Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

An Improved Generalized Class of Estimator for Finite Population Mean in Stratified Systematic Sampling using Auxiliary Information

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract: This paper introduces, the generalized class of exponential-type estimator in stratified systematic sampling scheme is proposed for estimating the population mean $\bar{Y}$ of the study variable y using auxiliary information x. Theoretical expressions for bias and mean square error (MSE) are derived up to the first order of approximation. A simulation study, along with analysis of three real data sets, is conducted to evaluate the performance of these estimators, where the percent relative efficiency (PRE) is considered as a performance criterion. The result indicate that the proposed estimator is outperforms the traditional mean, product and regression estimators in term of efficiency. The simulation study was performed using R software.

Keywords: Efficiency, Exponential estimator, PRE, Stratified sampling, Systematic sampling.

1. Introduction

In statistical surveys, when subpopulations within a larger population differ, it is beneficial to sample each subpopulation (or stratum) independently. Stratification involves dividing the population into homogeneous subgroups before sampling. These strata must be mutually exclusive, meaning each element belongs to only one stratum, and collectively exhaustive, ensuring no population element can be excluded. Simple random sampling or systematic sampling is applied within each stratum. This approach typically enhances the representativeness of the sample by reducing sampling error.

Systematic sampling, first explored by Madow and Madow [14], is commonly used in surveys of finite populations. This method involves selecting sample members from a larger population based on a random starting point and a fixed periodic interval. Typically, every "n^{th}" individual is chosen from the population for inclusion in the sample population. Despite the fixed interval, systematic sampling is still considered random, provided that the starting point is chosen randomly and the interval is predetermined.

Systematic sampling has the advantage of selecting the whole sample with just one random starting point. In addition to its simplicity, which is of considerable importance, this method often yields more efficient estimators compared to simple random sampling or stratified random sampling for certain types of population, as noted by (Cochran [7], Gautschi [9], Hajeck [11]).

The primary limitation of the ratio and product estimators is that of having efficiency not exceeding that of the regression estimator. Consequently, many researchers have focused on modifying these estimators to develop more efficient alternatives estimators. Notable contributions in this area come from; Singh and Vishwakarma [20,21], Sharma and Tailor [17], Onyeka [16], Tailor [29], Choudhury and Singh [4], Khare and Sinha [12], Singh and Audu [23].

Clement [5] addressed this issue by using systematic sampling and proposed a calibration ratio-type estimator under a stratified systematic sampling framework. Chaudhary and Dutta [3] addressed this issue by using some calibration estimators of finite population mean under stratified systematic sampling in the presence of non- response.

In the following sections, we have introduced a generalized class of exponential-type estimator for the population mean in stratified systematic sampling, utilizing information from a single auxiliary variable for the first time. The mean square error (MSE) expression for the proposed exponential estimator has been derived. Additionally, an empirical comparison between the proposed estimator and traditional estimators is provided, based on both simulation studies and real data analysis.

2. Notation [estimation procedure]

Consider a finite population Ω of N elements. suppose $\Omega = (\Omega_1, \Omega_2, \dots, \Omega_N)$ is divided into of L strata with N_h units in the h^{th}stratum from which a systematic random sample of size n_h is taken, and a random sample of n_h unit is selected from the first k_h units, and then every k_h^{th} subsequent unit is included in the sample of size n_h. where $k_h = \frac{N_h}{n_h}$, k_h a positive integer, and this method provides k_h samples, each of size n_h. Both the study variate (y) and auxiliary variate (x) are observed for each and every unit selected in the sample. Subsequently, the above scheme is stratified systematic sampling and the notations are defined as follows:

The total population size be $N = \sum_{h=1}^{L} N_h$ and sample size $n = \sum_{h=1}^{L} n_h$, respectively. Associated with the i^{th} element of the h^{th} stratum are y_{hi} and x_{hi} with $x_{hi} > 0$ being the covariate; where y_{hi} is the y value of the i^{th} element in stratum h, and x_{hi} is the x value of the i^{th} element in stratum h, $h = 1,2, \dots., L$ and $i = 1,2, \dots, N_h$. For the h^{th} stratum, let $W_h = \frac{N_h}{N}$ be the stratum weight and $f_h = \frac{n_h}{N_h}$, the sample fraction.

Let the h^{th} stratum means of the study variable y and auxiliary variable x.

$\bar{y}_h = \sum_{i=1}^{n_h} \frac{y_{hi}}{n_h}$ and $\bar{x}_h = \sum_{i=1}^{n_h} \frac{x_{hi}}{n_h}$ are the unbiased estimators of stratified systematic sample means corresponding to the population means $\bar{Y}_h = \sum_{i=1}^{N_h} \frac{y_{hi}}{N_h}$ and $\bar{X}_h = \sum_{i=1}^{N_h} \frac{x_{hi}}{N_h}$ of (y, x) respectively. Based on n_h observations.

Let $\bar{y}_{h.sys}$ be the mean of systematic sample in the h stratum, then the estimate of the sample mean and population mean $\bar{Y}$ in stratified systematic sampling scheme is given by [Cochran (1997)] as:

$\bar{y}_{st.sys} = \sum_{i=1}^{n_i} W_h \, \bar{y}_{h.sys}$, $\bar{Y}_{st.sys} = \bar{Y} = \sum_{i=1}^{n_i} W_h \, \bar{Y}_{h.sys}$

The following error terms are also defined.

$\in_y = \frac{\bar{y}_{h.sys} - \bar{Y}_h}{\bar{Y}_h}$, $\in_x = \frac{\bar{x}_{h.sys} - \bar{X}_h}{\bar{X}_h}$

$\bar{Y}_{h.sys} = \bar{Y}(1 + \epsilon_y)$, $\bar{X}_{h.sys} = \bar{X}(1 + \epsilon_x)$

Such that $E(\in_y) = E(\in_x) = 0$

Where $\bar{y}_{st.sys}$ and $\bar{x}_{st.sys}$ are usual unbiased estimators of population mean $\bar{Y}, \bar{X}$ respectively.

