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Exploring Quadruple Fixed Points in Fuzzy Metric Spaces for Occasionally Weakly Compatible Mappings

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Abstract: In this research paper we prove some quadruple fixed point theorems for occasionally weakly compatible mappings in fuzzy metric space.

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1. Introduction

1 Introduction

Zadeh [14] introduced the concept of fuzzy sets, while Kramosil and Michalek [11] developed the idea of fuzzy metric spaces. Later, George and Veermani [5] refined this concept by proposing a new framework for fuzzy metric spaces using continuous t-norms. Numerous researchers have since established common fixed point theorems for mappings under various commutativity conditions.

The study of fixed point theorems involving four self-maps initially relied on the assumption of commutativity. Sessa [13] relaxed this assumption by introducing the notion of pairwise weakly commuting maps. Jungck extended this further to pairwise compatible [6] and pairwise weakly compatible mappings [7]. Subsequently, Jungck and Rhoades [8] introduced the concept of occasionally weakly compatible (owc) mappings. The research in works [1], [3], [9], and [10] on quadruple fixed points is truly noteworthy.

In this paper we introduce some quadruple fixed point theorems for occasionally weakly compatible mappings in fuzzy metric space.

2 Preliminary Notes

Definition 2.1 A fuzzy set A in X is a function with domain X and values in [0,1].

Definition 2.2 A binary operation $*: [0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ is satisfying conditions:

(i) $*$ is an commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all a$\in$ [0,1];

(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ and $a, b, c, d \in [0,1]$.

Definitions 2.3 A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set $,*$ is a continuous $t - norm$ and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X, s, t > 0$,

(i) $M(x, y, t) > 0$;

(ii) $M(x, y, t) = 1$ $if\ and\ only\ if$ $x = y$;

(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;

(v) $M(x, y, \cdot) : (0, \infty) \to (0,1]$ is continuous.

Then M is called a $fuzzy\ metric$ on X. Then $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Example 2.4 Let (X, d) be a metric space. Denote $a * b = ab$ for all $a, b \in [0,1]$ and let M_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, *)$ is a fuzzy metric space.

Lemma 2.5 Let (X, M,*) be a fuzzy metric space. If there exists $q \in (0,1)$ such that M(x, y, qt) $\ge$ M(x ,y ,t) for all x, y $\in$ X and t>0, then x = y.

Definition 2.6 Let X be a non-empty set. An element $(x, y, z, t) \in X \times X \times X \times X$ is called a quadruple fixed point of a given mapping $f: X \times X \times X \times X \to X$ if $x = f(x, y, z, t), y = f(y, z, t, x), z = f(z, t, x, y), t = f(t, x, y, z)$.

Definition 2.7 An element $(x, y, z, t) \in X \times X \times X \times X$ is called a quadruple coincidence point of a mapping $f: X \times X \times X \times X \to X$ and $g: X \to X$ if $gx = f(x, y, z, t), gy = f(y, z, t, x), gz = f(z, t, x, y), gt = f(t, x, y, z)$ in this case (gx, gy, gz, gt) is called a quadruple point of coincidence.

Definition 2.8 The mappings $f: X \times X \times X \times X \to X$ and $g: X \to X$ of a set X are occasionally weakly compatible (owc) iff there is a point $(x, y, z, t) \in X \times X \times X \times X$ which is a coincidence point of f and g at which f and g commute i.e. (f, g) are occasionally weakly compatible maps iff $f(x, y, z, t) = g(x),\ f(y, z, t, x) = g(y), f(z, t, x, y) = g(z), f(t, x, y, z) = g(t)$

implies $gf(x, y, z, t) = f(gx, gy, gz, gt), gf(y, z, t, x) = f(gy, gz, gt, gx), gf(z, t, x, y) = f(gz, gt, gx, gy), gf(t, x, y, z) = f(gt, gx, gy, gz)$ for $(x, y, z, t) \in X \times X \times X \times X$.

