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On Class of Analytic Function Defined by Generalized Ruscheweyh Derivative

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Abstract: The aim of this paper is to introduce a class of analytic functions defined by using generalized Ruscheweyh derivative. The coefficient bound, inclusion result and a radius problem has been discussed in this paper.

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Conclusion: Here, in our present investigation, we have successfully introduced a new subclass of analytic functions $\mathcal{V}_{k,\lambda}^{m}[A, B, \alpha, b]$ using the Generalized Ruscheweyh derivative operator. Many properties and characteristics of this newly defined function class such as coefficient estimates, inclusion bounds and radius problem have been studied.

1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

(1.1) $$f(z)= z+\sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. If f and g are analytic in $\mathcal{U}$, we say that f is subordinate to g, written f $\prec$ g or f(z) $\prec$ g(z), if there exists a Schwarz function ω(z) in $\mathcal{U}$ such that f(z) = g(ω (z)).

Let P[A, B] be the class of functions h, analytic in $\mathcal{U}$ with h(0) = 1 and

$$h(z) \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1.$$

This class was introduced by Janowski [18]. The class P [A, B] is connected with the class P of functions with positive real parts by the relation

(1.2) $$h \in \mathrm{P[A, B]} \Leftrightarrow \frac{(B-1)h-(A-1)}{(B+1)h-(A+1)} \in P.$$

Later Polatoǧlu [19] defined the class P[A,B,α] as:

Let P[A,B,α] be the class of functions p_1, analytic in $\mathcal{U}$ with p_1 (0) = 1 and

(1.3) $$p_1(z) \prec \frac{1+\{(1-\alpha)A+\alpha B\}z}{1+Bz}, \quad \text{where} \quad -1 \le B < A \le 1, \quad 0 \le \alpha < 1.$$

From (1.3), it can easily be seen that, $p_1 \in P[A,B,\alpha]$, if and only if, there exists $h \in P[A,B]$ such that

(1.4) $$p_1(z) = (1-\alpha)h(z) + \alpha, \; 0 \le \alpha < 1, \; z \in \mathcal{U}.$$

It is also noted that $P[1,-1,0] \equiv P$, the well-known class of analytic functions in $\mathcal{U}$ with positive real part. Noor [5] considered the generalized class $P_k[A,B,\alpha]$ of Janowski functions which is defined as follows.

A function p is said to be in the class $P_k[A,B,\alpha]$, if and only if,

(1.5) $$p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)$$

where

$p_1, \; p_2 \in P[A, B, \alpha]$, $-1 \le B < A \le 1$, $k \ge 2$ and $0 \le \alpha < 1$. It is clear that $P_2[A,B,\alpha] \equiv P[A,B,\alpha]$ and $P_k[1,-1,0] \equiv P_k$, the well-known class given and studied by Pinchuk[3].

For any two analytic functions

$f_1(z)=\sum_{n=0}^{\infty} a_n z^n$ and $f_2(z)=\sum_{n=0}^{\infty} b_n z^n \quad (z \in \mathcal{U})$

the convolution of f_1 and f_2 is defined by

(1.6) $$((f_1 * f_2)(z) = \sum_{n=1}^{\infty} a_n b_n z^n$$

The Generalized Ruscheweyh Derivative $\mathcal{D}_\lambda^m$ [4] is defined as follows,

For $f \in \mathcal{A}$, $\lambda \ge 0$ and $m \in \mathbb{R}$, $m > -1$, we have

(1.7) $$D_\lambda^m f(z) = \frac{z}{(1-z)^{m+1}} * D_\lambda f(z), \; z \in \mathcal{U}.$$

(1.8) $$(m+1)\mathcal{D}_\lambda^{m+1} f(z) = m\mathcal{D}_\lambda^m f(z) + z(\mathcal{D}_\lambda^m f(z))'$$

For function $f \in \mathcal{A}$ of the form (1.1), we obtain the power series expansion of the form,