$E(\in_y^2) = \sum_{h=1}^{L} \theta_h \rho_{hy}^* C_{hy}^2 = V_{20}$; $E(\in_x^2) = \sum_{h=1}^{L} \theta_h \rho_{hx}^* C_{hx}^2 = V_{02}$

$E(\in_y \in_x) = \sum_{h=1}^{L} \theta_h \sqrt{\rho_{hy}^* \rho_{hx}^*} \rho_{yx} C_{hy} C_{hx} = \sum_{h=1}^{L} \theta_h \sqrt{\rho_{hy}^* \rho_{hx}^*} C_{hyx} = V_{11}$

where, $C_{hyx} = \rho_{hyx} C_{hy} C_{hx}$, and $\theta_h = \left(\frac{N_h - 1}{N_h n_h}\right)$

Let $S_{hy}^2 = \frac{1}{(N_h - 1)} \sum_{i=1}^{N_h} (y_{hi} - \bar{Y}_h)^2$ and $S_{hx}^2 = \frac{1}{(N_h - 1)} \sum_{i=1}^{N_h} (x_{hi} - \bar{X}_h)^2$ represent the population variances of the study variable and the auxiliary variable respectively, with the corresponding population covariance $S_{hyx} = \frac{1}{(N_h - 1)} \sum_{i=1}^{N_h} (y_{hi} - \bar{Y}_h)(x_{hi} - \bar{X}_h)$. Also $C_{hy}^2 = \frac{S_{hy}^2}{\bar{Y}^2}$ and $C_{hx}^2 = \frac{S_{hx}^2}{\bar{X}^2}$ are the population coefficient of variation of the variate (y, x) respectively.

here, $\rho_{hy}^* = [1 + (n_h - 1)\rho_{hy}]$, $\rho_{hx}^* = [1 + (n_h - 1)\rho_{hx}]$

where, $\rho_{hx} = \frac{E(x_{hi} - \bar{X}_h)(x'_{hi} - \bar{X}_h)}{E(x_{hi} - \bar{X}_h)^2}$; $\rho_{hy} = \frac{E(y_{hi} - \bar{Y}_h)(y'_{hi} - \bar{Y}_h)}{E(y_{hi} - \bar{Y}_h)^2}$

And $\rho_{hyx} = \frac{E(x_{hi} - \bar{X}_h)(y_{hi} - \bar{Y}_h)}{\sqrt{E(x_{hi} - \bar{X}_h)^2 (y_{hi} - \bar{Y}_h)^2}}$ be the usual population correlation coefficient between (y, x) respectively.

$C_{st.sys} = C_x = \sum_{h=1}^{L} W_h C_{hx}$: represent the population coefficient of variation of X for the h^{th} stratum, $\beta_{2st.sys} = \beta_{2x} = \sum_{h=1}^{L} W_h \beta_{2x(h)}$: represent the population coefficient of kurtosis of X for the h^{th} stratum, $\beta_{1st.sys} = \beta_{1x} = \sum_{h=1}^{L} W_h \beta_{1x(h)}$: represent the population coefficient of skewness of X for the h^{th} stratum, $\rho_{yxst.sys} = \rho_{yx} = \sum_{h=1}^{L} W_h \rho_{hyx}$: represent the population correlation coefficient between Y and X for h^{th} stratum

3. Existing estimators

In the context of stratified systematic sampling, several existing estimators of the population mean used in survey sampling are presented, along with their adaptations for stratified systematic sampling using auxiliary information. Their corresponding variance or mean square error (MSE) expressions are also provided.

1. The usual sample mean estimator in stratified systematic sampling is defined as given by

$\tilde{Y}_{0st.sys} = \bar{y}_{h.sys}$ … (1)

The variance of $\tilde{Y}_0$ is given by

$Var(\tilde{Y}_{0st.sys}) = MSE(\tilde{Y}_{0st.sys}) = \bar{Y}^2 \sum_{h=1}^{L} \theta_h \rho_{hy}^* C_{hy}^2 = \bar{Y}^2 V_{20}$ … (2)

2. The stratified systematic version of Cochran [6] classical ratio estimator for population mean is:

$$\tilde{Y}_{R.st.sys} = \bar{y}_{h.sys}\left(\frac{\bar{X}}{\bar{x}_{h.sys}}\right) \quad \ldots (3)$$

The MSE of estimator is given by

$$MSE(\tilde{Y}_{R.st.sys}) = \bar{Y}^2 \sum_{h=1}^{L} \theta_h \left[\rho_{hy}^* C_{hy}^2 + \rho_{hx}^* C_{hx}^2 - 2\sqrt{\rho_{hy}^* \rho_{hx}^*} C_{hyx}\right] = \bar{Y}^2[V_{20} + V_{02} - 2V_{11}] \quad .. (4)$$

3. The stratified systematic version of Murthy [15] classical product estimator for population mean is:

$$\tilde{Y}_{P.st.sys} = \bar{y}_{h.sys}\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right) \quad \ldots (5)$$

The MSE of estimator is given by

$$MSE(\tilde{Y}_{P.st.sys}) = \bar{Y}^2 \sum_{h=1}^{L} \theta_h \left[\rho_{hy}^* C_{hy}^2 + \rho_{hx}^* C_{hx}^2 + 2\sqrt{\rho_{hy}^* \rho_{hx}^*} C_{hyx}\right] = \bar{Y}^2[V_{20} + V_{02} + 2V_{11}] \quad .. (6)$$

4. The stratified systematic version of regression estimator for population mean is:

$$\tilde{Y}_{Reg.st.sys} = \bar{y}_{h.sys} + Q(\bar{X} - \bar{x}_{h.sys}) \quad \ldots (7)$$

The MSE of estimator is given by

$$MSE(\tilde{Y}_{Reg.st.sys}) = \frac{\bar{Y}^2(V_{20}V_{02} - V_{11}^2)}{V_{02}} \quad \ldots (8)$$

Equation (8) will become

$$MSE(\tilde{Y}_{Reg.st.sys}) = \bar{Y}^2 V_{20}(1 - \rho_{hyx}^2) \quad \ldots (9)$$

5. The stratified systematic version of Bahl and Tuteja [2] ratio-type and product-type exponential estimators for population mean is:

$$\tilde{Y}_{RE.st.sys} = \bar{y}_{h.sys} \exp\left(\frac{\bar{X} - \bar{x}_{h.sys}}{\bar{X} + \bar{x}_{h.sys}}\right) \quad \ldots (10)$$

$$\tilde{Y}_{PE.st.sys} = \bar{y}_{h.sys} \exp\left(\frac{\bar{x}_{h.sys} - \bar{X}}{\bar{x}_{h.sys} + \bar{X}}\right) \quad \ldots (11)$$

The MSE of estimators $\tilde{Y}_{RE}$ and $\tilde{Y}_{PE}$ are respectively given by

$$MSE(\tilde{Y}_{RE.st.sys}) = \bar{Y}^2 \sum_{h=1}^{L} \theta_h \left[\rho_{hy}^* C_{hy}^2 + \frac{\rho_{hx}^* C_{hx}^2}{4} - \sqrt{\rho_{hy}^* \rho_{hx}^*} C_{hyx}\right] = \bar{Y}^2\left[V_{20} + \frac{V_{02}}{4} - V_{11}\right] \quad .. (12)$$

$$MSE(\tilde{Y}_{PE.st.sys}) = \bar{Y}^2 \sum_{h=1}^{L} \theta_h \left[\rho_{hy}^* C_{hy}^2 + \frac{\rho_{hx}^* C_{hx}^2}{4} + \sqrt{\rho_{hy}^* \rho_{hx}^*} C_{hyx}\right] = \bar{Y}^2\left[V_{20} + \frac{V_{02}}{4} + V_{11}\right] \quad \ldots (13)$$

