

Closely Connected Domination Number in Corona Product of Graphs

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Abstract:

Let $G = (V, E)$ be a simple connected graph. The vertices $u, v \in V$ are closely connected if atleast one of the shortest paths connecting them is not a cut path. A set S of vertices of a simple graph $G = (V, E)$ is a CC-dominating set (closely connected dominating set) if for every vertex $v \in V \setminus S$ there exist a vertex $u \in S$ such that $\Gamma_{cc}(u, v) \geq 1$, where $\Gamma_{cc}(u, v)$ is the number of shortest paths connecting u and v except the cut paths. The minimum cardinality of a CC-dominating set is called the CC-domination number, denoted by $\gamma_{cc}(G)$. This paper evaluates CC- domination Number in Corona product of some standard graphs.

Objectives: Closely connected domination is new direction of Domination in Graphs, here find closely connected domination number for corona product of path, cycle with some graphs

Methods: Consider the Graph G is undirected connected simple graph. The closely connected domination number (CC- domination number) for product of graphs, which is represented as $\gamma_{cc}(G)$ is the minimum cardinality of Closely connected dominating set.

Keywords: closely-connected vertices, CC-degree of a vertex, CC-Domination number, Corona product of graphs

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1. Introduction

In our day-to-day life, shortest route is a path between two destinations which traverses the minimal distance over the network. Shortest route between different location cannot always be preferable and we prefer routes that optimize the cost and time. This is possible only if another path exists between the locations, even if the shortest route is not approachable. In graph theoretically, places are considered as vertices and paths are edges in a graph. Then our aim is to find for those vertex pairs which does not alter the connectivity of the graph. More precisely the vertex pairs do not disconnect the graph even if the shortest path between them is deleted. For this idea K. Priya and V. Anilkumar introduced closely connected vertices in [4]. Let $G = (V, E)$, the vertices $u, v \in V$ are closely connected if atleast one of the shortest paths connecting them is not a cut path.

The concept of domination discussed in [2,3] entirely depending upon the adjacency property of vertices in a graph. But adjacency is not at all sufficient to characterize the vertex pairs in a graph as the deletion of the edges linked by adjacent vertices may or may not disconnect the graph. A set $S \subseteq V(G)$ is called a dominating set of G if every vertex in $V(G) - S$ is adjacent to some vertex in S . The domination number $\gamma(G)$ of G is the minimum cardinality of its dominating sets.

The concept of closely connected domination introduced by K. Priya and V. Anilkumar in [4].

A subset S of V is closely connected dominating set(abbreviated as CC-dominating set) if for every vertex $v \in V \setminus S$ there exist a vertex $u \in S$ such that $\Gamma_{CC}(u, v) \geq 1$ where $\Gamma_{CC}(u, v)$ is cardinality of $\{p|P(u, v)|p \text{ is a not a cut path in } G\}$, where $P(u, v)$ is the set of all shortest paths linking u and v in G . The minimum cardinality of CC-dominating set is called CC-domination number, denoted by $\gamma_{CC}(G)$. The open CC-neighbourhood of a vertex $v \in V$ is the set $N_{CC}(v) = \{u \in V: \Gamma_{CC}(u, v) \geq 1\}$, whereas the closed CC-neighbourhood of V is defined as $N_{CC}[v] = N_{CC}(v) \cup \{v\}$. The cardinality of $N_{CC}(v)$ is the CC-degree of v , denoted by $\deg_{CC}(v)$. The vertex v is said to be CC-isolated if $N_{CC}(v) = \phi$.

2. Definitions and Previous Results

Definition 2.1 [4]

A subset S of V is closely connected dominating set(abbreviated as CC-dominating set) if for every vertex $v \in V \setminus S$ there exist a vertex $u \in S$ such that $\Gamma_{CC}(u, v) \geq 1$ where $\Gamma_{CC}(u, v)$ is cardinality of $\{p|P(u, v)|p \text{ is a not a cut path in } G\}$, where $P(u, v)$ is the set of all shortest paths linking u and v in G .

Definition 2.2[4]

The minimum cardinality of CC-dominating set is called CC-domination number, denoted by $\gamma_{CC}(G)$.

In this paper, we investigate the CC-domination number of Corona product of paths, cycles and some standard graphs. For a graph G of order n , the following are some basic results of $\gamma_{CC}(G)$ in [4].

1. $1 \leq \gamma_{CC}(G) \leq n$.
2. $\gamma_{CC}(G) = n$ iff G is acyclic.
3. For path P_n , $\gamma_{CC}(P_n) = n$.
4. If G has no cut edge, then $\gamma_{CC}(G) \leq \gamma(G)$.

