Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

A General Characteristic Equation for Eigen Values and Energy of Cycle Graph

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract: The characteristic equation of a graph frequently appears in mathematical sciences, chemistry and physics. As to graph theorists, the characteristic equation tells information about the structural properties of a graph. In this paper, we find out the characteristic equations & eigen value of cycle C_n of n vertices.

Keywords: Characteristic Equation, Eigen value, Cycle graph, Structural properties.

1. Introduction

All considered graph in this paper are simple, undirected, and connected. Assume that G is a simple, finite, undirected graph with vertex set $V(G)$ and edge set $E(G)$. G has n vertices, and each vertex is identified by the label $v_1, v_2, \dots, v_n$. The adjacency matrix X(G) of the graph G is a square matrix of order n, whose (i, j) entry is equal to 1 if the vertices v_i and v_j are adjacent and is equal to zero otherwise. Adjacent matrix will be symmetric binary matrix for simple graph and all entries along principal diagonal of X(G) are all 0's for simple graph. **(Brooks)** The characteristic polynomial of the graph G along the adjacency matrix X(G) is $\det(\lambda I_n - X(G))$, where I_n is the unit matrix of order n and will be denoted by $\Delta(G, \lambda)$. $\det\left(\lambda I_n - X(G)\right) = 0$ is the characteristic equation of graph G and the eigenvalues of a graph G are defined as the eigenvalues of its adjacency matrix X(G). **(Kumar et. al**) So they are just the roots of the characteristic equation $\det\left(\lambda I_n - X(G)\right) = 0$ i.e. $\Delta(G, \lambda) = 0$. Since X(G) is a real symmetric matrix, so its eigenvalues are all real with sum equal to zero. Denoting them by $\lambda_1, \lambda_2, \dots, \lambda_n$ and as a whole, they are called the spectrum of G. For specifics, spectral aspects of graphs, including characteristics of the characteristic polynomial, have been thoroughly investigated, see **(Brouwer & Haemers)**

Characteristic Polynomials

$$\Delta(G, \lambda) = \lambda^n - S_{1,n}\lambda^{n-1} + S_{2,n}\lambda^{n-2} - S_{3,n}\lambda^{n-3} + \dots + (-1)^k S_{k,n}\lambda^{n-k} + \dots + (-1)^n S_{n,n}$$

Where $S_{k,n}$ is the sum of the principal minors of order k in a matrix of order n.

Since cycle graphs have so many uses in domains like computer science, biology, and chemistry, graph theory has explored cycle graphs extensively. Even with its significance, figuring out cycle graph eigenvalues and energy has proven to be a difficult undertaking. In this study, we provide an equation for the general characteristics that can be applied to any cycle graph to find its energy and eigenvalues. The goal of this research is to present a thorough explanation of the equation's development and its numerous uses. This work aims to address the research challenge of not having a generic equation for calculating the energy and eigenvalues of cycle graphs. We can improve our knowledge of cycle graphs and their applications by proving this equation. The steps taken to construct the characteristic equation, its derivation, and its applications in diverse sectors will be the key points of contention in this work. In summary, this study will advance the subject of graph theory by offering a general formula for calculating cycle graph energy and eigenvalues, which can have important applications across a range of domains. A graph's energy can be expressed as the total of the absolute values of its adjacency matrix's eigenvalues. Stated otherwise, the total of the eigenvalues' magnitudes represents it. In mathematical terms, the energy E(G) of the graph G is given by: if the eigenvalues of the adjacency matrix A are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then graph energy $E(G) = |\lambda_1| + |\lambda_2| + \ldots + |\lambda_n|$ **(Estrada & Benzi)**

