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Optimality and Duality for Multiobjective Semiinfinite Programming Problems Involving Generalized $(C, \alpha, \eta, \rho, d)$-Invexity

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Abstract

In this paper, we formulate generalized $(C, \alpha, \eta, \rho, d)$-invexity and based on these definitions, we derive several sufficient conditions for optimality in multiobjective semi-infinite programming problems. Further, under the assumptions of dual model we solve corresponding weak, strong and strict converse duality theorems for these multiobjective semiinfinite programming problem.

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1 Introduction

The theory surrounding semi-infinite programming involves minimizing a function with a finite number of variables while adhering to an arbitrary number of inequalities. When there are multiple objective functions, this is referred to as a multiobjective semi-infinite programming problem. Shapiro [7] provided an overview of the foundational theory of semi-infinite programming, exploring various methods for establishing duality, discretization, and both first and second-order optimality conditions. For further information and applications related to semi-infinite programming, please consult the cited references. [2, 3, 4, 5, 6, 8, 11, 12, 33, 34].

In [28], Preda introduced the idea of (F, ρ)-convexity, expanding upon the concepts of F-convexity [18] and ρ-convexity [16], and derived several duality results. Liang *et al.* in [31] later presented (F, α, ρ, d)-convexity to address nonlinear fractional programming problems, which encompasses the (F, ρ)-convex functions. Subsequently, Liang *et al.* in [30] broadened the findings from [31] to include a specific category of multiobjective fractional programming problems. Yuan *et al.* defined (C, α, ρ, d)-convexity in [13], as a generalization of (F, α, ρ, d)-convexity and established optimality conditions along with duality results for nondifferentiable minimax fractional programming problems that utilize generalized convex functions. Additionally, Long in [29] and Mishra *et al.* in [25] derived sufficient optimality conditions and duality theorems employing (C, α, ρ, d)-convexity for nondifferentiable multiobjective fractional programming and nondifferentiable multiobjective semi-infinite programming problems, respectively. Mishra *et al.* [21] also achieved optimality and duality results for minimax fractional programming involving support functions within the framework of (C, α, ρ, d)-convexity.

Invexity is crucial for establishing optimality conditions and duality results across a range of optimization problems. Invex functions represent a broader category than convex functions, retaining many of their key properties. Additionally, invexity and its generalizations can be viewed as alternatives to convexity. For further information on invex functions, please refer to the relevant literature [1, 9, 10, 17, 22, 23, 32].

Weir [27] examined a multiobjective programming problem that incorporates invex functions and derived duality results under the condition that the multipliers for all objective functions are strictly positive. Kuk *et al.* in [15] formulated generalized K-K-T necessary and sufficient optimality conditions, along with duality theorems, for nonsmooth multiobjective fractional programming problems that involve $V-\rho-$ invex functions. Caristi *et al.* [14] focused on multiobjective programming with a set of constraints defined within a compact set, obtaining Kuhn-Tucker type optimality criteria under relaxed invexity conditions. They also introduced dual problems where both weak and strong duality properties are maintained in the same framework. Additionally, Mishra *et al.* [24] developed Wolfe and Mond-Weir-type dual models, establishing duality theorems for the nonsmooth semi-infinite programming problem discussed in [20]. Motivated by the research of Ben-Israel and Mond [1], Caristi *et al.* [14] and Jaiswal and Mishra [19], we extend the definition of generalized (C, α, ρ, d)-convexity to generalized $(C, \alpha, \eta, \rho, d)$-invexity. Additionally, we present sufficient conditions for optimality and duality theorems. The structure of this paper is organized as follows:

In Section 2, we provide some preliminaries, our problem and some definitions. Section 3 is dedicated to establishing definitions that will be essential for our theorems. In Section 4, we derive sufficient conditions for optimality. Finally, Section 5 presents weak, strong, and strict converse duality theorems that connect the primal problem with the Mond-Weir dual problem under the framework of generalized $(C, \alpha, \eta, \rho, d)$-invexity.

2 Preliminaries

In an n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbb{R}^n_+$ is the non-negative orthant . Consider the nonlinear multiobjective semiinfinite programming problem defined as:

$$\text{(SP)} \qquad \text{Min } f(x)$$

$$\text{Subject to} \quad g_j(x) \leqq 0, \; j \in J,$$

Let $f(x) = (f_1(x), ..., f_p(x))$, where each $f_i(i \in \underline{p} \equiv \{1, 2, ..., p\})$ and $g_j(j \in J)$ are differentiable functions defined on a non-empty open subset $X \subseteq \mathbb{R}^n$ and map to $\mathbb{R}$. The set J is an index set that may be infinite.

Here we take the non-empty feasible set S of (SP):

$$S = \{x \in X : g_j(x) \leqq 0, j \in J\}.$$

and

$$I = \{j \in J : g_j(x_0) = 0\}$$

where I represents the index set of active constraints for $x_0 \in S$.

Let $C : X \times X \times \mathbb{R}^n \to \mathbb{R}$ be a function such that for any $(x, x_0) \in X \times X$, it satisfies $C_{(x,x_0)}(0) = 0$. Additionally, let $\alpha : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $\rho \in \mathbb{R}$, and $d : X \times X \to \mathbb{R}_+$, $(d(x, x_0) = 0$ iff $x = x_0)$.