(1.9) $$\mathcal{D}_\lambda^m f(z) = z + \sum_{n=2}^{\infty}[1+(n-1)\lambda]\frac{(m+1)_{(n-1)}}{(1)_{(n-1)}} a_n z^n, \quad z \in \mathcal{U}$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{for } n = 0 \\ a(a+1)(a+2)\dots(a+n-1), & \text{for } n \in \mathbb{N} \end{cases}$$

Definition 1.1. A function $f \in \mathcal{A}$ is in the class $\mathcal{V}_{k,\lambda}^{m}[A,B,\alpha,b]$ if and only if ,

$$\left(1-\frac{2}{b}+\frac{2}{b}\frac{D_\lambda^{m+1}f(z)}{D_\lambda^{m}f(z)}\right) \in P_k[A,B,\alpha], \quad z \in \mathcal{U},$$

where $k \geq 2$, $m \geq 0$, $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$ and $b \in \mathbb{C}-\{0\}$.

Assigning certain values to different parameters, we have different well-known classes of analytic functions as can be seen below.

Special cases

(i) For a parametric value λ = 1; we get the class studied by S.N. Malik, M. Arif, K.I. Noor and M. Raza.[15]

(ii) $\mathcal{V}_k^{\lambda}[1,-1,\alpha,b] \equiv V_k(a,b,\lambda)$, the well-known class defined by Latha and Nanjunda Rao in [14].

(iii) $\mathcal{V}_2^{1}[A,B,\alpha,1] \equiv \mathrm{C}[\mathrm{A},\mathrm{B},\alpha]$

$\mathcal{V}_2^{0}[A,B,\alpha,2] \equiv S^*[A,B,\alpha]$,the well-known class defined by Polatoǧlu [19]

(iv) $\mathcal{V}_k^{1}[A,B,0,1] \equiv V_k[A,B]$,

$\mathcal{V}_2^{0}[A,B,0,2] \equiv R_k[A,B]$, where $V_k[A,B]$ and $R_k[A,B]$ denote the class of janowski functions with bounded boundary and bounded radius rotations respectively, given by Noor [9].

2. PRELIMINARY RESULTS

We need the following results to obtain our main results.

Lemma 2.1. Let $p(z) = 1+\sum_{n=1}^{\infty} q_n z^n \in P_k[A,B,\alpha]$. Then, for all n ≥ 1,

$$|q_n| \leq \frac{k(A-B)(1-\alpha)}{2} \tag{2.1}$$

This inequality is sharp.

The proof follows from (1.4), (1.5) and the coefficient bound of h ∈ P[A, B] given by Aouf [13].

Lemma 2.2. [17] Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\psi(u,v)$ be a complex valued function satisfying the conditions:

(i) $\psi(u,v)$ is continuous in a domain , $D \subset \mathbb{C}^2$

(ii) (1,0) $\in D$ and Re $\psi(1,0) > 0$,

(iii) Re $\psi(iu_2, v_1) \leq 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1+u^2)$. .

If h(z) = 1 + $c_1 z$ + ... is a function analytic in $\mathcal{U}$ such that (h(z),zh′(z)) ∈ D and Re ψ (h(z),zh′(z)) > 0 for z ∈ $\mathcal{U}$, then Reh(z) > 0 in $\mathcal{U}$.

Lemma 2.3. Let $p \in P_k[A,B,0]$ with k ≥ 2. Then, for |z| = r < 1, (2.2) $$\frac{2-(A-B)kr-2ABr^2}{2(1-B^2r^2)} \leq Re\, p(z) \leq |p(z)| \leq \frac{2+(A-B)kr-2ABr^2}{2(1-B^2r^2)}$$

The proof is immediate by using (1.5) and the growth result of h ∈ P [A, B], see [13].

Lemma 2.4. Let ∈ $P_k[A, B, 0]$ with k ≥ 2. Then, for |z| = r < 1.