Example 2.10.1 Let $(X, \mathcal{F}, *)$ be a fuzzy metric space, where $X = [0,1]$ with $a * b = \min\{a, b\}$ and

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

Let $f: X \times X \times X \to X$ & $g: X \to X$ be defined by

$$f(x,y,z,w) = \frac{2x+2y+2z+w}{2}$$

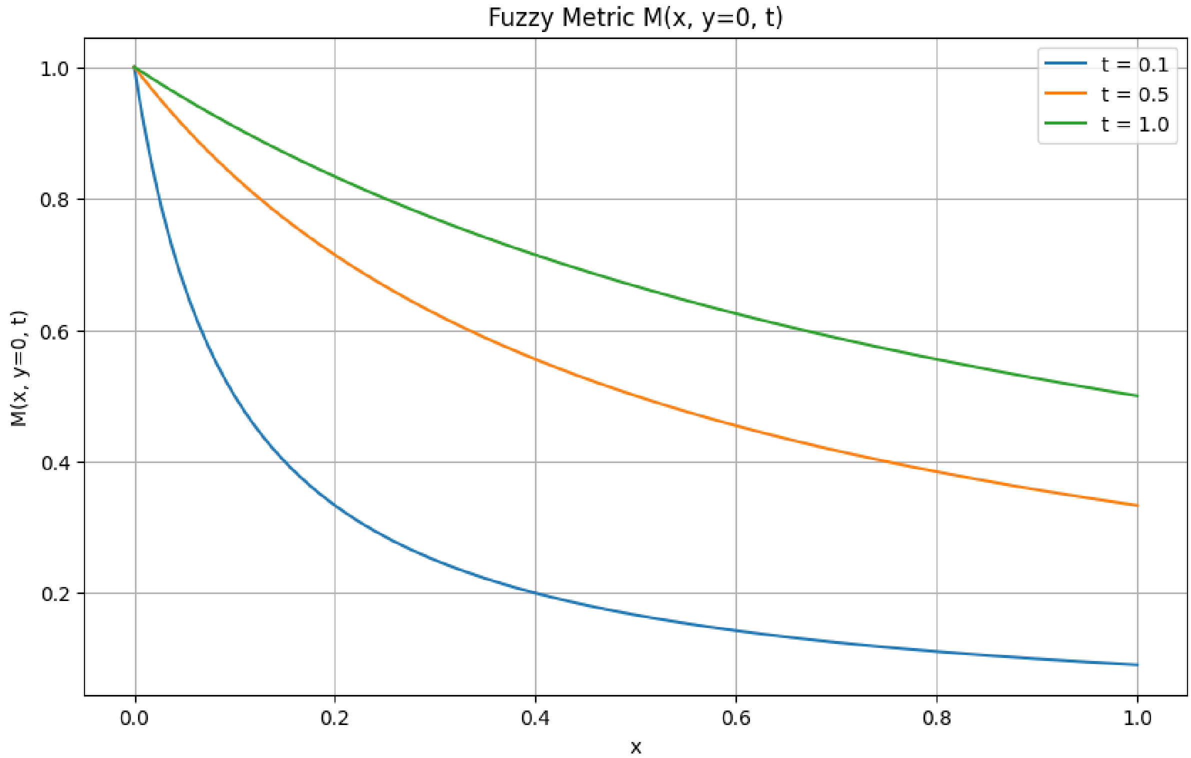
$$g(x) = \begin{cases} x, & \text{if } 0 \le x < 1; \\ \dfrac{7}{2}, & \text{if } x \ge 1. \end{cases}$$

Here, (0,0,0,0) and (1,1,1,1) are two coincidence points of f and g. That is $f(0,0,0,0) = 0 = g(0), f(1,1,1,1) = 1 = g(1)$ but $gf(0,0,0,0) = 0 = f(g0, g0, g0, g0)$, $gf(1,1,1,1) \neq f(g1, g1, g1, g1)$. Thus f and g are owc but not weakly compatible.

Coincidence points verification:

$f(0,0,0,0) = 0.0, g(0) = 0, gf(0,0,0,0) = 0.0, f(g0, g0, g0, g0) = 0.0$

$f(1,1,1,1) = 3.5, g(1) = 3.5, gf(1,1,1,1) = 3.5, f(g1, g1, g1, g1) = 12.25$



f and g are occasionally weakly compatible (owc) but not weakly compatible.

3 Main Results

Theorem: 3.1 Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A, B: X \times X \times X \times X \to X$ and $S, T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M(A(x,y,z,p),B(u,v,w,r),qt) \geq$
$min\left\{\begin{matrix} M(Sx,Tu,t), M(A(x,y,z,p),Sx,t), M(B(u,v,w,r),Tu,t), M(Sx,B(u,v,w,r),t), \\ M(A(x,y,z,p),Tu,t) \end{matrix}\right\}$

for all $x,y,z,p,u,v,w,r \in X$

(ii) $y = B(x,y,z,p)$

Moreover if the pairs (A,S) and (B,T) are owc, then there exists a unique point x in X such that $A(x,x,x,x) = T(x) = B(x,x,x,x) = S(x) = x$.