6. The stratified systematic version of difference type estimator for population mean is:

$$\tilde{Y}_{D.st.sys} = [Q_1 \bar{y}_{h.sys} + Q_2(\bar{X} - \bar{x}_{h.sys})] \quad \ldots (14)$$

Where $Q_1$and Q_2 are unknown constants. The optimal values of $Q_1$and Q_2 are given as

$$Q_1 = \frac{V_{02}}{[V_{20}V_{02} - V_{11}^2 + V_{02}]} \quad ; \quad Q_2 = \sum_{h=1}^{L} \frac{\bar{Y} V_{11}}{\bar{X}[V_{20}V_{02} - V_{11}^2 + V_{02}]}$$

The MSE of estimator is given by

$MSE(\tilde{Y}_{D.st.sys}) = \frac{\bar{Y}^2[V_{20}V_{02}-V_{11}^2]}{[V_{20}V_{02}-V_{11}^2+V_{02}]}$ … (15)

Equation (15) can also be written as

$MSE(\tilde{Y}_{D.st.sys}) = \sum_{h=1}^{L} \frac{\bar{Y}^2\theta_h\rho_{hy}^*C_{hy}^2(1-\rho_{hyx}^2)}{[1+\theta_h\rho_{hy}^*C_{hy}^2(1-\rho_{hyx}^2)]} = \sum_{h=1}^{L} \frac{\bar{Y}^2V_{20}(1-\rho_{hyx}^2)}{[1+V_{20}(1-\rho_{hyx}^2)]}$ … (16)

7. The stratified systematic version of Singh and Vishwakarma [22] ratio-product estimator for population mean is:

$\tilde{Y}_{RP.st.sys} = \bar{y}_{st.sys}\left[Q_3\left(\frac{\bar{X}}{\bar{x}_{h.sys}}\right) + (1-Q_3)\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right]$ … (17)

Where Q_3 is suitably constant

$Q_3 = \frac{1}{2}\left(\frac{V_{11}}{V_{02}} + 1\right)$

The MSE of estimator is given by

$MSE(\tilde{Y}_{RP.st.sys}) = \bar{Y}^2\sum_{h=1}^{L}\theta_h\rho_{hy}^*C_{hy}^2(1-\rho_{hyx}^2) = \bar{Y}^2V_{20}(1-\rho_{hyx}^2)$ … (18)

8. The stratified systematic version of Yadav and Kadilar [31] Exponential ratio-type estimator for population mean is:

$\tilde{Y}_{ER.st.sys} = Q_4\bar{y}_{h.sys}\exp\left(\frac{\bar{X}-\bar{x}_{h.sys}}{\bar{X}+\bar{x}_{h.sys}}\right)$ … (19)

Where k_o is suitably constant

$Q_4 = \sum_{h=1}^{L}\frac{\left[1+\frac{3}{8}V_{02}-\frac{1}{2}V_{11}\right]}{[1+V_{20}+V_{02}-2V_{11}]}$

$MSE(\tilde{Y}_{ER.st.sys}) = \bar{Y}^2\sum_{h=1}^{L}\left[1-\frac{\left(1+\frac{3}{8}\theta_h\rho_{hx}^*C_{hx}^2-\frac{1}{2}\theta_h\sqrt{\rho_{hy}^*\rho_{hx}^*}C_{hyx}\right)^2}{\left(1+\theta_h\left\{\rho_{hy}^*C_{hy}^2+\rho_{hx}^*C_{hx}^2-2\sqrt{\rho_{hy}^*\rho_{hx}^*}C_{hyx}\right\}\right)}\right] = \bar{Y}^2\left[1-\frac{\left(1+\frac{3}{8}V_{02}-\frac{1}{2}V_{11}\right)^2}{(1+V_{20}+V_{02}-2V_{11})}\right]$ …

(20)

9. The stratified systematic version of Singh, R. et al. [24] exponential estimator for population mean is:

$\tilde{Y}_{E.st.sys} = \bar{y}_{h.sys}\exp\left[\frac{\alpha(\bar{X}-\bar{x}_{h.sys})}{\alpha(\bar{X}+\bar{x}_{h.sys})+2\beta}\right]$ … (21)

The MSE of estimator is given by

$MSE(\tilde{Y}_{E.st.sys}) = \frac{\bar{Y}^2}{4}\sum_{h=1}^{L}\theta_h\left[4\rho_{hy}^*C_{hy}^2 + \delta^2\rho_{hx}^*C_{hx}^2 - 4\delta\sqrt{\rho_{hy}^*\rho_{hx}^*}C_{hyx}\right] = \frac{\bar{Y}^2}{4}[4V_{20} + \delta^2V_{02} - 4V_{11}]$ … (22)

Where $\delta = \frac{\alpha\bar{X}}{\alpha\bar{X}+\beta}$

10. The stratified systematic version of Grover and Kaur [10] a generalized class of ratio type exponential estimators for population mean is:

$\tilde{Y}_{G.st.sys} = \{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X}-\bar{x}_{h.sys})\}\exp\left[\frac{\alpha(\bar{X}-\bar{x}_{h.sys})}{\alpha(\bar{X}+\bar{x}_{h.sys})+2\beta}\right]$ … (23)

Where $Q_4$and Q_5 are unknown constants.