(2.3) $$|zp'(z)| \leq \frac{r\{(A-B)k-4B(A-B)r+B^2(A-B)kr^2\}\, Re\, p(z)}{(1-B^2r^2)(2+(A-B)kr-2ABr^2)}$$

The result follows directly by using Lemma 2.3

3. MAIN RESULTS

Theorem 3.1. Let f ∈ $\mathcal{V}_{k,\lambda}^{m}[A, B, \alpha, b]$ with -1≤ $B < A$ ≤1, m ≥0, 0 ≤ α<1 and b ∈ $\mathbb{C}-\{0\}$. Then

(3.1) $$|a_n| \leq \frac{(\sigma)_{n-1}}{(n-1)!\, \phi_n(m)}, \quad \forall\, n \geq 2,$$

Where

$\sigma = \frac{k|b|(A-B)(1-\alpha)(m+1)}{4}$ and $\phi_n(m)$=[1+(n-1)λ] $\frac{(m+1)_{n-1}}{(1)_{n-1}}$.

This result is sharp

Proof: Let

(3.2) $$1-\frac{2}{b}+\frac{2}{b}\frac{D_\lambda^{m+1}f(z)}{D_\lambda^{m}f(z)} = p(z)$$

so that $p \in P_k[A, B, \alpha]$. Let $p(z) = 1+\sum_{n=1}^{\infty} q_n z^n$. Then (3.2) can be written as

$$2(D_\lambda^{m+1}f(z) - D_\lambda^{m}f(z)) = bD_\lambda^{m}f(z)\sum_{n=1}^{\infty} q_n z^n$$

which implies that

$$\frac{2\phi_n(m)(n-1)a_n}{(m+1)} = b(q_{n-1}+\phi_2(m)a_2q_{n-2}+\cdots+\phi_{n-1}(m)a_{n-1}q_1).$$

Using Lemma 2.1, we obtain

$$|a_n| \leq \frac{k|b|(A-B)(1-\alpha)(m+1)}{4(n-1)\phi_n(m)}(1+\phi_2(m)|a_2|+\cdots+\phi_{n-1}(m)|a_{n-1}|)$$

$$= \frac{\sigma}{(n-1)\phi_n(m)}(1+\sum_{i=2}^{n-1}\phi_i(m)|a_i|).$$

For n=2, $|a_2| \leq \frac{\sigma}{\phi_2(m)} = \frac{(\sigma)_{2-1}}{(2-1)!\, \phi_2(m)}$

Therefore (3.1) holds for n=2. Assume that (3.1) is true for $n = l$ and consider

$$|a_{l+1}| \leq \frac{\sigma}{l\,\phi_{l+1}(m)}\left(1+\sum_{i=2}^{l}\phi_i(m)|a_i|\right)$$
$$\leq \frac{\sigma}{l\,\phi_{l+1}(m)}\left(1+\sum_{i=2}^{l}\frac{(\sigma)_{i-1}}{(i-1)!}\right)$$
$$= \frac{\sigma}{l\,\phi_{l+1}(m)}\left(1+\sum_{i=2}^{l}\sigma\,\prod_{j=1}^{i-1}\left(1+\frac{\sigma}{j}\right)\right)$$
$$= \frac{\sigma}{l\,\phi_{l+1}(m)}\prod_{j=1}^{l-1}\left(1+\frac{\sigma}{j}\right)$$
$$= \frac{(\sigma)_l}{l\,!\,\phi_{l+1}(m)}.$$

Therefore, the result is true for $n = l + 1$. Using mathematical induction, (3.1) holds true for all $n \geq 2$.

This result is sharp for $m \geq 0$, $0 \leq \alpha < 1$, $b \in \mathbb{C} - \{0\}$ and $k \geq 2$ as can be seen

from the functions $f_0(z)$ which are given as

$$1-\frac{2}{b}+\frac{2}{b}\frac{D_\lambda^{m+1}f_0(z)}{D_\lambda^{m}f_0(z)} = (1-\alpha)\left[\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+Az}{1+Bz}-\left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-Az}{1-Bz}\right]+\alpha.$$

For different values of A, B, α, b and λ, we obtain the following results [16].