Proof: Since the pairs (A,S) and (B,T) are owc so there are points a,b,c,d,a',b',c',d' in X such that

$A(a,b,c,d) = Sa,\ A(b,c,d,a) = Sb, A(c,d,a,b) = Sc, A(d,a,b,c) = Sd$ and

$B(a',b',c',d') = Ta',\ \ B(b',c',d',a') = Tb', B(c',d',a',b') = Tc', B(d',a',b',c') = Td'$

$$Sx = SA(a,b,c,d) = A(Sa,Sb,Sc,Sd) = A(x,y,z,p)$$

$$Sy = SA(b,c,d,a) = A(Sb,Sc,Sd,Sa) = A(y,z,p,x)$$

$$Sz = SA(c,d,a,b) = A(Sc,Sd,Sa,Sb) = A(z,p,x,y)$$

$$Sp = SA(d,a,b,c) = A(Sd,Sa,Sb,Sc) = A(p,x,y,z)$$

We claim that $Sa = Ta'$. If not, by inequality (i) we get

$M(A(a,b,c,d),B(a',b',c',d'),qt) \geq$
$min\left\{\begin{matrix} M(Sa,Ta',t), M(A(a,b,c,d),Sa,t), M(B(a',b',c',d'),Ta',t), M(Sa,B(a',b',c',d'),t), \\ M(A(a,b,c,d),Ta',t) \end{matrix}\right\}$

or $M(Sa,Ta',qt) \geq min\{M(Sa,Ta',t), M(Sa,Sa,t), M(Ta',Ta',t), M(Sa,Ta',t)M(Sa,Ta',t)\}$

$= min\{M(Sa,Ta',t), 1,1, M(Sa,Ta',t), M(Sa,Ta',t)\}$

$= M(Sa,Ta',t)$

$$\Rightarrow Sa = Ta'$$

Therefore $A(a,b,c,d) = Ta' = Sa = B(a',b',c',d')$

Similarly $A(b,c,d,a) = Tb' = Sb = B(b',c',d',a')$

$$A(c,d,a,b) = Tc' = Sc = B(c',d',a',b')$$

$$A(d,a,b,c) = Td' = Sd = B(d',a',b',c')$$

Thus the pairs (A,S) and (B,T) have common coincidence points.

Let $A(a,b,c,d) = Ta' = Sa = B(a',b',c',d') = x$

and $A(b,c,d,a) = Tb' = Sb = B(b',c',d',a') = y$

$$A(c,d,a,b) = Tc' = Sc = B(c',d',a',b') = z$$

$$A(d,a,b,c) = Td' = Sd = B(d',a',b',c') = p$$

Since (A,S) and (B,T) are owc

$$Sx = SA(a,b,c,d) = A(Sa,Sb,Sc,Sd) = A(x,y,z,p)$$

$$Sy = SA(b,c,d,a) = A(Sb,Sc,Sd,Sa) = A(y,z,p,x)$$

$$Sz = SA(c,d,a,b) = A(Sc,Sd,Sa,Sb) = A(z,p,x,y)$$

$$Sp = SA(d,a,b,c) = A(Sd,Sa,Sb,Sc) = A(p,x,y,z)$$

Also $Tx = TB(a',b',c',d') = B(Ta',Tb',Tc',Td') = B(x,y,z,p)$

and $Ty = TB(b',c',d',a') = B(Tb',Tc',Td',Ta') = B(y,z,p,x)$

$$Tz = TB(c',d',a',b') = B(Tc',Td',Ta',Tb') = B(z,p,x,y)$$

$$Tp = TB(d',a',b',c') = B(Td',Ta',Tb',Tc') = B(p,x,y,z)$$

Next we show that $x = y$, for this

putting $x = a$, $y = b$, $z = c$, $p = d, u = b'$, $v = c', w = d', r = a'$ in (i),

$M(x,y,qt) = M(A(a,b,c,d),B(b',c',d',a'),qt) \geq$
$min\begin{Bmatrix} M(Sa,Tb',t), M(A(a,b,c,d),Sa,t), M(B(b',c',d',a'),Tb',t), M(Sa,B(b',c',d',a'),t), \\ M(A(a,b,c,d),Tb',t) \end{Bmatrix}$

$$= min\{M(x,y,t), M(x,x,t), M(y,y,t), M(x,y,t), M(x,y,t)\}$$

$$= M(x,y,t)$$

$$\Longrightarrow x = y$$

Next we show that $x = z$, for this

putting $x = a$, $y = b$, $z = c$, $p = d, u = c'$, $v = d', w = a', r = b'$ in (i),

