{ij} is the energy-momentum tensor of the matter distribution and

$$S_{ij} = \frac{3}{a^2}\left(\gamma_{ij} - \frac{1}{2} g_{ij} g^{lk} \gamma_{lk}\right), \tag{2}$$

where, a is a constant scale parameter, of the order of the size of the universe.

From this one sees that, in general, if one is dealing with a physical system that is small compared to the size of the universe, e.g., the solar system, then the term S_{ij} is negligible in the field equations (1) and the present theory gives agreement with Einstein general relativity theory.

In recent years, the study of higher dimensional space-time has gained momentum because of the fact that the cosmos at eats early stage of evolution might have had a higher dimensional era. The extra space reduced to volume with passage of time, which is beyond the ability of experimental observations at the moment. Also, the possibility that the world may have more than 4-dimensions is due to Kaluza [7] and Klein [8], who used one extra dimension to unify gravity and electromagnetism in a theory that was essentially 5-dimensional in general relativity. Sabbatta [9], Lee [10], Appelquist and Chodos [11], and Collins et al. [12] accepted this idea and constructed cosmological models in higher dimensions by using various phenomena of particle physics and cosmology. Overduin and Wesson [13] presented an excellent review of Kaluza Klein theory and higher-dimensional unified theories, in which the cosmological and astrophysical implications of extra dimensions were studied. Mohanty and Mahanta [14, 15], Mohanty and Samanta [16] are some of the authors who have studied higher-dimensional cosmology in various theories of gravitation. There is now extensive literature dealing with different aspects of higher-dimensional cosmology. Particularly, Karade and khadekar [17] have shown non-existence of higher dimensional axially symmetric field in Rosen's theory of gravitation. Reddy and Venkateswara [18] have shown non-existence of higher dimensional anisotropic cosmological model in bimetric theory of gravitation. Katore et al. [19] have shown the non-existence of n-dimensional static plane symmetric solutions in bimetric relativity theory. Khadekar and Tade [20] studied string cosmological models in five dimensional bimetric theory of gravitation. Herrera et al. [21] studied the consequences of inclusion of one parameter group of conformal motion of anisotropic matter in the Einstein's general relativity and obtained analytical solutions of field equations for static and spherically symmetric distribution of isotropic and anisotropic matter. The study of static anisotropic spheres is important in relativistic astrophysics [22].

Sah and Prakash Chandra [23] have studied class of charged fluid balls in general relativity. Hasmani and Pandya [24] have investigated cosmological models for the static spherically symmetric spacetime with charged anisotropic fluid distribution in (n+2)-dimensions in context of Rosen's Bimetric General Relativity. Mukesh Kumar Madhukar [25] presented the study of some charged fluid spheres in General Relativity solutions and conditions for charged fluid spheres in General Relativity. Francesco Bajardi and et. al. [26] studied the higher dimensional static and spherically symmetric solutions in Extended Gauss–Bonnet Gravity. Recently, Zahraa and Mardanb [27] have presented five-dimensional analysis of electromagnetism with heat flow in the post-quasi-static approximation.

With the above motivation, in this paper, we have presented the field equations in higher dimensional static spherically symmetric space-time in the framework of bimetric theory of relativity proposed by Rosen [2, 3] for anisotropic distribution of matter and also obtain the solutions for the physical metric with the assumption that the space-time admits a one-parameter group of conformal motions. This work is a generalization of the results of Hererra et al. [21] obtained in general relativity to the higher dimensional bimetric theory of relativity proposed by Rosen. The solution gives agreement with Einstein's general relativity for physical system known in the universe, such as the solar system.

The work done, in this paper, is organized as follows: In section-2, we obtain the field equations in the presence of an anisotropic charged spherically symmetric matter in the frame work of higher dimensional bimetric theory of relativity proposed by Rosen [2, 3]. Section-3 deals with the solutions of the field equations under the assumption that the space time admits a one-parameter group of conformal motions [Herrera et al. (1985)] generated by the vector field ξ^{μ} if

$$\mathrm{L}\xi g_{\mu\nu} = \psi \; g_{\mu\nu} \;, \tag{1}$$

where L.H.S. of (1) is the Lie derivative operator and ψ is conformal factor. Section-4 contains some conclusions.