Corollary 3.2. *If* $f \in \mathcal{V}_{k,\lambda}^{0}[1,-1,\alpha,2] = R_k(\alpha)$, then

$$|a_n| \leq \frac{\big(k(1-\alpha)\big)_{n-1}}{(n-1)!}, \quad \forall\, n \geq 2$$

this result is sharp.

Corollary 3.3. *If* $f \in \mathcal{V}_{k,\lambda}^{1}[1,-1,\alpha,1] = V_k(\alpha)$, then

$$|a_n| \leq \frac{\big(k(1-\alpha)\big)_{n-1}}{(n)!}, \quad \forall\, n \geq 2.$$

this result is sharp.

Theorem 3.4. *For real b > 0,*

$\mathcal{V}_{k,\lambda}^{m+1}[A,B,\alpha,b] \subseteq \mathcal{V}_{k,\lambda}^{m}[1,-1,\beta,b+1]$, $z \in \mathcal{U}$,

where β (0 ≤ β < 1) is one of the roots of

(3.3) $$\eta_1\eta_2 b^2(m+2)^2(1-\alpha)^2 - b(m+2)(1-\alpha)[\eta_1(B+1)+\eta_2(B-1)] + (B^2-1) = 0,$$

where

(3.4) $$\eta_1 = \frac{(1-b)+\beta(1+b)}{(1+b)(1-\beta)}[\Delta(\mathrm{B}-1)-(\mathrm{A}-1)]$$

(3.5) $$\eta_2 = \frac{(1-b)+\beta(1+b)}{(1+b)(1-\beta)}[\Delta(\mathrm{B}+1)-(\mathrm{A}+1)]$$

and

$$\Delta = 1 - \frac{(1+b)(m+1)(1-\beta)}{b(m+2)(1-\alpha)}.$$

Proof: Suppose f $\in \mathcal{V}_{k,\lambda}^{m+1}[A,B,\alpha,b]$ and set

(3.6) $$p(z) = 1 - \frac{2}{b+1} + \frac{2}{b+1}\frac{D_\lambda^{m+1}f(z)}{D_\lambda^{m}f(z)}$$

where p is analytic in $\mathcal{U}$ with $p(0) = 1$. Then, by simple computations together with (3.6) and (1.9) yield

(3.7) $$1 - \frac{2}{b} + \frac{2}{b}\frac{D_\lambda^{m+2}f(z)}{D_\lambda^{m+1}f(z)} = (1-\mu_1) + \mu_1\left[p(z) + \frac{\mu_2 zp'(z)}{p(z)+\mu_3}\right],$$

where $\mu_1 = \frac{m+1}{m+2}\frac{b+1}{b}$, $\mu_2 = \frac{2}{(m+1)(b+1)}$, $\mu_3 = \frac{2}{b+1} - 1$.

Since f $\in \mathcal{V}_{k,\lambda}^{m+1}[A,B,\alpha,b]$, it follows that $(1-\mu_1) + \mu_1\left[p(z) + \frac{\mu_2 zp'(z)}{p(z)+\mu_3}\right] \in P_k[A,B,\alpha]$,

Or, equivalently

(3.8) $$\frac{(1-\alpha-\mu_1)}{(1-\alpha)} + \frac{\mu_1}{1-\alpha}\left[p(z) + \frac{\mu_2 zp'(z)}{p(z)+\mu_3}\right] \in P_k[A,B].$$

Define

$$\phi(z) = \frac{1}{(1+\mu_3)}\frac{z}{(1-z)^{\mu_2}} + \frac{\mu_3}{(1+\mu_3)}\frac{z}{(1-z)^{\mu_2+1}},$$

and by using convolution techniques given by Noor[3], we have

$$p(z) + \frac{\mu_2 zp'(z)}{p(z)+\mu_3} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(p_1(z) + \frac{\mu_2 zp_1'(z)}{p_1(z)+\mu_3}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(p_2(z) + \frac{\mu_2 zp_2'(z)}{p_2(z)+\mu_3}\right)$$

By using (3.8), we see that

$$\frac{(1-\alpha-\mu_1)}{(1-\alpha)} + \frac{\mu_1}{1-\alpha}\left[p_i(z) + \frac{\mu_2 zp_i'(z)}{p_i(z)+\mu_3}\right] \in P[A,B],$$

where $z \in \mathcal{U}$, i=1,2.