The optimal values of $Q_1$and Q_2 are given as

$Q_5 = \sum_{h=1}^{L} \frac{V_{02}[\delta^2 V_{02} - 8]}{8[V_{11}^2 - V_{20}V_{02} - V_{02}]}$; $Q_6 = \frac{\bar{Y}[\delta^3 V_{02}^2 - \delta^2 V_{02}V_{11} + 4\delta V_{20}V_{02} - 4\delta^2 V_{11}^2 - 4\delta V_{02} + 8V_{11}]}{8\bar{X}[V_{20}V_{02} - V_{11}^2 + V_{02}]}$

The MSE of estimator is given by

$$MSE(\tilde{Y}_{G.st.sys}) = \sum_{h=1}^{L} \frac{\bar{Y}^2}{64}\left[64 - 16\delta^2\theta_h\rho_{hx}^* C_{hx}^2 - \frac{(\delta^2\theta_h\rho_{hx}^* C_{hx}^2 - 8)^2}{1 + \theta_h\rho_{hy}^* C_{hy}^2(1 - \rho_{hyx}^2)}\right] = \frac{\bar{Y}^2}{64}\left[64 - 16\delta^2 V_{02} - \frac{V_{02}(\delta^2 V_{02} - 8)^2}{V_{02}(1 + V_{20}) - V_{11}^2}\right]$$

… (24)

Table 1. Some members of our proposed and existing estimator

α	β	$\tilde{Y}_{E.st.sys}$	$\tilde{Y}_{G.st.sys}$	$\tilde{Y}_{AB.st.sys}$
1	0	$\bar{y}_{h.sys} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys})}\right]$	$\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X} - \bar{x}_{h.sys})\} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys})}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys})}\right]$
1	1	$\bar{y}_{h.sys} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2}\right]$	$\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X} - \bar{x}_{h.sys})\} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2}\right]$
1	C_x	$\bar{y}_{h.sys} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2C_x}\right]$	$\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X} - \bar{x}_{h.sys})\} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2C_x}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2C_x}\right]$
1	ρ_{yx}	$\bar{y}_{h.sys} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2\rho_{yx}}\right]$	$\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X} - \bar{x}_{h.sys})\} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2\rho_{yx}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2\rho_{yx}}\right]$
C_x	ρ_{yx}	$\bar{y}_{h.sys} \exp\left[\frac{C_x(\bar{X} - \bar{x}_{h.sys})}{C_x(\bar{X} + \bar{x}_{h.sys}) + 2\rho_{yx}}\right]$	$\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X} - \bar{x}_{h.sys})\} \exp\left[\frac{C_x(\bar{X} - \bar{x}_{h.sys})}{C_x(\bar{X} + \bar{x}_{h.sys}) + 2\rho_{yx}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{C_x(\bar{X} - \bar{x}_{h.sys})}{C_x(\bar{X} + \bar{x}_{h.sys}) + 2\rho_{yx}}\right]$
ρ_{yx}	C_x	$\bar{y}_{h.sys} \exp\left[\frac{\rho_{yx}(\bar{X} - \bar{x}_{h.sys})}{\rho_{yx}(\bar{X} + \bar{x}_{h.sys}) + 2C_x}\right]$	$\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X} - \bar{x}_{h.sys})\} \exp\left[\frac{\rho_{yx}(\bar{X} - \bar{x}_{h.sys})}{\rho_{yx}(\bar{X} + \bar{x}_{h.sys}) + 2C_x}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{\rho_{yx}(\bar{X} - \bar{x}_{h.sys})}{\rho_{yx}(\bar{X} + \bar{x}_{h.sys}) + 2C_x}\right]$
1	β_{1x}	$\bar{y}_{h.sys} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2\beta_{1x}}\right]$	$\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X} - \bar{x}_{h.sys})\} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2\beta_{1x}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{(\bar{X} - \bar{x}_{h.sys})}{(\bar{X} + \bar{x}_{h.sys}) + 2\beta_{1x}}\right]$
β_{1x}	C_x	$\bar{y}_{h.sys} \exp\left[\frac{\beta_{1x}(\bar{X} - \bar{x}_{h.sys})}{\beta_{1x}(\bar{X} + \bar{x}_{h.sys}) + 2C_x}\right]$	$\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X} - \bar{x}_{h.sys})\} \exp\left[\frac{\beta_{1x}(\bar{X} - \bar{x}_{h.sys})}{\beta_{1x}(\bar{X} + \bar{x}_{h.sys}) + 2C_x}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{\beta_{1x}(\bar{X} - \bar{x}_{h.sys})}{\beta_{1x}(\bar{X} + \bar{x}_{h.sys}) + 2C_x}\right]$

C_x	β_{1x}	$\bar{y}_{h.sys} \exp\left[\frac{C_x(\bar{X}-\bar{x}_{h.sys})}{C_x(\bar{X}+\bar{x}_{h.sys})+2\beta_{1x}}\right]$	$\left\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X}-\bar{x}_{h.sys})\right\} \exp\left[\frac{C_x(\bar{X}-\bar{x}_{h.sys})}{C_x(\bar{X}+\bar{x}_{h.sys})+2\beta_{1x}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{C_x(\bar{X}-\bar{x}_{h.sys})}{C_x(\bar{X}+\bar{x}_{h.sys})+2\beta_{1x}}\right]$
β_{1x}	ρ_{yx}	$\bar{y}_{h.sys} \exp\left[\frac{\beta_{1x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{1x}(\bar{X}+\bar{x}_{h.sys})+2\rho_{yx}}\right]$	$\left\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X}-\bar{x}_{h.sys})\right\} \exp\left[\frac{\beta_{1x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{1x}(\bar{X}+\bar{x}_{h.sys})+2\rho_{yx}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{\beta_{1x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{1x}(\bar{X}+\bar{x}_{h.sys})+2\rho_{yx}}\right]$
ρ_{yx}	β_{1x}	$\bar{y}_{h.sys} \exp\left[\frac{\rho_{yx}(\bar{X}-\bar{x}_{h.sys})}{\rho_{yx}(\bar{X}+\bar{x}_{h.sys})+2\beta_{1x}}\right]$	$\left\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X}-\bar{x}_{h.sys})\right\} \exp\left[\frac{\rho_{yx}(\bar{X}-\bar{x}_{h.sys})}{\rho_{yx}(\bar{X}+\bar{x}_{h.sys})+2\beta_{1x}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{\rho_{yx}(\bar{X}-\bar{x}_{h.sys})}{\rho_{yx}(\bar{X}+\bar{x}_{h.sys})+2\beta_{1x}}\right]$
1	β_{2x}	$\bar{y}_{h.sys} \exp\left[\frac{(\bar{X}-\bar{x}_{h.sys})}{(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$	$\left\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X}-\bar{x}_{h.sys})\right\} \exp\left[\frac{(\bar{X}-\bar{x}_{h.sys})}{(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{(\bar{X}-\bar{x}_{h.sys})}{(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$
β_{2x}	C_x	$\bar{y}_{h.sys} \exp\left[\frac{\beta_{2x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{2x}(\bar{X}+\bar{x}_{h.sys})+2C_x}\right]$	$\left\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X}-\bar{x}_{h.sys})\right\} \exp\left[\frac{\beta_{2x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{2x}(\bar{X}+\bar{x}_{h.sys})+2C_x}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{\beta_{2x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{2x}(\bar{X}+\bar{x}_{h.sys})+2C_x}\right]$
C_x	β_{2x}	$\bar{y}_{h.sys} \exp\left[\frac{C_x(\bar{X}-\bar{x}_{h.sys})}{C_x(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$	$\left\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X}-\bar{x}_{h.sys})\right\} \exp\left[\frac{C_x(\bar{X}-\bar{x}_{h.sys})}{C_x(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{C_x(\bar{X}-\bar{x}_{h.sys})}{C_x(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$
β_{2x}	ρ_{yx}	$\bar{y}_{h.sys} \exp\left[\frac{\beta_{2x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{2x}(\bar{X}+\bar{x}_{h.sys})+2\rho_{yx}}\right]$	$\left\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X}-\bar{x}_{h.sys})\right\} \exp\left[\frac{\beta_{2x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{2x}(\bar{X}+\bar{x}_{h.sys})+2\rho_{yx}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{\beta_{2x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{2x}(\bar{X}+\bar{x}_{h.sys})+2\rho_{yx}}\right]$
ρ_{yx}	β_{2x}	$\bar{y}_{h.sys} \exp\left[\frac{\rho_{yx}(\bar{X}-\bar{x}_{h.sys})}{\rho_{yx}(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$	$\left\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X}-\bar{x}_{h.sys})\right\} \exp\left[\frac{\rho_{yx}(\bar{X}-\bar{x}_{h.sys})}{\rho_{yx}(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{\rho_{yx}(\bar{X}-\bar{x}_{h.sys})}{\rho_{yx}(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$
β_{2x}	β_{1x}	$\bar{y}_{h.sys} \exp\left[\frac{\beta_{2x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{2x}(\bar{X}+\bar{x}_{h.sys})+2\beta_{1x}}\right]$	$\left\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X}-\bar{x}_{h.sys})\right\} \exp\left[\frac{\beta_{2x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{2x}(\bar{X}+\bar{x}_{h.sys})+2\beta_{1x}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{\beta_{2x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{2x}(\bar{X}+\bar{x}_{h.sys})+2\beta_{1x}}\right]$
β_{1x}	β_{2x}	$\bar{y}_{h.sys} \exp\left[\frac{\beta_{1x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{1x}(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$	$\left\{Q_5\,\bar{y}_{h.sys} + Q_6(\bar{X}-\bar{x}_{h.sys})\right\} \exp\left[\frac{\beta_{1x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{1x}(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$	$\left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{\beta_{1x}(\bar{X}-\bar{x}_{h.sys})}{\beta_{1x}(\bar{X}+\bar{x}_{h.sys})+2\beta_{2x}}\right]$