2. Metric and Field Equations

We consider the higher dimensional static distribution of matter represented by charged spherically symmetric fluid which may be anisotropic. In terms of Schwarzschild coordinates the physical metric takes the form

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\Omega^2, \tag{2}$$

where

$$d\Omega^2 = d\theta_1^2 + sin^2\theta_1\, d\theta_2^2 + sin^2\theta_1 \;\; sin^2\theta_3\, d\theta_3^2 + \cdots + \left[\prod_{n-1}^{i=1} sin^2\theta_i\right] d\theta_n^2$$

and λ, ν are the functions of r only.

With this choice of coordinates the total energy-momentum tensor T_{ij} is assumed to be the sum of two parts, $\overset{A}{T}_{ij}$ and $\overset{E}{T}_{ij}$,

i.e.
$$T_{ij} = \overset{A}{T}_{ij} + \overset{E}{T}_{ij}. \tag{3}$$

Here $\overset{\boldsymbol{A}}{\boldsymbol{T}}_{\boldsymbol{ij}}$ is an anisotropic spherically symmetric matter distribution and is given by

$$\overset{A}{T}_{ij} = (\rho_m + P_\perp)\; U_i \; U_j - P_\perp g_{ij} + (P_r - P_\perp)\; \chi_i \; \chi_j, \tag{4}$$

where U^i is the time like $(n+2)$ -velocity $U^i = \delta^i_{(n+2)} e^{-\nu/2}$, χ^i a unit space-like vector in the radial direction $\chi^i = \delta^i_1 e^{-\lambda/2}$, ρ_m the energy density, P_r the radial pressure in the direction of χ_i and $P_\perp$ the pressure orthogonal to χ_i.

Similarly, the electromagnetic contribution can be written as

$$\overset{\boldsymbol{E}}{\boldsymbol{T}}_{\boldsymbol{ij}} = -\frac{1}{4\pi}\left(\boldsymbol{F}_i^{\alpha}\boldsymbol{F}_{j\alpha} - \frac{1}{4}\boldsymbol{g}_{ij}\,\boldsymbol{F}_{lk}\boldsymbol{F}^{lk}\right), \tag{5}$$

where $\boldsymbol{F}_{ij}$ is the electromagnetic field tensor defined in terms of the $(\boldsymbol{n}+\boldsymbol{2})$-potential $\boldsymbol{A}_i$ as

$$\boldsymbol{F}_{ij} = \boldsymbol{A}_{i,\,j} - \boldsymbol{A}_{j,i}\,. \tag{6}$$

As the frame is considered to be the rest frame, we adopt the gauge

$$\boldsymbol{A_i = A_i(0,\ 0,\ 0,\ \cdots (n+1)\ times\ ,\varphi(r))}.$$

Also, $\boldsymbol{F_{ij}}$ satisfy the Einstein-Maxwell field equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0, \tag{7}$$

$$F^{ij}_{;j} = -4\pi\ J^i, \tag{8}$$

whereJ^i being the charge-current density that becomes $J^i = \bar{\rho}_e u_i$ ($\bar{\rho}_e$ is the proper charge density).

In the commoving system, we choose

$$\left.\begin{array}{c} U^i = (\ 0,\ 0,\ 0,\ \cdots,(n+1)0, U^{n+2}) \\ \chi^i = (\chi^1,\ 0,\ 0,\ \cdots\ ,0(n+1)) \end{array}\right\} \tag{9}$$

From $U^i\ U_i = -\chi^i\ \chi_i = 1$, we obtain

$$U^{(n+2)} = e^{-\nu/2},\quad \chi^1 = e^{-\lambda/2}. \tag{10}$$

The non-vanishing components of energy momentum tensors T^i_j are

$$\left.\begin{array}{l} \overset{A}{T}{}^1_1 = -P_r,\ \overset{A}{T}{}^2_2 = \overset{A}{T}{}^3_3 = \cdots = \overset{A}{T}{}^{n+1}_{n+1} = -P_\perp,\ \overset{A}{T}{}^{n+2}_{n+2} = \rho \\ \text{and}\quad \overset{E}{T}{}^1_1 = \overset{E}{T}{}^{n+2}_{n+2} = E^2,\ \overset{E}{T}{}^2_2 = \overset{E}{T}{}^3_3 = \overset{E}{T}{}^4_4 = \cdots = \overset{E}{T}{}^{n+1}_{n+1} = -E^2. \end{array}\right\} \tag{11}$$