Definition 2.1. *A function $C : X \times X \times \mathbb{R}^n \to \mathbb{R}$ is convex on $\mathbb{R}^n$ if and only if for any fixed point $(x, x_0) \in X \times X$ and for any $y_1, y_2 \in \mathbb{R}^n$, the following condition holds:*

$$C_{(x,x_0)}(\lambda y_1 + (1-\lambda)y_2) \leqq \lambda C_{(x,x_0)}(y_1) + (1-\lambda)C_{(x,x_0)}(y_2), \quad \forall \; \lambda \in (0, 1).$$

Definition 2.2. *A feasible point $x_0 \in X$ is considered an efficient solution for problem (SP) if and only if there is no point $x \in X$ such that:*

$$f(x) \leq f(x_0).$$

Definition 2.3. *A feasible point $x_0 \in X$ is a weak efficient solution for problem (SP) iff there is no point $x \in X$ such that:*

$$f(x) < f(x_0).$$

The necessary optimality conditions for (SP) presented below are taken from [19]

Theorem 2.4. (*Necessary Optimality Conditions*) *Let x_0 be an efficient solution for (SP) and $I(x_0) \neq \phi$. If (SP) satisfies the suitable constraint qualification* (*see* [26]) *at x_0 then there exist $\overline{u} \in \mathbb{R}^p$, $\overline{v} = (\overline{v}_j)_{j\in J}$, such that*

$$\overline{u}^T \nabla f(x_0) + \overline{v}_I^T \nabla g_I(x_0) = 0, \tag{2.1}$$

$$\overline{v}^T g(x_0) = 0,$$

$$\overline{u} \geq 0, \overline{v} \geq 0 \text{ and } \overline{v}_j \neq 0 \text{ for finitely many } j \in I.$$

3 Definitions

In 2006, Yuan and Liu (*see* [13]) gave the definition of (C, α, ρ, d) convex function. We see that this definition is not hold when $\alpha \leq 0$. In this paper, we introduce the definitions of $(C, \alpha, \eta, \rho, d)$ invex function and its generalizations, which will also hold when $\eta \leq 0$. These definitions will be used in the sequal.

Definition 3.1. *The vector-valued function $f : X \to \mathbb{R}^p$ is called $(C, \alpha, \eta, \rho, d)$-invex at $x_0 \in X$ if, for each $f_i : X \to \mathbb{R}$, there exists a vectorial function $\eta_i : X \times X \to \mathbb{R}$ such that for every $x \in X$ and $i \in \underline{p}$, the following condition holds:*

$$\frac{f_i(x) - f_i(x_0)}{\alpha_i(x, x_0)}(>) \geqq C_{(x,x_0)}(\nabla f_i(x_0))\eta_i(x, x_0) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x, x_0)}.$$

The function f is said to be $(C, \alpha, \eta, \rho, d)$-invex on X if and only if it is $(C, \alpha, \eta, \rho, d)$-invex at every point in X.

By replacing $\eta(x, x_0)$ with 1 in the above definition, we will get the definition of (C, α, ρ, d) convexity.

In this example, we try to illustrate the importance of invex functions and its generalization.

Exmple. Let $X = \{x : \frac{\pi}{4} \leq x \leq \frac{\pi}{2}\}$, $\rho = -1$, $\alpha(x, x_0) = 1$, $d(x, x_0) = (x - x_0)$, $C_{(x,x_0)}(a) = a^2(x - x_0)$ for any $(x, x_0) \in X \times X$ and let $f(x) = cos^2 x$. Then, we see that $f(x)$ is not (C, α, ρ, d)-convex at $x = \frac{\pi}{4}$, but it is $(C, \alpha, \eta, \rho, d)$-invex at $x = \frac{\pi}{4}$ with $\eta(x, x_0) = (1 - \frac{cos(x - x_0)}{(x - x_0)})$.

As we know that the definition of (C, α, ρ, d) convexity is not hold when $\alpha \leq 0$. But on the other hand we see that, the definition of $(C, \alpha, \eta, \rho, d)$ invexity holds for all real values of η.

Definition 3.2. *The vector-valued function $f : X \to \mathbb{R}^p$ is called $(C, \alpha, \eta, \rho, d)$-pseudo-invex at $x_0 \in X$ if, for each $f_i : X \to \mathbb{R}$, there exists a vectorial function $\eta_i : X \times X \to \mathbb{R}$ such that for every $x \in X$ and $i \in \underline{p}$, the following condition holds:*

$$f_i(x)(\leqq) < f_i(x_0) \Rightarrow C_{(x,x_0)}(\nabla f_i(x_0))\eta_i(x, x_0) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x, x_0)} < 0.$$

The function f is said to be $(C, \alpha, \eta, \rho, d)$-pseudo-invex on X iff it is $(C, \alpha, \eta, \rho, d)$-pseudo-invex at each point in X.

Definition 3.3. *The vector-valued function $f : X \to \mathbb{R}^p$ is called weak strictly $(C, \alpha, \eta, \rho, d)$-pseudo-invex at $x_0 \in X$ if, for each $f_i : X \to \mathbb{R}$ there exists a vectorial function $\eta_i : X \times X \to \mathbb{R}$ such that for every $x \in X$ and $i \in \underline{p}$, the following condition holds:*

$$f_i(x) \leq f_i(x_0) \Rightarrow C_{(x,x_0)}(\nabla f_i(x_0))\eta_i(x, x_0) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x, x_0)} < 0.$$

The function f is said to be weak strictly $(C, \alpha, \eta, \rho, d)$-pseudo-invex on X iff it is weak strictly $(C, \alpha, \eta, \rho, d)$-pseudo-invex at each point in X.