Now, want to show that $p_i \in P[A,B,\beta]$, where $\beta(0 \le \beta < 1)$ is one of the root of (3.3).

Let $p_i(z) = (1-\beta)h_i(z) + \beta$, i=1,2.

Then,

$$\frac{(1-\alpha-\mu_1)(1-\beta)}{(1-\alpha)} + \frac{\mu_1(1-\beta)}{1-\alpha}\left[h_i(z) + \frac{\frac{\mu_2}{(1-\beta)}zh_i'(z)}{h_i(z) + \frac{\mu_3+\beta}{(1-\beta)}}\right] \in P[A,B]$$

Using the fact illustrated in (1.2), we have

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$$\left\{\frac{(B-1)\big[\big(\eta+\mu h_i(z)\big)(h_i(z)+\omega_2)+\omega_1\mu z h_i'(z)\big]-(A-1)(h_i(z)+\omega_2)}{(B+1)\big[\big(\eta+\mu h_i(z)\big)(h_i(z)+\omega_2)+\omega_1\mu z h_i'(z)\big]-(A+1)(h_i(z)+\omega_2)}\right\}\in P,$$

where $\omega_1=\frac{\mu_2}{1-\beta}$, $\omega_2=\frac{\mu_3+\beta}{1-\beta}$, $\eta=\frac{1-\alpha-\mu_1(1-\beta)}{1-\alpha}$ and $\mu=\frac{\mu_1(1-\beta)}{1-\alpha}$. We now form the functional $\psi(u,v)$ by choosing $u=h_i(z), v=zh_i'(z)$ and note that the first two conditions of Lemma 2.2 are clearly satisfied. We check condition (iii) as follows.

$$\psi(u,v)=\frac{\{(B-1)[(\eta+\mu u)(u+\omega_2)+\omega_1\mu v]-(A-1)(u+\omega_2)\}}{\{(B+1)[(\eta+\mu u)(u+\omega_2)+\omega_1\mu v]-(A+1)(u+\omega_2)\}}$$

$$=\frac{\{\eta_1+\omega_1\mu(B-1)v+[\eta+\mu(u+\omega_2))(B-1)-(A-1)]u\}}{\{\eta_2+\omega_1\mu(B+1)v+[(\eta+\mu(u+\omega_2))(B+1)-(A+1)]u\}}$$

where $\eta_1=\ \omega_2[\eta(B-1)-(A-1)]$ and $\eta_2=\ \omega_2[\eta(B+1)-(A+1)]$.

Now,

$$\psi(iu_2,v_1)=\frac{\{\eta_1+\mu(\omega_1v_1-u_2^2)(B-1)+[(\eta+\mu\omega_2)(B-1)-(A-1)]iu_2\}}{\{\eta_2+\mu(\omega_1v_1-u_2^2)(B+1)+[(\eta+\mu u_2)(B+1)-(A+1)]iu_2\}}$$

Taking real part of $\psi(iu_2,v_1)$, we have

$$\mathrm{Re}\,\psi(iu_2,v_1)$$
$$=\frac{[-\eta_1+\mu(\omega_1v_1-u_2^2)(1-B)][\eta_2+\mu(\omega_1v_1-u_2^2)(B+1)]-[(\eta+\mu\omega_2)(B-1)-(A-1)][(\eta+\mu\omega_2)(B}{-[\eta_2+\mu(\omega_1v_1+u_2)(B+1)]^2-[(\eta+\mu\omega_2)(B+1)-(A+1)]^2u_2^2}$$