Now let us tern to the gravitational field equations. In bimetric theory of reletivity they have the form [Rosen, (1980)]

$$G_{ij} \equiv R_{ij} - \frac{1}{2}g_{ij}R - S_{ij} = -8\pi\left(\overset{A}{T}_{ij} + \overset{E}{T}_{ij}\right), \tag{12}$$

where the additional term S_{ij} is given by

$$S_{ij} = \frac{3}{a^2}\left(\gamma_{ij} - \frac{1}{2}g_{ij}g^{\alpha\beta}\gamma_{\alpha\beta}\right). \tag{13}$$

If we take the background metric γ_{ij} in a higher dimensional static de-Sitter form, the line element in terms of polar coordinates

$$(x^1, x^2 \cdots, x^{n+1},\ x^{n+2}) = (r, \theta_1,\ \theta_2,\ \cdots, \theta_n,\ t)$$

is given by

$$d\sigma^2 = \left(1 - \frac{r^n}{a^2}\right)\ dt^2 - \left(1 - \frac{r^n}{a^2}\right)^{-1} dr^2 - \ r^2 d\Omega^2. \tag{14}$$

For a region very small as compared to a (i.e. for $r << a$), this line element has the flat space form

$$d\sigma^2 = dt^2 - dr^2 - r^2 d\Omega^2. \tag{15}$$

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Let us consider a region with $r << a$ and neglect the terms, which are small throughout this region, we can write the non-vanishing components of S_i^j in higher dimensions (Falik and Rosen, 1980)

$$-S_1^1 = -S_2^2 = -S_3^3 = \cdots = -S_{n+1}^{n+1} = S_{n+2}^{n+2} = \frac{3}{2a}e^{-\nu}. \qquad (16)$$

Rosen's (1980) bimetric field equations given by equation (12) for the for the line-element (2) (for $r << a$) can be written as

$$-e^{-\lambda}\left[\frac{n(n-1)}{2r^2} - \frac{n\lambda'}{2r}\right] + \frac{n(n-1)}{2r^2} = -\frac{3}{2a^2}e^{-\nu} + .8\pi\rho + E^2 \qquad (17)$$

$$-e^{-\lambda}\left[\frac{n(n-1)}{2r^2} + \frac{n\nu'}{2r}\right] + \frac{n(n-1)}{2r^2} = \frac{3}{2a^2}e^{-\nu} - 8\pi P_r + E^2 \qquad (18)$$

$$e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{(n-1)(\lambda'-\nu')}{2r} - \frac{\lambda'\nu'}{4} + \frac{(n-1)(n-2)}{2r^2}\right] - \frac{(n-1)(n-2)}{2r^2}$$
$$= -\frac{3}{2a^2}e^{-\nu} + 8\pi P_\perp + E^2, \qquad (19)$$

where a prime denotes a derivative with respect to r.

We now define the effective density and pressures ρ_{m_e} , P_{r_e} and $P_{\perp_e}$ (Harpaz and Rosen, 1985) as

$$\left.\begin{aligned} \rho_{m_e} = \rho_m - \frac{3}{16\pi a^2}e^{-\nu}, \quad P_{r_e} = P_r - \frac{3}{16\pi a^2}e^{-\nu} \\ \text{and} \quad P_{\perp_e} = P_\perp - \frac{3}{16\pi a^2}e^{-\nu}, \end{aligned}\right\} \qquad (20)$$

so that the field equations (17) – (19) take the form

$$-e^{-\lambda}\left[\frac{n(n-1)}{2r^2} - \frac{n\lambda'}{2r}\right] + \frac{n(n-1)}{2r^2} = 8\pi\rho_{m_e} + E^2 \qquad (21)$$

$$-e^{-\lambda}\left[\frac{n(n-1)}{2r^2} + \frac{n\nu'}{2r}\right] + \frac{n(n-1)}{2r^2} = -8\pi P_{r_e} + E^2 \qquad (22)$$

$$e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{(n-1)(\lambda'-\nu')}{2r} - \frac{\lambda'\nu'}{4} + \frac{(n-1)(n-2)}{2r^2}\right] - \frac{(n-1)(n-2)}{2r^2}$$
$$= 8\pi P_{\perp_e} + E^2. \qquad (23)$$

These equations look like the usual Einstein equations with ρ_m, p_r and $p_\perp$ replaced by ρ_{m_e}, p_{r_e} and $p_{\perp_e}$ respectively.