Theorem 2.3.[4] *Let G be a graph and $u \in V(G)$ be CC-isolated. Then u belongs to every CC-dominating set of G .*

Theorem 2.4.[4] *For the complete graph K_n . $\gamma_{CC}(K_n) = \begin{cases} 2 & \text{if } n = 2 \\ 1 & \text{if } n \neq 2 \end{cases}$*

Theorem 2.5. *For the cycle C_n , $\gamma_{CC}(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil, \forall n \in N$.*

Proof. In C_n every vertex is closely connected only with its adjacent vertices, so we conclude the above result. \square

3. Closely connected Domination number in Corona product of graphs

The Corona of two graphs G_1 and G_2 has been defined by Frucht and Harary in [1] to be the graph G formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 , where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2 and is denoted by $G = G_1 \circ G_2$.

Theorem 3.1[2] *For the Path $G = P_n$, $\gamma(G) = \lceil \frac{n}{2} \rceil$.*

Theorem 3.2.[4] For the path $G = P_n$, $\gamma_{cc}(G) = n$.

Theorem 3.3. For $n \geq 1$, $\gamma_{cc}(P_n \circ K_1) = 2n$.

Proof. Let $G = P_n \circ K_1$, let $V(P_n) = \{s_1, s_2, \dots, s_n\}$ and t_i be the vertices of the i^{th} copy of K_1 joined to s_i . Then $V(G) = \{s_1, s_2, \dots, s_n\} \cup \{t_1, t_2, \dots, t_n\}$. Every vertex in $V(G)$ is CC-isolated vertex, By the Theorem 1 we have all the CC-isolated vertices must belong to every CC-dominating set. The only one CC-dominating set is $V(G)$. Therefore $\gamma_{cc}(G) = |V(G)| = 2n$. \square

Theorem 3.4. For $n \geq 1$, $\gamma_{cc}(P_n \circ K_2) = n$.

Proof. Let $G = P_n \circ K_2$. Let $V(P_n) = \{s_1, s_2, \dots, s_n\}$ and $\{t_{i1}, t_{i2}\}$ be the vertex set of the i^{th} copy of K_2 attached with s_i . Then $V(G) = \{s_1, s_2, s_3, \dots, s_n\} \cup \{t_{11}, t_{12}\} \cup \{t_{21}, t_{22}\} \cup \dots \cup \{t_{n1}, t_{n2}\}$ and $|V(G)| = 3n$. In Figure 1, Every vertex s_i of P_n is closely connected with t_{i1} and t_{i2} but not closely connected with s_{i-1} and s_{i+1} . i.e., $\Gamma_{cc}(s_i, t_{i1}) \geq 1$, $\Gamma_{cc}(s_i, t_{i2}) \geq 1$ and $\Gamma_{cc}(s_i, s_{i-1}) = 0$ and $\Gamma_{cc}(s_i, s_{i+1}) = 0$. So, one of the minimums CC-dominating set $D = \{s_1, s_2, \dots, s_n\}$. Since all vertices in G is closely connected to at least one vertex in D . On removing one vertex in D , it is not a CC-dominating set. Hence $\gamma_{cc}(G) = |D| = n$.

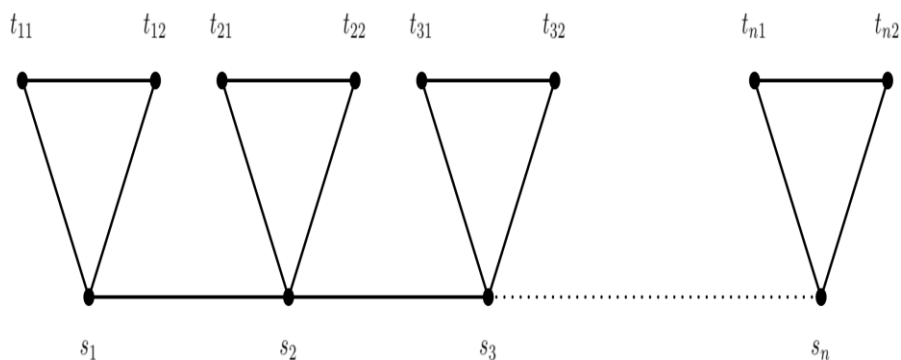


Figure 1 The Corona product $P_n \circ K_2$

Theorem 3.6. The CC-domination number of $P_n \circ K_m$ is $\gamma_{cc}(P_n \circ K_m) = n$ for $n \geq 1$ and $m > 1$.

Proof. Let $G = P_n \circ K_m$, $V(P_n) = \{s_1, s_2, \dots, s_n\}$. Let $\{t_{i1}, t_{i2}, t_{i3}, \dots, t_{im}\}$ be the vertex set of the i^{th} copy of K_m joined with the vertex s_i .