Definition 3.4. *The vector-valued function $f : X \to \mathbb{R}^p$ is said to be strong $(C, \alpha, \eta, \rho, d)$-pseudo-invex at $x_0 \in X$ if, for each $f_i : X \to \mathbb{R}$ there exists a vectorial function $\eta_i : X \times X \to \mathbb{R}$ such that for each $x \in X$ and $i \in \underline{p}$, the following condition holds:*

$$f_i(x) \leq f_i(x_0) \Rightarrow C_{(x,x_0)}(\nabla f_i(x_0))\eta_i(x, x_0) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x, x_0)} \leq 0.$$

The function f is said to be strong (C,α,η,ρ,d)-pseudo-invex on X iff it is strong (C,α,η,ρ,d)-pseudo-invex at each point in X.

Definition 3.5. *The vector-valued function $f : X \to \mathbb{R}^p$ is said to be (C,α,η,ρ,d)-quasi-invex at $x_0 \in X$ if, for each $f_i : X \to \mathbb{R}$ there exists a vectorial function $\eta_i : X \times X \to \mathbb{R}$ such that for each $x \in X$ and $i \in \underline{p}$, the following condition holds:*

$$f_i(x) \leqq f_i(x_0) \Rightarrow C_{(x,x_0)}(\nabla f_i(x_0))\eta_i(x,x_0) + \rho_i \frac{d_i(x,x_0)}{\alpha_i(x,x_0)} \leqq 0.$$

The function f is said to be (C,α,η,ρ,d)-quasi-invex on X iff it is (C,α,η,ρ,d)-quasi-invex at each point in X.

Definition 3.6. *The vector-valued function $f : X \to \mathbb{R}^p$ is said to be weak (C,α,η,ρ,d)-quasi-invex at $x_0 \in X$ if, for each $f_i : X \to \mathbb{R}$ there exists a vectorial function $\eta_i : X \times X \to \mathbb{R}$ such that for each $x \in X$ and $i \in \underline{p}$, the following condition holds:*

$$f_i(x) \leq f_i(x_0) \Rightarrow C_{(x,x_0)}(\nabla f_i(x_0))\eta_i(x,x_0) + \rho_i \frac{d_i(x,x_0)}{\alpha_i(x,x_0)} \leqq 0.$$

The function f is said to be weak (C,α,η,ρ,d)-quasi-invex on X if and only if it is weak (C,α,η,ρ,d)-quasi-invex at each point in X.

4 Optimality

For the nonlinear multiobjective semiinfinite programming problem (SP), we state the following optimality conditions.

Theorem 4.1. *If there exist a feasible solution x_0 for (SP) and, $\overline{u} \in \mathbb{R}^p$ and $\overline{v} = (\overline{v}_j)_{j\in J}$ are vectors which satisfies*

$$\begin{gathered}\overline{u}^T \nabla f(x_0) + \overline{v}_I^T \nabla g_I(x_0) = 0,\\ \overline{v}^T g(x_0) = 0, \qquad (4.1)\\ \overline{u} \geq 0, \overline{v} \geq 0 \text{ and } \overline{v}_j \neq 0 \text{ for finite many } j \in I.\end{gathered}$$

Also, if f is strong $(C,\alpha^1,\eta^1,\rho^1,d^1)$-pseudoinvex function at x_0 and g_I is $(C,\alpha^2,\eta^2,\rho^2,d^2)$-quasi-invex function at x_0 w.r.t. the same $\eta(x,x_0)$ (i.e., $\eta^1 = \eta^2 = \eta$) with

$$\sum_{i=1}^{p} \overline{u}_i \rho_i^1 \frac{d_i^1(x,x_0)}{\alpha_i^1(x,x_0)} + \sum_{j\in I} \overline{v}_j \rho_j^2 \frac{d_j^2(x,x_0)}{\alpha_j^2(x,x_0)} \geqq 0. \qquad (4.2)$$

Then, x_0 is an efficient solution of (SP)

Proof. Assume that x_0 is not an efficient solution of (SP), then for a feasible solution $x \in S$, we have

$$f_i(x) \leq f_i(x_0).$$

Since $g_I(x_0) = 0$, hence

$$g_I(x) \leqq g_I(x_0).$$

Since f is strong $(C,\alpha^1,\eta^1,\rho^1,d^1)$-pseudo-invex at x_0 and g_I is $(C,\alpha^2,\eta^2,\rho^2,d^2)$-quasi-invex at x_0, therefore we have

$$C_{(x,x_0)}(\nabla f_i(x_0))\eta(x,x_0) + \rho_i^1 \frac{d_i^1(x,x_0)}{\alpha_i^1(x,x_0)} \leq 0,$$

and

$$C_{(x,x_0)}(\nabla g_I(x_0))\eta(x,x_0) + \rho_I^2 \frac{d_I^2(x,x_0)}{\alpha_I^2(x,x_0)} \leqq 0.$$