4. The proposed estimator

Motivated by Koyuncu [13] we propose the following estimator in stratified systematic sampling.

$$\tilde{Y}_{AB.st.sys} = \left\{Q_7\,\bar{y}_{h.sys} + Q_8\left(\frac{\bar{x}_{h.sys}}{\bar{X}}\right)\right\} \exp\left[\frac{\alpha(\bar{X}-\bar{x}_{h.sys})}{\alpha(\bar{X}+\bar{x}_{h.sys})+2\beta}\right] \quad \ldots (25)$$

Here Q_7 and Q_8 are appropriate constants that should be selected to minimize the mean square error (MSE) of the estimator $\tilde{Y}_{AB.st.sys}$. It can be observed that the proposed estimator $\tilde{Y}_{AB.st.sys}$ encompasses all the aforementioned estimators for various values of α and β.

In this context, Q_i where $(i = 7{,}8)$ represents the constants for reducing the bias within this class of estimators, and α and β have been previously explained.

In Table 1, we present specific members of both the proposed and existing classes of estimators that utilize different combination of α and β.

By expressing equation (25) in terms of $e's$, we obtain

$$\tilde{Y}_{AB.st.sys} = \left\{Q_7\bar{Y}\left(1+\epsilon_y\right) + Q_8\left(\frac{\bar{X}(1+\epsilon_x)}{\bar{X}}\right)\right\}\exp\left[\frac{\alpha\left(\bar{X}-\bar{X}(1+\epsilon_x)\right)}{\alpha\left(\bar{X}+\bar{X}(1+\epsilon_x)\right)+2\beta}\right]$$

$$\tilde{Y}_{AB.st.sys} = \left\{Q_7\bar{Y}\left(1+\epsilon_y\right) + Q_8(1+\epsilon_x)\right\}\exp\left[\frac{-\alpha\bar{X}\epsilon_x}{2(\alpha\bar{X}+\beta)+\alpha\bar{X}\epsilon_x}\right]$$

$$\delta = \frac{\alpha\bar{X}}{\alpha\bar{X}+\beta}$$

$$\tilde{Y}_{AB.st.sys} = \left\{Q_7\bar{Y}\left(1+\epsilon_y\right) + Q_8(1+\epsilon_x)\right\}\exp\left[\frac{-\delta\epsilon_x}{2}\left(1+\frac{\delta\epsilon_x}{2}\right)^{-1}\right]$$

$$\tilde{Y}_{AB.st.sys} = \left\{Q_7\bar{Y}\left(1+\epsilon_y\right) + Q_8(1+\epsilon_x)\right\}\left(1-\frac{1}{2}\,\delta\in_x+\frac{3}{8}\,\delta^2\in_x^2\right)$$

$$\tilde{Y}_{AB.st.sys} = \left\{Q_7\bar{Y}\left(1+\epsilon_y-\frac{\delta\epsilon_x}{2}+\frac{3}{8}\,\delta^2\in_x^2-\frac{\delta}{2}\epsilon_y\epsilon_x\right) + Q_8\left(1+\epsilon_x-\frac{\delta\epsilon_x}{2}+\frac{3}{8}\,\delta^2\in_x^2-\frac{\delta\in_x^2}{2}\right)\right\} \quad \ldots (26)$$

Subtracting $\bar{Y}$ from both the sides of equation (26)

$$\left(\tilde{Y}_{AB.st.sys}-\bar{Y}\right) = \bar{Y}(Q_7-1) + Q_7\bar{Y}\left(\epsilon_y-\frac{\delta\epsilon_x}{2}+\frac{3}{8}\,\delta^2\in_x^2-\frac{\delta}{2}\epsilon_y\epsilon_x\right) + Q_8\left(1+\epsilon_x-\frac{\delta\epsilon_x}{2}+\frac{3}{8}\,\delta^2\in_x^2-\frac{\delta\in_x^2}{2}\right) \quad \ldots (27)$$

By taking the expectation of both sides in equation (27), we derive the bias of the estimator $\tilde{Y}_{AB.st.sys}$, up to the first order of approximation as follows:

$$B\left(\tilde{Y}_{AB.st.sys}\right) = \bar{Y}(Q_7-1) + Q_7\frac{\bar{Y}\delta}{2}\left(\frac{3}{4}\delta V_{02}-V_{11}\right) + Q_8\left(1+\frac{\delta V_{02}}{2}\left(\frac{3\delta}{4}-1\right)\right) \quad \ldots (28)$$