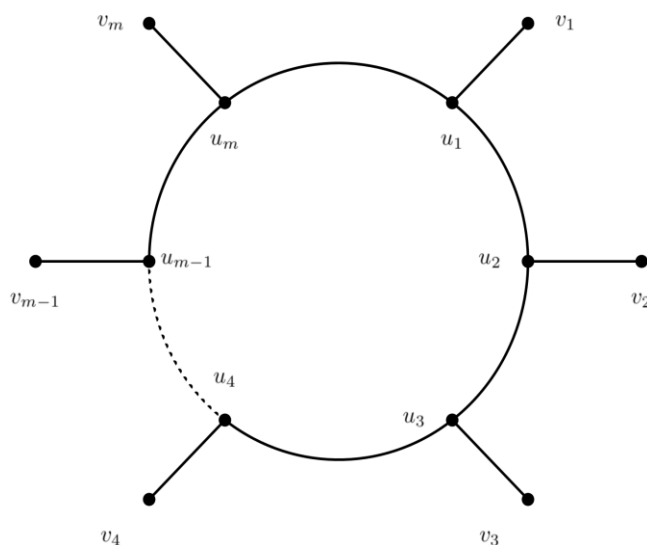
$$\therefore V(P_n \circ K_m) = \{s_1, s_2, \dots, s_n, t_{11}, t_{12}, \dots, t_{1m}, t_{21}, t_{22}, \dots, t_{2m}, \dots, t_{n1}, t_{n2}, t_{n3}, \dots, t_{nm}\}$$

For $j = 1$ to m , $S_j = \{t_{1j}, t_{2j}, \dots, t_{nj}\}$ and $S = \{s_1, s_2, \dots, s_n\}$ are some CC-dominating sets of G and $|S| = |S_j| = n$, since all vertices in G is closely connected to atleast one vertex in S (or) S_j . On removing one vertex in S (or) S_j , it is not CC-dominating set. Hence, we conclude $\gamma_{cc}(G) = n$. \square

Theorem 3.7 The CC-domination number of $C_m \circ K_1$ is $\gamma_{cc}(C_m \circ K_1) = m + \left\lceil \frac{m}{3} \right\rceil$

Proof. Let $G = C_m \circ K_1$ and let $V(G) = \{u_1, u_2, \dots, u_m\} \cup \{v_1, v_2, \dots, v_m\}$. Let v_i be the vertices of the i^{th} copy of K_1 is joined to u_i , then $|V(G)| = 2m$. Let $D = \{v_1, v_2, \dots, v_m\}$ be the set of all pendent vertices of G and is also CC-isolated vertices. So D is subset of every CC-dominating set of G . Further

each vertex u_i in G is closely connected with its adjacent vertices. i.e., $N_{cc}(u_i) = \{u_{i-1}, u_{i+1}\}$ for $2 \leq i \leq n - 1$. So, from the Figure.2, the set $S = \{v_1, v_2, \dots, v_m, u_2, u_5, \dots, u_{m-1}\}$ be the one of the minimum CC-dominating set of G . By using theorem 1.3, the cardinality of S is $m + \left\lceil \frac{m}{3} \right\rceil$. So, every vertex in G is closely connected to atleast one of the vertices in S . Hence, we conclude $\gamma_{cc}(G) = m + \left\lceil \frac{m}{3} \right\rceil$



Theorem 3.8. The CC-domination number of $C_m \circ K_2$ is $\gamma_{cc}(C_m \circ K_2) = \left\lceil \frac{m}{3} \right\rceil$ for $m \geq 2$.

Proof. Let $G = C_m \circ K_2, V(C_m) = \{u_1, u_2, \dots, u_m\}$. Let $\{v_{i1}, v_{i2}\}$ be the vertex set of the i^{th} copy of K_2 joined with the vertex u_i .

$$\therefore V(G) = \{u_1, u_2, \dots, u_m\} \cup \{v_{11}, v_{12}\} \cup \{v_{21}, v_{22}\} \cup \dots \cup \{v_{m1}, v_{m2}\}$$

and $|V(G)| = 3m$. Each vertex u_i of C_m is closely connected with $u_{i-1}, v_{i+1}, v_{i1}, v_{i2}, v_{(i-1)1}, v_{(i-1)2}, v_{(i+1)1}$ and $v_{(i+1)2}$. So $|N_{cc}(u_i)| = 8, \forall i = 1, 2, \dots, m$. Then the set $D_1 = \{u_1, u_4, \dots, u_{m-2}\}$ and $D_2 = \{u_2, u_5, \dots, u_{m-1}\}$ are some CC-dominating sets of G and it is also minimum CC-dominating set. Hence $\gamma_{cc}(G) = \left\lceil \frac{m}{3} \right\rceil$. □

Theorem 3.9. For $n \geq 1$ and $m > 1, \gamma_{cc}(C_m \circ K_n) = \left\lceil \frac{m}{3} \right\rceil$.