Since C is convex, thus from the above inequalities, we get

$$C_{(x,x_0)}\left(\sum_{i=1}^{p} \frac{1}{\tau}\overline{u}_i \nabla f_i(x_0) + \sum_{j\in I} \frac{1}{\tau}\overline{v}_j \nabla g_j(x_0)\right)\eta(x,x_0)$$

$$+\sum_{i=1}^{p}\frac{1}{\tau}\overline{u}_i\rho_i^1\frac{d_i^1(x,x_0)}{\alpha_i^1(x,x_0)}+\sum_{j\in I}\frac{1}{\tau}\overline{v}_j\rho_j^2\frac{d_j^2(x,x_0)}{\alpha_j^2(x,x_0)}<0,$$

where

$$\tau=\sum_{i=1}^{p}\overline{u}_i+\sum_{j\in I}\overline{v}_j.$$

Since $C_(x,x_0)(0)=0$, thus from equation (4.1) we have

$$\sum_{i=1}^{p}\overline{u}_i\rho_i^1\frac{d_i^1(x,x_0)}{\alpha_i^1(x,x_0)}+\sum_{j\in I}\overline{v}_j\rho_j^2\frac{d_j^2(x,x_0)}{\alpha_j^2(x,x_0)}<0,$$

which go against our assumption (4.2). Thus x_0 is an efficient solution of (SP). □

Example: Consider the semiinfinite problem as follows:
(SP 1) Minimum $f(x)$

$$\text{s.t. } g_j(x)\leqq 0, j\in J,\ x\in\mathbb{R},$$

where the functions $F:X\to\mathbb{R}^2$ and $g_j:X\to\mathbb{R}$, $(X=\mathbb{R})$ are defined as:

$$F(x)=(f_1(x),f_2(x))=(x^2-2x,x^3-x^2)$$

and

$$g_1(x)=x^2(x-2),$$

$$g_2(x)=x^3+x,$$

$$g_k(x)=x+\frac{1}{k},\ k=3,4,...$$

Also the function $C:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is defined by $C_{(x,x_0)}(a)=-a^2(x+x_0)$, the function $\eta:X\times X\to\mathbb{R}$ is defined by $\eta(x,x_0)=(2x-x_0)$ and the set S of feasible solutions for (SP 1) is defined by $S=\{x\in\mathbb{R}:g_j(x)\leqq 0\}=\{x\in\mathbb{R}:x\leqq-\frac{1}{3}\}$

We can check that $f_i(i=1,2)$ are strong $(C,\alpha_i,\eta_i,\rho_i,d_i)$-pseudo-invex at $x_0=-\frac{1}{3}$, with $\alpha_i(x,x_0)=1$, $\eta_i=\eta$, $\rho_i=0$ and $d_i(x,x_0)=1$, also $g_j(j\in I)$ are $(C,\alpha_j,\eta_j,\rho_j,d_j)$-quasi-invex at $x_0=-\frac{1}{3}$, with $\alpha_j(x,x_0)=1$, $\eta_j=\eta$, $\rho_j=0$ and $d_j(x,x_0)=1$. Clearly, $\eta_i=\eta_j=\eta$ and $x_0=-\frac{1}{3}$ is a feasible solution for (SP 1) that meets the requirements of Theorem 4.1, in which $\overline{u}=(\frac{3}{4},1)$ and $\overline{v}=(0,0,1,0,...,0,...)$. For $x_0=-\frac{1}{3}$, $I=\{j\in J:g_j(x_0)=0\}=\{3\}$ is the index set of active constraints. We conclude that there is no point $x\in S$ such that $f(x)\leq f(x_0)$. Hence, $x_0=-\frac{1}{3}$ is an efficient solution of (SP 1).

Theorem 4.2. *If there exist a feasible solution x_0 for (SP) and, $\overline{u}\in\mathbb{R}^p$ and $\overline{v}=(\overline{v}_j)_{j\in J}$ are vectors which satisfies*

$$\overline{u}^T\nabla f(x_0)+\overline{v}_I^T\nabla g_I(x_0)=0,$$

$$\overline{v}^Tg(x_0)=0,$$

$$\overline{u}\geq 0,\overline{v}\geq 0 \text{ and } \overline{v}_j\neq 0 \text{ for finitely many } j\in I.$$

Also, if f is weak strictly $(C,\alpha^1,\eta^1,\rho^1,d^1)$-pseudo-invex at x_0 and g_I is $(C,\alpha^2,\eta^2,\rho^2,d^2)$-quasi-invex at x_0 with respect to same $\eta(x,x_0)$ (i.e., $\eta^1=\eta^2=\eta$) with

$$\sum_{i=1}^{p}\overline{u}_i\rho_i^1\frac{d_i^1(x,x_0)}{\alpha_i^1(x,x_0)}+\sum_{j\in I}\overline{v}_j\rho_j^2\frac{d_j^2(x,x_0)}{\alpha_j^2(x,x_0)}\geqq 0.$$

Then, x_0 is an efficient solution of (SP).

Proof. Assume that x_0 is not an efficient solution of (SP). Then for a feasible solution $x \in S$, we have

$$f_i(x) \leq f_i(x_0).$$

Since $g_I(x_0) = 0$, hence

$$g_I(x) \leqq g_I(x_0).$$

Since f is weak strictly $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at x_0 and g_I is $(C, \alpha^2, \eta^2, \rho^2, d^2)$-quasi-invex at x_0, therefore we have

$$C_{(x,x_0)}(\nabla f_i(x_0))\eta(x, x_0) + \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} < 0,$$

and

$$C_{(x,x_0)}(\nabla g_I(x_0))\eta(x, x_0) + \rho_I^2 \frac{d_I^2(x, x_0)}{\alpha_I^2(x, x_0)} \leqq 0.$$