As $\omega_1>0,\mu>0$, so applying $v_1\le-\frac{1}{2}(1+u_2^2)$ and after a little simplification, we have

(3.9) $$\mathrm{Re}\,\psi(iu_2,v_1)\le\frac{A_1+B_1u_2^2+C_1u_2^4}{D_1}$$

where

$$A_1=\frac{1}{4}[2\eta_1-\omega_1\mu(B-1)][2\eta_2-\omega_1\mu(B+1],$$

$$B_1=-\frac{1}{2}\mu(\omega_1+2)[\eta_1(B+1)-\omega_1\mu(B^2-1)+\eta_2(B-1)]+(\eta+\mu\omega_2)^2(B^2-1)$$
$$-2(\eta+\mu\omega_2)(AB-1)+(A^2-1),$$

$$C_1=-\frac{1}{4}\mu^2(1-B^2)(\omega_1+2)^2,$$

and

$$D_1=[\eta_2+\mu(\omega_1v_1+u_2)(B+1)]^2+[(\eta+\mu\omega_2)(B+1)-(A+1)]^2u_2^2.$$

The right hand side of (3.9) is negative if $A_1\le 0$ and $B_1\le 0$. From $A_1\le 0$, we have β to be one of the roots of

$$\eta_1\eta_2b^2(m+2)^2(1-\alpha)^2-b(m+2)(1-\alpha)[\eta_1(B+1)+\eta_2(B-1)]+(B^2-1)=0$$

With 0 ≤ β < 1 and also for 0 ≤ β < 1, we have $B_1 \leq 0$.

Since all the conditions of Lemma 2.2 are satisfied, it follows that $h_i \in P,$ i=1,2 and consequently $p \in P_k[1,-1,\beta]$. Hence from (3.6),

$$f \in \mathcal{V}_{k,\lambda}^{m}[1,-1,\beta,b+1].$$

By choosing the parameters $\mathrm{A}=1, \mathrm{B}=-1, \mathrm{b}=1, \mathrm{m}=0$, we obtain the following known result, proved in [11].

Corollary 3.5. Let $f \in V_{k,\lambda}(\alpha)$. Then $f \in R_{k,\lambda}(\beta)$, where β is a root of

$$2\beta^2-(2\alpha-1)\beta-1=0$$

With 0 ≤ β < 1, which is

$$\beta=\frac{1}{4}\left[(2\alpha-1)+\sqrt{4\alpha^2-4\alpha+9}\right].$$

For $\alpha=0, k=2$ in Corollary 3.5, we have the following well known results [2].

$V_{2,\lambda}(0)=C \subseteq R_{2,\lambda}\left(\frac{1}{2}\right)=S^*\left(\frac{1}{2}\right),$ for z ∈ $\mathcal{U}$.

Theorem 3.6. Let f ∈ $\mathcal{V}_{k,\lambda}^{m}[A,B,0,b]$, $m \geq 0$, b>0(real), k≥ 2 and $0<a=\frac{b(m+1)}{2} \leq 1$. Then $D_\lambda^m f(z)$ maps $|z|<r_0$ onto a convex domain, where r_0 is the least positive root of the equation

(3.10) $a_1 r^4+a_2 r^3+a_3 r^2+a_4 r+4(2a-1)=0$ with $0 \leq r<1$,

where

$$a_1=4a^2A^2B^2-4(a-1)^2B^4$$

$$a_2=2\mathrm{a}(2\mathrm{a}-1)(\mathrm{B}-\mathrm{A})B^2k$$

$$a_3=8a^2(a-2)+8a(1-a)AB-a^2(A-B)^2k^2$$

$$a_4=2a(2a-3)(A-B)k.$$

This result is sharp.