Now, by squaring both sides of equation (27) and taking the expectation, we obtain the MSE of the estimator to the first order of approximation as follow:

$$MSE\left(\tilde{Y}_{AB.st.sys}\right) = \left[\bar{Y}^2 + Q_7^2\bar{Y}^2(1+V_{20}+\delta^2V_{02}-2\delta V_{11}) + Q_8^2(1+\{\delta-1\}^2V_{02}) - 2Q_7\bar{Y}^2\left\{1+\frac{3\delta^2}{8}V_{02}-\frac{\delta}{2}V_{11}\right\} - 2Q_8\bar{Y}\left\{1+\frac{\delta}{2}\left(\frac{3\delta}{4}-1\right)V_{02}\right\} + 2Q_7Q_8\bar{Y}\{1+(\delta-1)(\delta V_{02}-V_{11})\}\right] \quad \ldots (29)$$

Optimality condition for the proposed class of estimator:

To explore the optimal conditions for this proposed class of estimator, let us consider:

$\frac{\partial MSE(\tilde{Y}_{AB.st.sys})}{\partial Q_7} = 0$; $\frac{\partial MSE(\tilde{Y}_{AB.st.sys})}{\partial Q_8} = 0$

So that

$$Q_{7(opt)} = \frac{[V_{02}-V_{11}][8+\delta^2(\delta-1)V_{02}-4\delta]}{8[V_{20}+V_{02}-2V_{11}+(\delta-1)^2(V_{20}V_{02}-V_{11}^2)]} \quad \ldots (30)$$

$$Q_{8(opt)} = \frac{\bar{Y}\left[V_{20}+\frac{\delta V_{20}V_{02}}{2}\left(\frac{3\delta}{4}-1\right)-\left(\frac{\delta}{2}+1\right)V_{11}-\frac{\delta}{2}(\delta-1)V_{11}^2-\frac{\delta V_{02}}{2}\left(\frac{\delta^2 V_{02}}{4}-1\right)+\frac{\delta^2 V_{02}V_{11}}{8}(\delta+1)\right]}{[V_{20}+V_{02}-2V_{11}+(\delta-1)^2(V_{20}V_{02}-V_{11}^2)]} \quad \ldots (31)$$

Substituting the value of $Q_{7(opt)}$ and $Q_{8(opt)}$ into equation (29) yields the optimal MSE of the estimator as follows:

$$MSE(\tilde{Y}_{AB.st.sys})_{opt} = \frac{\bar{Y}^2\left[\left(\frac{\delta}{2}-1\right)^2(V_{20}V_{02}-V_{11}^2)+\frac{\delta^2 V_{02}\left\{V_{11}^2\left(\frac{\delta^2}{2}+1-\frac{3\delta}{2}\right)-V_{20}V_{02}\left(\frac{3\delta}{4}-1\right)^2\right\}}{4}-\frac{\delta^4 V_{02}^2}{32}\left(\frac{\delta^2 V_{02}}{2}+V_{11}\right)\right]}{[V_{20}+V_{02}-2V_{11}+(\delta-1)^2(V_{20}V_{02}-V_{11}^2)]} \quad \ldots (32)$$

5. *Empirical study*

5.1 Real population

To assess the theoretical findings and empirically evaluate the efficiency and optimality of the proposed exponential-type estimator $(\tilde{Y}_{AB.st.sys})$ in comparison to other estimators. In this study, we have used three different real populations that we have collected from various sources, along with a simulation study.

Population 1: The first population is sourced from Model Assisted Survey Sampling by "Carl-Erik Sarndal, Bengt Swensson, Jan Wretman" [Appendix-B "The MU284 Population"]

This dataset includes 125 municipalities, where the total population in (1985) is used as study variable y and the number of Conservative seats in municipal council serves as auxiliary variables x.

Now the entire population of 125 municipalities are divided into 5 strata based on geographic region indicator, we have,

Strata	Geographic region indicator	Total region
1.	1	25
2.	2	25
3.	3	25
4.	4	25
5.	5	25

Table 2. Present data statistics for population (1)

Stratum	N_h	n_h	$\bar{Y}_h$	$\bar{X}_h$	$C_{x(h)}$	$C_{y(h)}$	$\rho_{x(h)}$	$\rho_{y(h)}$	$\rho_{yx(h)}$	$\beta_{1x(h)}$	$\beta_{2x(h)}$
1	25	5	62.44	16	0.1908	0.7881	0.856	0.034	0.677	1.141	4.270
2	25	5	34.72	9.72	0.1701	0.2586	0.898	0.846	0.900	1.059	3.307
3	25	5	21.04	9	0.0836	0.0916	0.815	0.442	0.856	0.953	4.004
4	25	5	34.32	12.4	0.1511	0.2842	0.902	0.3	0.514	0.561	2.694
5	25	5	45.88	11.32	0.1300	0.5173	0.896	0.254	0.671	0.798	3.092
			$\bar{Y} = 39.68$	$\bar{X} = 11.68$	$C_x = 0.1451$				$\rho_{yx} = 0.724$	$\beta_{1x} = 0.902$	$\beta_{2x} = 3.473$

Population 2: The second population is sourced from Theory and Analysis of Sample Survey DESIGNS - Second Edition by “Daroga Singh, F S Chaudhary” [chapter-4]

This dataset comprises 70 villages in India, along with their 1981 population and cultivated area (in acres). In this study, the cultivated area (in acres) from (1985) is used as study variable y, while the population of the village serves as the auxiliary variables x.

Now the entire population of 70 villages is divided into two equal strata.

Table 3. Present data statistics for population (2)

Stratum	N_h	n_h	$\bar{Y}_h$	$\bar{X}_h$	$C_{x(h)}$	$C_{y(h)}$	$\rho_{x(h)}$	$\rho_{y(h)}$	$\rho_{yx(h)}$	$\beta_{1x(h)}$	$\beta_{2x(h)}$
1	35	7	1355.91	2238.8	0.9186	0.6410	0.717	0.911	0.832	1.504	5.420
2	35	7	609.51	1272.25	0.5496	0.2842	0.352	0.863	0.526	2.371	8.404
			$\bar{Y} = 982.71$	$\bar{X} = 1755.51$	$C_x = 0.7341$				$\rho_{yx} = 0.679$	$\beta_{1x} = 1.937$	$\beta_{2x} = 6.912$

Population 3: The third population is sourced from Multivariate Statistical Methods- A Primer 4rd Edition by “Bryan F. J. Manly, Jorge A. Navarro Alberto” [Chapter-1]

The considered data relates to total 50 body measurements of female sparrows. we consider the total length as study variable y and length of beak and head as auxiliary variables x.