The remaining part of the proof is similar to proof of Theorem 4.1. □

Theorem 4.3. *If there exist a feasible solution x_0 for (SP) and, $\overline{u} \in \mathbb{R}^p$ and $\overline{v} = (\overline{v}_j)_{j \in J}$ are vectors which satisfies*

$$\overline{u}^T \nabla f(x_0) + \overline{v}_I^T \nabla g_I(x_0) = 0,$$

$$\overline{v}^T g(x_0) = 0,$$

$$\overline{u} \geq 0, \overline{v} \geq 0 \text{ and } \overline{v}_j \neq 0 \text{ for finitely many } j \in I.$$

Also, if f is weak $(C, \alpha^1, \eta^1, \rho^1, d^1)$-quasi-invex at x_0 and g_I is strictly $(C, \alpha^2, \eta^2, \rho^2, d^2)$-pseudo-invex at x_0 with respect to same $\eta(x, x_0)$ (i.e., $\eta^1 = \eta^2 = \eta$) with

$$\sum_{i=1}^{p} \overline{u}_i \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} + \sum_{j \in I} \overline{v}_j \rho_j^2 \frac{d_j^2(x, x_0)}{\alpha_j^2(x, x_0)} \geqq 0.$$

Then, x_0 is an efficient solution of (SP).

Proof. Assume that x_0 is not an efficient solution of (SP). Then for a feasible solution $x \in S$, we have

$$f_i(x) \leq f_i(x_0).$$

Since $g_I(x_0) = 0$, hence

$$g_I(x) \leqq g_I(x_0).$$

Since f is weak $(C, \alpha^1, \eta^1, \rho^1, d^1)$-quasi-invex at x_0 and g_I is strictly $(C, \alpha^2, \eta^2, \rho^2, d^2)$-pseudo-invex at x_0, therefore we have

$$C_{(x,x_0)}(\nabla f_i(x_0))\eta(x, x_0) + \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} \leqq 0,$$

and

$$C_{(x,x_0)}(\nabla g_I(x_0))\eta(x, x_0) + \rho_I^2 \frac{d_I^2(x, x_0)}{\alpha_I^2(x, x_0)} < 0.$$

The remaining part of the proof is similar to proof of Theorem 4.1. □

Theorem 4.4. *If there exist a feasible solution x_0 for (SP) and, $\overline{u} \in \mathbb{R}^p$ and $\overline{v} = (\overline{v}_j)_{j \in J}$ are vectors which satisfies*

$$\overline{u}^T \nabla f(x_0) + \overline{v}_I^T \nabla g_I(x_0) = 0,$$

$$\overline{v}^T g(x_0) = 0,$$

$$\overline{u} \geq 0, \overline{v} \geq 0 \text{ and } \overline{v}_j \neq 0 \text{ for finitely many } j \in I.$$

Also, if f is weak strictly $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at x_0 and g_I is strictly $(C, \alpha^2, \eta^2, \rho^2, d^2)$-pseudo-invex at x_0 with respect to same $\eta(x, x_0)$ (i.e., $\eta^1 = \eta^2 = \eta$) with

$$\sum_{i=1}^{p} \overline{u}_i \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} + \sum_{j \in I} \overline{v}_j \rho_j^2 \frac{d_j^2(x, x_0)}{\alpha_j^2(x, x_0)} \geqq 0.$$

Then, x_0 is an efficient solution of (SP).

Proof. Assume that x_0 is not an efficient solution of (SP). Then for a feasible solution $x \in S$, we have

$$f_i(x) \leq f_i(x_0).$$

Since $g_I(x_0) = 0$, hence

$$g_I(x) \leqq g_I(x_0).$$

Since f is weak strictly $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at x_0 and g_I is strictly $(C, \alpha^2, \eta^2, \rho^2, d^2)$-pseudo-invex at x_0, therefore we have

$$C_{(x,x_0)}(\nabla f_i(x_0))\eta(x, x_0) + \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} < 0,$$

and

$$C_{(x,x_0)}(\nabla g_I(x_0))\eta(x, x_0) + \rho_I^2 \frac{d_I^2(x, x_0)}{\alpha_I^2(x, x_0)} < 0.$$

The remaining part of the proof is similar to proof of Theorem 4.1. ☐

Theorem 4.5. *If there exist a feasible solution x_0 for (SP) and, $\overline{u} \in \mathbb{R}^p$ and $\overline{v} = (\overline{v}_j)_{j \in J}$ are vectors which satisfies*

$$\overline{u}^T \nabla f(x_0) + \overline{v}_I^T \nabla g_I(x_0) = 0,$$

$$\overline{v}^T g(x_0) = 0,$$

$$\overline{u} \geq 0, \overline{v} \geq 0 \text{ and } \overline{v}_j \neq 0 \text{ for finitely many } j \in I.$$

Also, if f is strong $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at x_0 and g_I is strictly $(C, \alpha^2, \eta^2, \rho^2, d^2)$-pseudo-invex at x_0 with respect to same $\eta(x, x_0)$ (i.e., $\eta^1 = \eta^2 = \eta$) with

$$\sum_{i=1}^{p} \overline{u}_i \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} + \sum_{j \in I} \overline{v}_j \rho_j^2 \frac{d_j^2(x, x_0)}{\alpha_j^2(x, x_0)} \geqq 0.$$

Then, x_0 is an efficient solution of (SP).