Proof. Let $G = C_m \circ K_n$, and $V(C_m) = \{u_1, u_2, \dots, u_m\}$. Let $\{v_{i1}, v_{i2}, v_{i3}, \dots, v_{in}\}$ be the vertex set of the i^{th} copy of K_n joined with the vertex u_i . So $V(G) = \{u_1, u_2, u_3, \dots, u_m, v_{11}, v_{12}, \dots, v_{1n}, v_{21}, v_{22}, \dots, v_{2n}, \dots, v_{m1}, v_{m2}, \dots, v_{mn}\}$ and $|V(G)| = mn + m$. For each u_i in C_m is closely connected with the following vertices $u_{i-1}, u_{i+1}, (i - 1)^{\text{th}}$ copy of $K_n, (i)^{\text{th}}$ copy of $K_n, (i + 1)^{\text{th}}$ copy of K_n , then $|N_{cc}(u_i)| = 3n + 2$. Then the set $D = \{u_2, u_5, \dots, u_{m-1}\}$ is CC-dominating set, on removing one vertex in D it is not CC-dominating set. Hence of D is the minimal CC-dominating set and $|D| = \left\lceil \frac{m}{3} \right\rceil$. So, we conclude $\gamma_{cc}(G) = \left\lceil \frac{m}{3} \right\rceil$. □

Theorem 3.10. *Let H_1, H_2 be two connected graphs and $H_2 \neq K_1$, then $\gamma_{cc}(H_1 \circ H_2) = \gamma_{cc}(H_1)$.*

Proof. Let $G = H_1 \circ H_2$. In G , the j^{th} vertex of H_1 is adjacent to every vertex in the j^{th} copy of H_2 . H_2 is non-trivial connected graph, and every pair of vertices of H_2 in the i^{th} copy of H_2 are closely connected. So, every vertex in the i^{th} copy of H_2 is closely connected with the j^{th} vertex of H_1 . Suppose D is the minimum CC-dominating set of H_1 , every vertex in $V(H_1) - D$ is closely connected with at least one vertex in D and also every vertex in $V(G) - D$ is closely connected with atleast one vertex in D .

Hence, we conclude that $\gamma_{cc}(G) = |D| = \gamma_{cc}(H_1)$. □

Theorem 3.11. *Let H_1 be a connected graph and $H_2 = \bar{K}_n$ (complement of complete graph K_n). Then $\gamma_{cc}(H_1 \circ H_2) = \gamma_{cc}(H_1) + |V(H_1)||V(H_2)|$.*

Proof. Let $G_1 = H_1 \circ H_2$. In G all vertices of H_2 are pendent vertices and also CC-isolated vertices. ie, the j^{th} vertex of H_1 is adjacent to every vertex in the i^{th} copy of H_2 . since H_2 is isolated graph, so we remove any edge between H_1 and H_2 , then the graph G is disconnected. Hence, we have $|V(H_1)||V(H_2)|$ number of isolated vertices and also H_2 does not alter the CC-domination of H_1 . By theorem 1.1, every CC-isolated vertex must belong to every CC-dominating set. Hence, we have

$$\gamma_{cc}(H_1 \circ H_2) = \gamma_{cc}(H_1) + |V(H_1)||V(H_2)|.$$

□ **Corollary 3.12.** *The following results are direct computation from Theorem 12.*

1. $\gamma_{cc}(C_m \circ P_n) = \left\lceil \frac{m}{3} \right\rceil = \gamma_{cc}(C_m)$
2. $\gamma_{cc}(P_n \circ P_m) = n = \gamma_{cc}(P_n)$
3. $\gamma_{cc}(C_n \circ C_m) = \left\lceil \frac{n}{3} \right\rceil = \gamma_{cc}(C_n)$
4. $\gamma_{cc}(P_n \circ C_m) = n = \gamma_{cc}(P_n)$

4. Applications

In traditional network is operated by all hosts for two-way communications. Every host has a service area and within this close-transmission covers the range, link failure of the network does not disconnect or disrupt the communication of the entire network. A pair of such hosts that are communication with one another are called as closely connected neighbors. Thus, closely connected neighbors serves as a new approach to design networks in [5].

5. Conclusion

In this paper, the concept of CC-domination number is studied using closely connected vertices of graphs for Corona product of graphs. Investigation of CC-domination of various other products is a significant direction for further research. Also determine $\gamma_{cc}(G)$ for the family of graphs.

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