When examining the properties of cycle graphs, many matrices are crucial. Among these are the seidel adjacency matrix Laplacian matrix, singles Laplacian matrix, and normalized Laplacian matrix. The invention of a general characteristic equation that makes it possible to calculate the energy and eigenvalues of cycle graphs is particularly intriguing. This emphasizes the generic formulas for a cycle graph's spectra, which are based on the cycle's length and vertex count **(Stin et. al)**. A crucial source of information about a graph's structure and connectivity is its energy and eigenvalues. Researchers can gain a better understanding of the connections between these attributes and the general dynamics of the graph by creating a general characteristic equation for cycle graphs**. (Rowlinson)** addresses spectral properties of non-bipartite graphs with three distinct eigenvalues, focusing on conditions for G to be the cone over a strongly regular graph and analyzing scenarios involving one non-main eigenvalue and certain degree conditions. With precisely three different eigenvalues and a maximum of one for the second greatest eigenvalue, **[(Cheng et. al), (Qi et al)]** categorize the connected graphs. By shedding light on the structure and features of connected regular graphs with four different eigenvalues, **(Huang & Huang)** advanced the knowledge of spectral qualities and validates some discoveries regarding their non-existence **(Li et al.)**.

2. Characteristics of Cyclic Graph:

Let C_n be a cycle of n vertices $v_1, v_2, \ldots, v_n$ and adjacent matrix of C_n denoted by $X(C_n)$ defined as:

$$X(Cn) = \begin{bmatrix} 0 & 1 & 0 & & 0 & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & & 0 & 1 & 0 \end{bmatrix}_{n\times n}$$

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
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Its Characteristic Polynomials is $\Delta(C_n,\lambda) = \det\,(\lambda I_n - X(C_n))$

$$\Delta(C_n,\lambda) = \begin{vmatrix} \lambda & -1 & 0 & & 0 & 0 & -1 \\ -1 & \lambda & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & \lambda & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & \lambda & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & \lambda & -1 \\ -1 & 0 & 0 & & 0 & -1 & \lambda \end{vmatrix}$$

$$\Delta(C_n,\lambda) = \lambda^n - S_{1,n}\lambda^{n-1} + S_{2,n}\lambda^{n-2} - S_{3,n}\lambda^{n-3} + \cdots + (-1)^k S_{k,n}\lambda^{n-k} + \cdots + (-1)^n S_{n,n}$$

And Characteristic equation is $\Delta(C_n,\lambda) = \det\,(\lambda I_n - X(C_n)) = 0$ i.e.

$$\lambda^n - S_{1,n}\lambda^{n-1} + S_{2,n}\lambda^{n-2} - S_{3,n}\lambda^{n-3} + \cdots + (-1)^k S_{k,n}\lambda^{n-k} + \cdots + (-1)^n S_{n,n} = 0$$

Where $S_{k,n}$ is the sum of the principal minors of order k of cycle C_n in an adjacent matrix X.

Now we will try to find the values of $S_{k,n}$.

There are four cases to find $S_{k,n}$.

Case I: If $n \equiv 0 (\mathrm{mod}\ 4)$, then

$$S_{k,n} = \begin{cases} 0, & \text{k is odd or } k = n, \\ -n & k = 2, \end{cases}$$

and

$$S_{k,n} = (-1)^{k/2}\left[(k+1) + (-1)^{(k-2)/2} \sum_{m=0}^{n-2-k} S(k-2, k+m)\right]$$

where k is even, $k \neq 2$ and $k < n-1$.

For example: Let C_4 be a cycle of 4 vertices, adjacent matrix of C_4 is.

$$X(C_4) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4\times 4}$$

Characteristic equation is $\Delta(C_4,\lambda) = \det\big(\lambda I_4 - X(C_4)\big) = 0$

$\Rightarrow \lambda^4 - S_{1,4}\lambda^3 + S_{2,4}\lambda^2 - S_{3,4}\lambda + S_{4,4} = 0$

$\Rightarrow \lambda^4 - 4\lambda^2 = 0$ as $S_{1,4} = 0, S_{2,4} = -4, S_{3,4} = 0$ *and* $S_{4,4} = 0$

$\Rightarrow \lambda = 0, 0, -2$ *and* 2

Case II: If, $n \equiv 1 (\mathrm{mod}\ 4)$, then

$$S_{k,n} = \begin{cases} 0, & \text{k is odd and } k \neq n \\ -n, & k = 2 \\ 2, & k = n \\ n, & k = n-1 \end{cases}$$

And

$$S_{k,n} = (-1)^{k/2}\left[(k+1) + (-1)^{(k-2)/2} \sum_{m=0}^{n-2-k} S(k-2, k+m)\right]$$

where k is even, $k \neq 2$ and $k < n - 1$.