Proof. Assume that x_0 is not an efficient solution of (SP). Then for a feasible solution $x \in S$, we have

$$f_i(x) \leq f_i(x_0).$$

Since $g_I(x_0) = 0$, hence

$$g_I(x) \leqq g_I(x_0).$$

Since f is strong $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at x_0 and g_I is strictly $(C, \alpha^2, \eta^2, \rho^2, d^2)$-pseudo-invex at x_0, therefore we have

$$C_{(x,x_0)}(\nabla f_i(x_0))\eta(x, x_0) + \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} \leq 0,$$

and

$$C_{(x,x_0)}(\nabla g_I(x_0))\eta(x, x_0) + \rho_I^2 \frac{d_I^2(x, x_0)}{\alpha_I^2(x, x_0)} < 0.$$

The remaining part of the proof is similar to proof of Theorem 4.1. ☐

Since it is clear from the definitions that, an efficient solution is also a weak efficient solution for (SP) but the converse need not be true, therefore Theorem 4.1 - Theorem 4.5 are still valid for weak efficiency.

Theorem 4.6. *If there exist a feasible solution x_0 for (SP) and, $\overline{u} \in \mathbb{R}^p$ and $\overline{v} = (\overline{v}_j)_{j \in J}$ are vectors which satisfies*

$$\overline{u}^T \nabla f(x_0) + \overline{v}_I^T \nabla g_I(x_0) = 0,$$

$$\overline{v}^T g(x_0) = 0,$$

$$\overline{u} \geq 0, \overline{v} \geq 0 \text{ and } \overline{v}_j \neq 0 \text{ for finitely many } j \in I.$$

Also, if f is $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at x_0 and g_I is strictly $(C, \alpha^2, \eta^2, \rho^2, d^2)$-pseudo-invex at x_0 with respect to same $\eta(x, x_0)$ (i.e., $\eta^1 = \eta^2 = \eta$) with

$$\sum_{i=1}^{p} \overline{u}_i \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} + \sum_{j \in I} \overline{v}_j \rho_j^2 \frac{d_j^2(x, x_0)}{\alpha_j^2(x, x_0)} \geqq 0.$$

Then, x_0 is a weak efficient solution of (SP).

Proof. Assume that x_0 is not a weak efficient solution of (SP). Then for a feasible solution $x \in S$, we have

$$f_i(x) < f_i(x_0).$$

Since $g_I(x_0) = 0$, hence

$$g_I(x) \leqq g_I(x_0).$$

Since f is $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at x_0 and g_I is strictly $(C, \alpha^2, \eta^2, \rho^2, d^2)$-pseudo-invex at x_0, therefore we have

$$C_{(x,x_0)}(\nabla f_i(x_0))\eta(x, x_0) + \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} < 0,$$

and

$$C_{(x,x_0)}(\nabla g_I(x_0))\eta(x, x_0) + \rho_I^2 \frac{d_I^2(x, x_0)}{\alpha_I^2(x, x_0)} < 0.$$

The remaining part of the proof is similar to proof of Theorem 4.1. ☐

Theorem 4.7. *If there exist a feasible solution x_0 for (SP) and, $\overline{u} \in \mathbb{R}^p$ and $\overline{v} = (\overline{v}_j)_{j \in J}$ are vectors which satisfies*

$$\overline{u}^T \nabla f(x_0) + \overline{v}_I^T \nabla g_I(x_0) = 0,$$

$$\overline{v}^T g(x_0) = 0,$$

$$\overline{u} \geq 0, \overline{v} \geq 0 \text{ and } \overline{v}_j \neq 0 \text{ for finitely many } j \in I.$$

Also, if f is $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at x_0 and g_I is $(C, \alpha^2, \eta^2, \rho^2, d^2)$-quasi-invex at x_0 with respect to same $\eta(x, x_0)$ (i.e., $\eta^1 = \eta^2 = \eta$) with

$$\sum_{i=1}^{p} \overline{u}_i \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} + \sum_{j \in I} \overline{v}_j \rho_j^2 \frac{d_j^2(x, x_0)}{\alpha_j^2(x, x_0)} \geqq 0.$$

Then, x_0 is a weak efficient solution of (SP).

Proof. Assume that x_0 is not a weak efficient solution of (SP). Then for a feasible solution $x \in S$, we have

$$f_i(x) < f_i(x_0).$$

Since $g_I(x_0) = 0$, hence

$$g_I(x) \leqq g_I(x_0).$$

Since f is $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at x_0 and g_I is $(C, \alpha^2, \eta^2, \rho^2, d^2)$-quasi-invex at x_0, therefore we have

$$C_{(x,x_0)}(\nabla f_i(x_0))\eta(x, x_0) + \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} < 0,$$

and

$$C_{(x,x_0)}(\nabla g_I(x_0))\eta(x, x_0) + \rho_I^2 \frac{d_I^2(x, x_0)}{\alpha_I^2(x, x_0)} \leqq 0.$$

The remaining part of the proof is similar to proof of Theorem 4.1. ☐

5 Duality

We now prove duality relations between nonlinear multiobjective semiinfinite programming problem (SP) and Mond-Weir-type dual problem (MWD):

(MWD) $\qquad \text{Max } f(y) = (f_1(y), ..., f_p(y)),$

$$\text{Subject to } \sum_{i=1}^{p} u_i \nabla f_i(y) + \sum_{j \in J} v_j \nabla g_j(y) = 0, \tag{5.1}$$

$$v^T g(y) \geqq 0, v = (v_j)_{j \in J}, v_j \in \mathbb{R}_+ \text{ and } v_j \neq 0 \text{ for finitely many } j \in J,$$

$$\sum_{i=1}^{p} u_i = 1, u_i > 0 \ (i = 1, ..., p), y \in X \subseteq \mathbb{R}^n,$$

where f_i and g_i are differnetiable functions from X to $\mathbb{R}$, and X is nonempty open subset of $\mathbb{R}^n$.