$M(x,z,qt) = M(A(a,b,c,d),B(c',d',a',b'),qt) \geq$
$min\begin{Bmatrix} M(Sa,Tc',t), M(A(a,b,c,d),Sa,t), M(B(c',d',a',b'),Tc',t), M(Sa,B(c',d',a',b'),t), \\ M(A(a,b,c,d),Tc',t) \end{Bmatrix}$

$$= min\{M(x,z,t), M(x,x,t), M(z,z,t), M(x,z,t), M(x,z,t)\}$$

$= M(x,z,t)$

$$\Longrightarrow x = z$$

Next we show that $x = p$, for this

putting $x = a$, $y = b$, $z = c$, $p = d, u = d'$, $v = c', w = a', r = b'$ in (i),

$M(x,z,qt) = M(A(a,b,c,d),B(d',c',a',b'),qt) \geq$
$min\begin{Bmatrix} M(Sa,Td',t), M(A(a,b,c,d),Sa,t), M(B(d',c',a',b'),Td',t), M(Sa,B(d',c',a',b'),t), \\ M(A(a,b,c,d),Td',t) \end{Bmatrix}$

$$= min\{M(x,p,t), M(x,x,t), M(p,p,t), M(x,p,t), M(x,p,t)\}$$

$= M(x,p,t)$

$$\Longrightarrow x = p$$

$$\Rightarrow x = y = z = p$$

Now we prove that $Sx = Tx$

Putting $u = y, v = z, w = t, r = x$

$M(Sx,Tx,qt) = M(A(x,y,z,p),B(y,z,t,x),qt) \geq$
$min\begin{Bmatrix} M(Sx,Ty,t), M(A(x,y,z,p),Sx,t), M(B(y,z,t,x),Ty,t), M(Sx,B(y,z,t,x),t), \\ M(A(x,y,z,p),Ty,t) \end{Bmatrix}$

$= min\{M(Sx,Ty,t), M(Sx,Sx,t), M(Ty,Ty,t), M(Sx,Ty,t), M(Sx,Ty,t)\}$

$= min\ \{M(Sx,Ty,t), 1,1, M(Sx,Ty,t), M(Sx,Ty,t)\}$

$= M(Sx,Ty,t)$

$$\Longrightarrow Sx = Ty$$

$$\Rightarrow Sx = Tx$$

Also by condition (ii) we have,

$$x = B(x,x,x,x)$$

Thus $A(x,x) = T(x) = B(x,x) = S(x) = x$.

Example 3.1.1 Let $X = [0,1]$ with the metric d defined by $d(x,y) = |x - y|$ and for each $t \in [0,1]$, define

$$M(x,y,t) = \begin{cases} \dfrac{t}{t + |x - y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0 \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, *)$ be a fuzzy metric space, with $a * b = \min\{a,b\}$. Let $S,T: X \to X$ and $A,B: X \times X \times X \times X \to X$ defined by