Now the whole population of 50 body measurements of female sparrows is divided into two equal strata. we have

Table 4. Present data statistics for population (3)

Stratum	N_h	n_h	$\bar{Y}_h$	$\bar{X}_h$	$C_{x(h)}$	$C_{y(h)}$	$\rho_{x(h)}$	$\rho_{y(h)}$	$\rho_{yx(h)}$	$\beta_{1x(h)}$	$\beta_{2x(h)}$
1	25	5	157.12	31.43	0.0236	0.0209	0.928	0.355	0.682	0.291	2.194
2	25	5	159.04	31.44	0.0273	0.0243	0.899	0.223	0.595	0.477	2.564
			$\bar{Y} = 158.08$	$\bar{X} = 31.44$	$C_x = 0.0255$				$\rho_{yx} = 0.639$	$\beta_{1x} = 0.384$	$\beta_{2x} = 2.379$

Table 5. The PREs of various estimators with respect to $\tilde{Y}_{0.st.sys}$, are calculated for different combination of α and β using population (1).

Estimators	PRE	Estimator	PRE	Estimator	PRE	Estimator	PRE
$\tilde{Y}_{0.st.sys}$	100.0	$\tilde{Y}_{E.st.sys}^{(1,0)}$	127.19	$\tilde{Y}_{G.st.sys}^{(1,0)}$	871.28	$\tilde{Y}_{AB.st.sys}^{(1,0)}$	3758.62
$\tilde{Y}_{R.st.sys}$	156.56	$\tilde{Y}_{E.st.sys}^{(1,1)}$	124.90	$\tilde{Y}_{G.st.sys}^{(1,1)}$	843.15	$\tilde{Y}_{AB.st.sys}^{(1,1)}$	3248.26
$\tilde{Y}_{P.st.sys}$	61.47	$\tilde{Y}_{E.st.sys}^{(1,C_x)}$	124.91	$\tilde{Y}_{G.st.sys}^{(1,C_x)}$	843.28	$\tilde{Y}_{AB.st.sys}^{(1,C_x)}$	3250.60
$\tilde{Y}_{Reg.st.sys}$	185.08	$\tilde{Y}_{E.st.sys}^{(1,\rho_{yx})}$	125.49	$\tilde{Y}_{G.st.sys}^{(1,\rho_{yx})}$	850.05	$\tilde{Y}_{AB.st.sys}^{(1,\rho_{yx})}$	3368.46
$\tilde{Y}_{RE.st.sys}$	127.19	$\tilde{Y}_{E.st.sys}^{(C_x,\rho_{yx})}$	123.22	$\tilde{Y}_{G.st.sys}^{(C_x,\rho_{yx})}$	824.97	$\tilde{Y}_{AB.St.sys}^{(C_x,\rho_{yx})}$	2946.07
$\tilde{Y}_{Pe.st.sys}$	78.11	$\tilde{Y}_{E.st.sys}^{(\rho_{yx},C_x)}$	125.91	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},C_x)}$	855.02	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},C_x)}$	3456.95
$\tilde{Y}_{D.st.sys}$	715.58	$\tilde{Y}_{E.st.sys}^{(1,\beta_{1x})}$	125.10	$\tilde{Y}_{G.st.sys}^{(1,\beta_{1x})}$	845.52	$\tilde{Y}_{AB.st.sys}^{(1,\beta_{1x})}$	3289.16
$\tilde{Y}_{RP.st.sys}$	185.08	$\tilde{Y}_{E.st.sys}^{(\beta_{1x},C_x)}$	126.15	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},C_x)}$	858.01	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},C_x)}$	3511.12
$\tilde{Y}_{ER.st.sys}$	579.44	$\tilde{Y}_{E.st.sys}^{(C_x,\beta_{1x})}$	122.41	$\tilde{Y}_{G.st.sys}^{(C_x,\beta_{1x})}$	816.86	$\tilde{Y}_{AB.st.sys}^{(C_x,\beta_{1x})}$	2817.80
		$\tilde{Y}_{E.st.sys}^{(\beta_{1x},\rho_{yx})}$	125.32	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},\rho_{yx})}$	848.04	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},\rho_{yx})}$	3333.11
		$\tilde{Y}_{E.st.sys}^{(\rho_{yx},\beta_{1x})}$	124.39	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},\beta_{1x})}$	837.44	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},\beta_{1x})}$	3151.2
		$\tilde{Y}_{E.st.sys}^{(1,\beta_{2x})}$	120.58	$\tilde{Y}_{G.st.sys}^{(1,\beta_{2x})}$	800.02	$\tilde{Y}_{AB.st.sys}^{(1,\beta_{2x})}$	2563.41
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},C_x)}$	126.91	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},C_x)}$	867.65	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},C_x)}$	3689.64
		$\tilde{Y}_{E.st.sys}^{(C_x,\beta_{2x})}$	114.88	$\tilde{Y}_{G.st.sys}^{(C_x,\beta_{2x})}$	758.86	$\tilde{Y}_{AB.st.sys}^{(C_x,\beta_{2x})}$	2004.35
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},\rho_{yx})}$	126.68	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},\rho_{yx})}$	864.65	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},\rho_{yx})}$	3633.40

		$\tilde{Y}_{E.st.sys}^{(\rho_{yx},\beta_{2x})}$	118.82	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},\beta_{2x})}$	785.63	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},\beta_{2x})}$	2358.43
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},\beta_{1x})}$	126.47	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},\beta_{1x})}$	861.93	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},\beta_{1x})}$	3583.00
		$\tilde{Y}_{E.st.sys}^{(\beta_{1x},\beta_{2x})}$	98.46	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},\beta_{2x})}$	716.07	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},\beta_{2x})}$	1401.76

Table 6. The PREs of various estimators with respect to $\tilde{Y}_{0.st.sys}$ are calculated for different combination of α and β using population (2).