Theorem 5.1. (*Weak Duality*) *Let x_0 and (y_0, u, v) be feasible solution for (SP) and (MWD) respectively, and*

$$\sum_{i=1}^{p} u_i \rho_i^1 \frac{d_i^1(x_0, y_0)}{\alpha_i^1(x_0, y_0)} + \sum_{j \in J} v_j \rho_j^2 \frac{d_j^2(x_0, y_0)}{\alpha_j^2(x_0, y_0)} \geqq 0. \tag{5.2}$$

(a) Let f be strong $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at y_0 and $v^T g$ be $(C, \alpha^2, \eta^2, \rho^2, d^2)$-quasi-invex at y_0 with respect to a common kernal $\eta(x, x_0)$ (i.e., $\eta^1 = \eta^2 = \eta$).

(b) Let f be weak strictly $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at y_0 and $v^T g$ be $(C, \alpha^2, \eta^2, \rho^2, d^2)$-quasi-invex at y_0 with respect to a common kernal $\eta(x, x_0)$ (i.e., $\eta^1 = \eta^2 = \eta$).

(c) Let f be weak $(C, \alpha^1, \eta^1, \rho^1, d^1)$-quasi-invex at y_0 and $v^T g$ be strictly $(C, \alpha^2, \eta^2, \rho^2, d^2)$-pseudo-invex at y_0 with respect to a common kernal $\eta(x, x_0)$ (i.e., $\eta^1 = \eta^2 = \eta$).

(d) Let f be weak strictly $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at y_0 and $v^T g$ be strictly $(C, \alpha^2, \eta^2, \rho^2, d^2)$-pseudo-invex at y_0 with respect to a common kernal $\eta(x, x_0)$ (i.e., $\eta^1 = \eta^2 = \eta$).

(e) Let f be strong $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invex at y_0 and $v^T g$ be strictly $(C, \alpha^2, \eta^2, \rho^2, d^2)$ - pseudo-invex at y_0 with respect to a common kernal $\eta(x, x_0)$ (i.e., $\eta^1 = \eta^2 = \eta$).

If any one of the following assumptions will hold, then the following inequality will not hold

$$f(x_0) \leq f(y_0). \tag{5.3}$$

Proof. (a) Suppose that the assumption (a) hold. Since x_0 and (y_0, u, v) are the feasible solution for (SP) and (MWD). Therefore,

$$v_j g_j(x_0) \leqq 0 \leqq v_j g_j(y_0).$$

Also, $v^T g$ is $(C, \alpha^2, \eta^2, \rho^2, d^2)$-quasi-invex at y_0, which implies that

$$C_{(x_0,y_0)}(v_j \nabla g_j(y_0))\eta(x_0, y_0) + \rho_j^2 \frac{d_j^2(x_0, y_0)}{\alpha_j^2(x_0, y_0)} \leqq 0. \tag{5.4}$$

Let (5.3) holds, then by the definition of strong $(C, \alpha^1, \eta^1, \rho^1, d^1)$-pseudo-invexity, we have

$$C_{(x_0,y_0)}(\nabla f_i(y_0))\eta(x_0, y_0) + \rho_i^1 \frac{d_i^1(x_0, y_0)}{\alpha_i^1(x_0, y_0)} \leq 0. \tag{5.5}$$

Let

$$\tau = \sum_{i=1}^{p} u_i + \sum_{j \in J} v_j.$$

Then, by the convexity of $C_{(x_0,y_0)}(\cdot)$ and equations (5.1),(5.2),(5.4),(5.5), we get

$$\begin{aligned}
0 &> \sum_{i=1}^{p} \frac{u_i}{\tau} C_{(x_0,y_0)}(\nabla f_i(y_0))\eta(x_0, y_0) + \sum_{j \in J} \frac{v_j}{\tau} C_{(x_0,y_0)}(\nabla g_j(y_0))\eta(x_0, y_0) \\
&+ \sum_{i=1}^{p} \frac{u_i}{\tau} \rho_i^1 \frac{d_i^1(x_0, y_0)}{\alpha_i^1(x_0, y_0)} + \sum_{j \in J} \frac{v_j}{\tau} \rho_j^2 \frac{d_j^2(x_0, y_0)}{\alpha_j^2(x_0, y_0)} \\
&\geqq C_{(x_0,y_0)} \left(\frac{1}{\tau} \left(\sum_{i=1}^{p} u_i \nabla f_i(y_0) + \sum_{j \in J} v_j \nabla g_j(y_0) \right) \right) \eta(x_0, y_0) \\
&+ \frac{1}{\tau} \left(\sum_{i=1}^{p} u_i \rho_i^1 \frac{d_i^1(x_0, y_0)}{\alpha_i^1(x_0, y_0)} + \sum_{j \in J} v_j \rho_j^2 \frac{d_j^2(x_0, y_0)}{\alpha_j^2(x_0, y_0)} \right) \\
&\geqq 0.
\end{aligned}$$

Which gives a contradiction. Thus, the proof of part (a) is complete.
Similarly we can prove part (b) - (e). □

Theorem 5.2. (*Strong Duality*) *Let f and $v^T g$ be satisfy any of the five assumptions specified in Theorem 5.1. If $x_0 \in S$ is an efficient solution for (SP) and (SP) satisfies a suitable constraint qualification (see[26]). Then, $\exists\ \overline{u} \in \mathbb{R}_+^p\ \overline{u} > 0$, $\overline{v} = (\overline{v}_j)_{(j \in J)}$, $\overline{v}_j \in \mathbb{R}_+$, such that $(x_0, \overline{u}, \overline{v})$ is an efficient solution of (MWD) and the respective objective values of (SP) and (MWD) are equal.*