For example: Let C_5 be a cycle of 5 vertices, adjacent matrix of C_5 is.

$$X(C_5) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}_{5\times 5}$$

Characteristic equation is $\Delta(C_5, \lambda) = \det\big(\lambda I_5 - X(C_5)\big) = 0$

$\Rightarrow \lambda^5 - S_{1,5}\lambda^4 + S_{2,5}\lambda^3 - S_{3,5}\lambda^2 + S_{4,5}\lambda - S_{5,5} = 0$

$\Rightarrow \lambda^5 - 5\lambda^3 + 5\lambda - 2 = 0$ as $S_{1,5} = 0, S_{2,5} = -5, S_{3,5} = 0, S_{4,5} = 5 \; and \; S_{5,5} = 2$

$\Rightarrow \lambda = -1.6180, -1.6180, 0.6180, 0.6180, 2$

Case III: If $n \equiv 2 \pmod 4$ and $n \neq 2$, then

$$S_{k,n} = \begin{cases} 0, k \text{ is odd and } k \neq n \\ -n, & k = 2 \\ -4, & k = n \end{cases}$$

And

$$S_{k,n} = (-1)^{k/2}\left[(k+1) + (-1)^{(k-2)/2} \sum_{m=0}^{n-2-k} S(k-2, k+m)\right]$$

where k is even, $k \neq 2$ and $k < n - 1$.

For example: Let C_6 be a cycle of 6 vertices, adjacent matrix of C_6 is

$$X(C_6) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{6\times 6}$$

Characteristic equation is $\Delta(C_6, \lambda) = \det\big(\lambda I_6 - X(C_6)\big) = 0$

$\Rightarrow \lambda^6 - S_{1,6}\lambda^5 + S_{2,6}\lambda^4 - S_{3,6}\lambda^3 + S_{4,6}\lambda^2 - S_{5,6}\lambda + S_{6,6} = 0$

$\Rightarrow \lambda^6 - 6\lambda^4 + 9\lambda^2 - 4 = 0$ as $S_{1,6} = 0, S_{2,6} = -6, S_{3,6} = 0, S_{4,6} = 9, S_{5,6} = 0$ and $S_{6,6} = -4$

$\Rightarrow \lambda = -2, -1, -1, 1, 1, 2$

Case IV: If $n \equiv 3 \pmod 4$, then

$$S_{k,n} = \begin{cases} 0, & k \text{ is odd and } k \neq n \\ -n, & k = 2 \\ 2, & k = n \\ -n, & k = n-1 \end{cases}$$

And

$$S_{k,n} = (-1)^{k/2}\left[(k+1) + (-1)^{(k-2)/2} \sum_{m=0}^{n-2-k} S(k-2, k+m)\right]$$

where k is even, $k \neq 2$ and $k < n - 1$.

For example: Let C_7 be a cycle of 7 vertices, adjacent matrix of C_7 is

$$X(C7) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{7\times 7}$$

Characteristic equation is $\Delta(C_7, \lambda) = \det\big(\lambda I_7 - X(C_7)\big) = 0$

$\Rightarrow \lambda^7 - S_{1,7}\lambda^6 + S_{2,7}\lambda^5 - S_{3,7}\lambda^4 + S_{4,7}\lambda^3 - S_{5,7}\lambda^2 + S_{6,7}\lambda - S_{7,7} = 0$

$\Rightarrow \lambda^7 - 7\lambda^5 + 14\lambda^3 - 7\lambda - 2 = 0$ as $S_{1,7} = 0, S_{2,7} = -7, S_{3,7} = 0, S_{4,7} = 14, S_{5,7} = 0, S_{6,7} = -7 \text{ and } S_{7,7} = 2$