It is very difficult to obtain the solution of equations (21)-(23) due to the non-linearity of the field equations and therefore one has to make certain simplifying assumption to derive the useful results.

3. Conformal motions and solutions of field equations

In this section, we assume that the physical metric (2) admits a one-parameter group of conformal motions and obtained solution of the field equations for static and spherically symmetric fluid distributions of anisotropic matter.

The field equations (21)-(23) resulting in the new form

$$\frac{n(n-1)}{2r^2}\left[1-\frac{\psi^2}{C_3^2}\right]-\frac{n\psi\psi'}{C_3^2 r}=8\pi\rho_e+E^2, \tag{24}$$

$$-\frac{n\psi^2}{C_3^2 r^2}\left[1-\frac{(n-1)}{2}\right]+\frac{n(n-1)}{2r^2}=8\pi P_{r_e}+E^2, \tag{25}$$

$$\frac{n\psi\psi'}{rC_3^2}+\frac{n(n-1)\psi^2}{2r^2C_3^2}-\frac{(n-1)(n-2)}{2r^2}=8\pi P_{\perp_e}+E^2, \tag{26}$$

We now define a function Δ, as a measure of anisotropy, such that

$$\Delta=4\pi(P_{\perp_e}-P_{r_e}). \tag{27}$$

Also, if we take $X=\frac{\psi^2}{{C_3}^2}$, then in terms of the function defined in (27), the equations (24) –(26) reduces to :

$$\frac{n(n-1)}{4r^2}-\frac{nX}{2r^2}+\frac{nX'}{4r}-\frac{(n-1)(n-2)}{4r^2}-\Delta=E^2 \tag{28}$$

$$\frac{n(n-1)}{4r^2}-\frac{3nX'}{4r}-\frac{n(n-2)X}{4r^2}-\frac{(n-1)(n-2)}{4r^2}+\Delta=8\pi\rho_{m_e} \tag{29}$$

$$\frac{nX}{2r^2}-\frac{n(n-1)}{4r^2}+\frac{nX'}{4r}-\frac{(n-1)(n-2)}{4r^2}-\Delta=8\pi P_{r_e}. \tag{30}$$

From which we obtain

$$\frac{n(n-1)}{4r^2}-\frac{nX}{2r^2}+\frac{nX'}{4r}-\frac{(n-1)(n-2)}{4r^2}-\Delta=E^2 \tag{31}$$

$$\frac{n(n-1)}{4r^2}-\frac{3nX'}{4r}-\frac{n(n-2)X}{4r^2}-\frac{(n-1)(n-2)}{4r^2}+\Delta=\frac{-3}{2a^2}e^{\nu}+8\pi\rho_{m_e} \tag{32}$$

$$\frac{nX}{2r^2}-\frac{n(n-1)}{4r^2}+\frac{nX'}{4r}-\frac{(n-1)(n-2)}{4r^2}-\Delta=\frac{3}{2a^2}e^{\nu}-8\pi P_{r_e}. \tag{33}$$

Therefore, the line element (2), in terms of $e^{\nu}={C_2}^2r^2$, $\psi={C_3}^2e^{-\lambda/2}$ and $X=\frac{\psi^2}{{C_3}^2}$ now becomes

$$ds^2={C_2}^2r^2dt^2-\left(\frac{1}{X}\right)dr^2-r^2d\Omega^2. \tag{34}$$

4. Conclusions

In this paper, we have presented the field equations for anisotropic charged spheres in higher dimensional Rosen’s bimetric theory of relativity. To get a determinate solution of the field equations, we have assumed that the space-time admits one parameter group of conformal motion. It is observed that the solutions obtained in bimetric relativity in higher dimensions will reduce to the corresponding solutions of Einstein’s field equations obtained by Herrera et al. [21] of a physical system, which is very small compared to the size of the universe, when the term $\frac{3}{2a^2}e^{-\nu}$ that appears in the field equations of bimetric relativity is negligible.

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