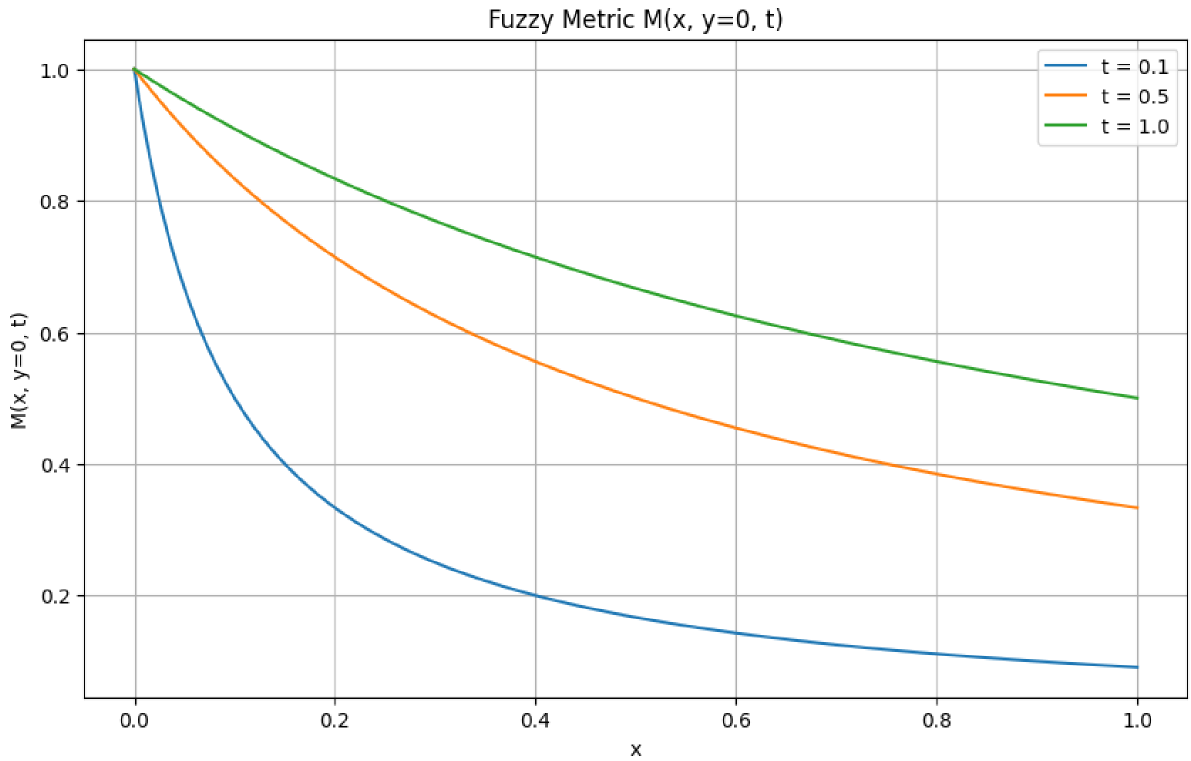
$$A(x,y,z,w) = \frac{2x+y+2z+w}{2} \qquad S(X) = \begin{cases} x, & \text{if } 0 \leq x < 1; \\ \frac{7}{2}, & \text{if } x \geq 1. \end{cases}$$

$$B(x,y,z,w) = y \qquad T(X) = \begin{cases} x, & \text{if } 0 \leq x < 1; \\ 5, & \text{if } x \geq 1. \end{cases}$$

Clearly all the conditions of the above theorem are satisfied. Also

$$SA(0,0,0,0) = A(S0,S0,S0,S0) \text{ and } TB(0,0,0,0) = B(T0,T0,T0,T0)$$

So, (A, S) and (B, T) are owc maps and (0, 0, 0, 0) is the common quadruple fixed point of A, B, S and T.



All conditions of the theorem are satisfied.

Common quadruple fixed point is (0, 0, 0, 0).

Theorem: 3.2 Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A, B: X \times X \times X \times X \rightarrow X$ and $S, T: X \rightarrow X$ be four self-mappings satisfying the following conditions:

(i) $$M(A(x,y,z,p), B(u,v,w,r), qt) \geq \left\{\frac{\frac{M(B(u,v,w,r),Sx,t).M(B(u,v,w,r),Tu,t)+M(Sx,B(u,v,w,r),t).}{M(B(u,v,w,r),Tu,t)}}{2}\right\}$$

for all $x, y, z, p, u, v, w, r \in X$

(ii) $y = B(x,y,z,p)$

Moreover if the pairs (A, S) and (B, T) are owc, then there exists a unique point x in X such that $A(x,x,x,x) = T(x) = B(x,x,x,x) = S(x) = x$.

Theorem: 3.3 Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A, B: X \times X \times X \times X \rightarrow X$ and $S, T: X \rightarrow X$ be four self-mappings satisfying the following conditions:

(i) $$M(A(x,y,z,p), B(u,v,w,r), qt) \geq min\left\{M(Sx, Tu, t), \left(\frac{1+M(A(x,y,z,p),Sx,t)}{1+M(B(u,v,w,r),Tu,t)}\right), \frac{M(A(x,y,z,p),Sx,t)}{M(B(u,v,w,r),Tu,t)}\right\}$$

for all $x, y, z, p, u, v, w, r \in X$

(ii) $y = B(x,y,z,p)$

Moreover if the pairs (A, S) and (B, T) are owc, then there exists a unique point x in X such that $A(x, x, x, x) = T(x) = B(x, x, x, x) = S(x) = x$.

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