$\Rightarrow \lambda = -1.8019, -1.8019, -0.4450, -0.4450, 1.2470, 1.2470, 2$

Table: For $S_{k,n}$ of cycle C_n

n	$S_{1,n}$	$S_{2,n}$	$S_{3,n}$	$S_{4,n}$	$S_{5,n}$	$S_{6,n}$	$S_{7,n}$	$S_{8,n}$	$S_{9,n}$	$S_{10,n}$	$S_{12,n}$	$S_{14,n}$	$S_{16,n}$	$S_{18,n}$	$S_{20,n}$
3	0	-3	2												
4	0	-4	0	0											
5	0	-5	0	5	2										
6	0	-6	0	9	0	-4									
7	0	-7	0	14	0	-7	2								
8	0	-8	0	20	0	-16	0	0							
9	0	-9	0	27	0	-30	0	9	2						
10	0	-10	0	35	0	-50	0	25	0	-4					

11	0	-11	0	44	0	-77	0	55	0	-11					
12	0	-12	0	54	0	-112	0	105	0	-36	0				
13	0	-13	0	65	0	-156	0	182	0	-91	13				
14	0	-14	0	77	0	-210	0	294	0	-196	49	-4			
15	0	-15	0	90	0	-275	0	450	0	-378	140	-15			
16	0	-16	0	104	0	-352	0	660	0	-672	336	-64	0		
17	0	-17	0	119	0	-442	0	935	0	-1122	714	-204	17		
18	0	-18	0	135	0	-546	0	1287	0	-1782	1386	-540	81	-4	
19	0	-19	0	152	0	-665	0	1729	0	-2717	2508	-1254	285	-19	
20	0	-20	0	170	0	-800	0	2275	0	-4004	4290	-2640	825	-100	0

Table: For Eigen values and Energy of cycle C_n

n	Eigen values	Energy
3	-1, 1, 2	4
4	-2, 0, 0, 2	4
5	-1.6180, -1.6180, 0.6180, 0.6180, 2	6.472
6	-2, -1, -1, 1, 1, 2	8
7	-1.8019, -1.8019, -0.4450, -0.4450, 1.2470, 1.2470, 2	8.9878
8	-2, -1.4142, -1.4142, 0, 0, 1.4142, 1.4142, 2	9.6568
9	-1.8794, -1.8794, -1, -1, 0.3473, 0.3473, 1.5321, 1.5321, 2	11.5176
10	-2, -1.6180, -1.6180, -0.6180, -0.6180, 0.6180, 0.6180, 1.6180, 1.6180, 2	12.944
11	-1.9190, -1.9190, -1.3097, -1.3097, -0.2846, -0.2846, 0.8308, 0.8308, 1.6825, 1.6825, 2	14.0532
12	-2, -1.732, -1.732, -1, -1, 0, 0, 1, 1, 1.732, 1.732, 2	14.928
15	-1.9563, -1.9563, -1.6180, -1.6180, -1, -1, -0.2091, -0.2091, 0.6180, 0.6180, 1.3383, 1.3383, 1.8271, 1.8271, 2	19.1336
16	-2, -1.8478, -1.8478, -1.4142, -1.4142, -0.7654, -0.7654, 0, 0, 0.7654, 0.7654, 1.4142, 1.4142, 1.8478, 1.8478, 2	20.1096

Conclusion

The characteristic polynomial in the above tables is easy to compute. When working with matrices that have complex expressions with parameters, which are frequently encountered in different mathematical models, the actual usefulness of this idea becomes evident. By expanding to first-order three-dimensional discrete dynamics, this method is useful in determining stability criteria for first-order two-dimensional discrete dynamics. Building upon these findings, our future work will expand these investigations to derive stability criteria for first-order four-dimensional discrete dynamics and, more generally, for first-order n-dimensional discrete dynamics.

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