Proof: Since f ∈ $\mathcal{V}_{k,\lambda}^{m}[A,B,0,b]$ then

(3.11) $$\frac{\mathrm{D}_\lambda^{m+1}\mathrm{f}(\mathrm{z})}{\mathrm{D}_\lambda^{m}\mathrm{f}(\mathrm{z})}=\frac{\mathrm{b}(\mathrm{p}(\mathrm{z})-1)+2}{2}$$

where p ∈ $P_k[A,B,0]$. Using the identity (1.9), we have from (3.11),

(3.12) $$\frac{z(D_\lambda^m f(z))'}{D_\lambda^m f(z)}=\frac{b(p(z)-1)(m+1)+2}{2}$$

Logarithmic differentiation of (3.12) yields

$$\frac{(\boldsymbol{z}(D_\lambda^m f(z))')'}{(D_\lambda^m f(z))'}=ap(z)-a+1+\frac{zp'(z)}{p(z)-1+\frac{1}{a}}$$

where $a = \frac{b(m+1)}{2}$. Then we have

$$\mathrm{Re}\left(1 + \frac{z(D_\lambda^m f(z))''}{(D_\lambda^m f(z))'}\right) \geq \mathrm{a\ Re\ p(z)} + (1 - \mathrm{a}) - \frac{|zp'(z)|}{\left|p(z) - 1 + \frac{1}{a}\right|},$$

and hence, by using Lemma 2.3 and Lemma 2.4,

$$\mathrm{Re}\left(1 + \frac{z(D_\lambda^m f(z))''}{(D_\lambda^m f(z))'}\right)$$

$$\geq Re\, p(z)\left\{a + \frac{2(1-a)(1-B^2r^2)}{2+(A-B)kr-2ABr^2} - \frac{2ar\{(A-B)k - 4B(A-B)r + B^2(A-B)kr^2\}}{(2+(A-B)kr-2ABr^2)\xi}\right\}$$

$$= Re\, p(z)\left\{\frac{a_1r^4 + a_2r^3 + a_3r^2 + a_4r + 4(2a-1)}{(2+(A-B)kr-2ABr^2)\xi}\right\} > 0,$$

provided

$T(r) = a_1r^4 + a_2r^3 + a_3r^2 + a_4r + 4(2a-1) > 0,$

where

$a_1 = 4a^2A^2B^2 - 4(a-1)^2B^4$

$a_2 = 2\mathrm{a}(2\mathrm{a}-1)(\mathrm{B}-\mathrm{A})B^2k$

$a_3 = 8a^2(a-2) + 8a(1-a)AB - a^2(A-B)^2k^2$

$a_4 = 2a(2a-3)(A-B)k$

and

$\xi = 2(2a-1) - a(A-B)kr + 2(B^2 - a(A+B)B)r^2.$

We have $T(0) > 0$ *and* $T(1) < 0$. Therefore, $D_\lambda^m f(z)$ maps $|z| < r_0$ onto a convex domain, where r_0 is the least positive root of the equation $T(r) = 0$, lying in (0,1).

For $D_\lambda^m f_1(z)$ such that

$$\frac{D_\lambda^{m+1} f_1(z)}{D_\lambda^m f_1(z)} = \frac{b(p_k(z)-1)+2}{2}$$

where $p_k(z) = \frac{2+(A-B)kz-2ABz^2}{2(1-B^2z^2)}$, we have

$$\frac{(\boldsymbol{z}(D_\lambda^m f_1(z))')'}{(D_\lambda^m f_1(z))'} = \frac{a_1r^4 + a_2r^3 + a_3r^2 + a_4r + 4(2a-1)}{(2+(A-B)kr-2ABr^2)\xi} = 0$$

for $z = r_0$. Hence this radius r_0 is sharp.

By choosing the parameters $A = 1, B = -1, k = 2, b = 2$ *and* $m = 0$, we obtain the following known result, see[2].

Corollary 3.7: Let $f \in S'$. Then f maps $|z| < r_0$ onto a convex domain, where r_0 is the least positive root of the equation

$$r^4 - 2r^3 - 6r^2 - 2r + 1 = 0$$

with $0 \leq r < 1$, which is $r_0 = 2 - \sqrt{3}$. This is also sharp.

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