Proof. Since $x_0 \in S$ is an efficient solution for (SP) and a suitable constraint qualification is satisfied $(see[26])$. Therefore by Theorem 4.1, there exist $\overline{u} \in \mathbb{R}^p_+$ $\overline{u} > 0$, $\overline{v} = (\overline{v}_j)_{(j\in J)}$, $\overline{v}_j \in \mathbb{R}_+$, such that $(x_0, \overline{u}, \overline{v})$ is a feasible solution for (MWD). Suppose that $(x_0, \overline{u}, \overline{v})$ is not an efficient solution for (MWD), this implies there exist a feasible solution $(y_0, \overline{u}_0, \overline{v}_0)$ for (MWD), s.t.

$$(f_1(x_0), ..., f_p(x_0)) \leq (f_1(y_0), ..., f_p(y_0)),$$

which contradicts Theorem 5.1. Hence, the proof is complete. □

Theorem 5.3. (*Strict Converse Duality*) *Let the assumptions of Theorem 5.2 be satisfied and* f *be strictly* $(C, \alpha^1, \eta^1, \rho^1, d^1)$*-pseudo-invex at* y_0*. If* x_0 *and* $(y_0, \overline{u}_0, \overline{v}_0)$ *are feasible solutions for (SP) and (MWD) respectively, then* $x_0 = y_0$*.*

Proof. Suppose that $x_0 \neq y_0$. By strong duality theorem $\exists\ \overline{u} \in \mathbb{R}^p_+$, $\overline{u} > 0$, $\overline{v} = (\overline{v}_j)_{(j\in J)}$, $\overline{v}_j \in \mathbb{R}_+$, such that $(x_0, \overline{u}, \overline{v})$ is an efficient solution for (MWD). Hence,

$$f(x_0) = f(y_0). \tag{5.6}$$

Since x_0 and $(y_0, \overline{u}_0, \overline{v}_0)$ are the feasible solution for (SP) and (MWD), this implies that

$$\overline{v}_{0j} g_j(x_0) \leqq 0 \leqq \overline{v}_{0j} g_j(y_0). \tag{5.7}$$

By definition of $(C, \alpha^2, \eta^2, \rho^2, d^2)$-quasi-invexity, we have

$$C_{(x_0,y_0)}(\overline{v}_{0j} \nabla g_j(y_0))\eta(x_0, y_0) + \rho^2_j \frac{d^2_j(x_0, y_0)}{\alpha^2_j(x_0, y_0)} \leqq 0, \qquad (\eta^2 = \eta). \tag{5.8}$$

Again, by the assumption on $f_i (i = 1, 2, ..., p)$, we have

$$C_{(x_0,y_0)}(\nabla f_i(y_0))\eta(x_0, y_0) + \rho^1_i \frac{d^1_i(x_0, y_0)}{\alpha^1_i(x_0, y_0)} < 0, \qquad (\eta^1 = \eta). \tag{5.9}$$

Let us denote

$$\tau = \sum_{i=1}^{p} \overline{u}_{0i} + \sum_{j\in I} \overline{v}_{0j}.$$

Therefore from equations (5.8)-(5.9) and the convexity of $C_{(x_0,y_0)}(\cdot)$, we get

$$\begin{aligned}
0 &> \sum_{i=1}^{p} \frac{\overline{u}_{0i}}{\tau} C_{(x_0,y_0)}(\nabla f_i(y_0))\eta(x_0, y_0) + \sum_{j\in J} \frac{\overline{v}_{0j}}{\tau} C_{(x_0,y_0)}(\nabla g_j(y_0))\eta(x_0, y_0) \\
&+ \sum_{i=1}^{p} \frac{\overline{u}_{0i}}{\tau} \rho^1_i \frac{d^1_i(x_0, y_0)}{\alpha^1_i(x_0, y_0)} + \sum_{j\in J} \frac{\overline{v}_{0j}}{\tau} \rho^2_j \frac{d^2_j(x_0, y_0)}{\alpha^2_j(x_0, y_0)} \\
&\geqq C_{(x_0,y_0)} \left(\frac{1}{\tau} \left(\sum_{i=1}^{p} \overline{u}_{0i} \nabla f_i(y_0) + \sum_{j\in J} \overline{v}_{0j} \nabla g_j(y_0) \right) \right) \eta(x_0, y_0) \\
&+ \frac{1}{\tau} \left(\sum_{i=1}^{p} \overline{u}_{0i} \rho^1_i \frac{d^1_i(x_0, y_0)}{\alpha^1_i(x_0, y_0)} + \sum_{j\in J} \overline{v}_{0j} \rho^2_j \frac{d^2_j(x_0, y_0)}{\alpha^2_j(x_0, y_0)} \right) \\
&\geqq 0.
\end{aligned}$$

Which gives a contradiction, i.e., our assumption is wrong. Therefore $x_0 = y_0$. □

6 conclusion

In this paper, we formulted generalized $(C, \alpha, \eta, \rho, d)$ invexities, which are the generalized version of $(C, \alpha, \eta, \rho, d)$ convexity. Using these definitions, we have stablished several optimality and duality results for multiobjective semiinfinite programming problems. The results of this paper are more general than the corresponding results present in the literature [19]. These definitions we can use to extend more results of literature.

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