Estimators	PRE	Estimator	PRE	Estimator	PRE	Estimator	PRE
$\tilde{Y}_{0.st.sys}$	100.0	$\tilde{Y}_{E.st.sys}^{(1,0)}$	247.04	$\tilde{Y}_{G.st.sys}^{(1,0)}$	396.02	$\tilde{Y}_{AB.st.sys}^{(1,0)}$	435.44
$\tilde{Y}_{R.st.sys}$	149.88	$\tilde{Y}_{E.st.sys}^{(1,1)}$	246.98	$\tilde{Y}_{G.st.sys}^{(1,1)}$	395.84	$\tilde{Y}_{AB.st.sys}^{(1,1)}$	434.92
$\tilde{Y}_{P.st.sys}$	20.19	$\tilde{Y}_{E.st.sys}^{(1,C_x)}$	246.99	$\tilde{Y}_{G.st.sys}^{(1,C_x)}$	395.89	$\tilde{Y}_{AB.st.sys}^{(1,C_x)}$	435.06
$\tilde{Y}_{Reg.st.sys}$	257.30	$\tilde{Y}_{E.st.sys}^{(1,\rho_{yx})}$	247.00	$\tilde{Y}_{G.st.sys}^{(1,\rho_{yx})}$	395.90	$\tilde{Y}_{AB.st.sys}^{(1,\rho_{yx})}$	435.09
$\tilde{Y}_{RE.st.sys}$	247.04	$\tilde{Y}_{E.st.sys}^{(C_x,\rho_{yx})}$	246.98	$\tilde{Y}_{G.st.sys}^{(C_x,\rho_{yx})}$	395.86	$\tilde{Y}_{AB.St.sys}^{(C_x,\rho_{yx})}$	434.96
$\tilde{Y}_{Pe.st.sys}$	40.76	$\tilde{Y}_{E.st.sys}^{(\rho_{yx},C_x)}$	246.97	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},C_x)}$	395.83	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},C_x)}$	434.88
$\tilde{Y}_{D.st.sys}$	301.11	$\tilde{Y}_{E.st.sys}^{(1,\beta_{1x})}$	246.93	$\tilde{Y}_{G.st.sys}^{(1,\beta_{1x})}$	395.68	$\tilde{Y}_{AB.st.sys}^{(1,\beta_{1x})}$	434.43
$\tilde{Y}_{RP.st.sys}$	257.30	$\tilde{Y}_{E.st.sys}^{(\beta_{1x},C_x)}$	247.02	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},C_x)}$	395.95	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},C_x)}$	435.25
$\tilde{Y}_{ER.st.sys}$	325.30	$\tilde{Y}_{E.st.sys}^{(C_x,\beta_{1x})}$	246.88	$\tilde{Y}_{G.st.sys}^{(C_x,\beta_{1x})}$	395.56	$\tilde{Y}_{AB.st.sys}^{(C_x,\beta_{1x})}$	434.06
		$\tilde{Y}_{E.st.sys}^{(\beta_{1x},\rho_{yx})}$	247.02	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},\rho_{yx})}$	395.96	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},\rho_{yx})}$	435.26
		$\tilde{Y}_{E.st.sys}^{(\rho_{yx},\beta_{1x})}$	246.87	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},\beta_{1x})}$	395.53	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},\beta_{1x})}$	433.95
		$\tilde{Y}_{E.st.sys}^{(1,\beta_{2x})}$	246.64	$\tilde{Y}_{G.st.sys}^{(1,\beta_{2x})}$	394.83	$\tilde{Y}_{AB.st.sys}^{(1,\beta_{2x})}$	431.85
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},C_x)}$	247.03	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},C_x)}$	396.00	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},C_x)}$	435.39
		$\tilde{Y}_{E.st.sys}^{(C_x,\beta_{2x})}$	246.49	$\tilde{Y}_{G.st.sys}^{(C_x,\beta_{2x})}$	394.41	$\tilde{Y}_{AB.st.sys}^{(C_x,\beta_{2x})}$	430.57
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},\rho_{yx})}$	247.03	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},\rho_{yx})}$	396.00	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},\rho_{yx})}$	435.39
		$\tilde{Y}_{E.st.sys}^{(\rho_{yx},\beta_{2x})}$	246.45	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},\beta_{2x})}$	394.28	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},\beta_{2x})}$	430.19
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},\beta_{1x})}$	247.02	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},\beta_{1x})}$	395.97	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},\beta_{1x})}$	435.30

		$\tilde{Y}_{E.st.sys}^{(\beta_{1x},\beta_{2x})}$	246.03	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},\beta_{2x})}$	395.40	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},\beta_{2x})}$	433.58

Table 7. The PREs of various estimators with respect to $\tilde{Y}_{0.st.sys}$ are calculated for different combination of α and β using population (3).

Estimators	PRE	Estimator	PRE	Estimator	PRE	Estimator	PRE
$\tilde{Y}_{0.st.sys}$	100.0	$\tilde{Y}_{E.st.sys}^{(1,0)}$	157.06	$\tilde{Y}_{G.st.sys}^{(1,0)}$	167.58	$\tilde{Y}_{AB.st.sys}^{(1,0)}$	400.90
$\tilde{Y}_{R.st.sys}$	60.11	$\tilde{Y}_{E.st.sys}^{(1,1)}$	159.46	$\tilde{Y}_{G.st.sys}^{(1,1)}$	167.58	$\tilde{Y}_{AB.st.sys}^{(1,1)}$	377.28
$\tilde{Y}_{P.st.sys}$	16.95	$\tilde{Y}_{E.st.sys}^{(1,C_x)}$	157.13	$\tilde{Y}_{G.st.sys}^{(1,C_x)}$	167.58	$\tilde{Y}_{AB.st.sys}^{(1,C_x)}$	400.25
$\tilde{Y}_{Reg.st.sys}$	167.48	$\tilde{Y}_{E.st.sys}^{(1,\rho_{yx})}$	158.64	$\tilde{Y}_{G.st.sys}^{(1,\rho_{yx})}$	167.58	$\tilde{Y}_{AB.st.sys}^{(1,\rho_{yx})}$	385.39
$\tilde{Y}_{RE.st.sys}$	157.06	$\tilde{Y}_{E.st.sys}^{(C_x,\rho_{yx})}$	159.68	$\tilde{Y}_{G.st.sys}^{(C_x,\rho_{yx})}$	167.54	$\tilde{Y}_{AB.St.sys}^{(C_x,\rho_{yx})}$	192.39
$\tilde{Y}_{Pe.st.sys}$	36.31	$\tilde{Y}_{E.st.sys}^{(\rho_{yx},C_x)}$	157.17	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},C_x)}$	167.58	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},C_x)}$	399.88
$\tilde{Y}_{D.st.sys}$	167.52	$\tilde{Y}_{E.st.sys}^{(1,\beta_{1x})}$	158.03	$\tilde{Y}_{G.st.sys}^{(1,\beta_{1x})}$	167.58	$\tilde{Y}_{AB.st.sys}^{(1,\beta_{1x})}$	391.39
$\tilde{Y}_{RP.st.sys}$	167.48	$\tilde{Y}_{E.st.sys}^{(\beta_{1x},C_x)}$	157.24	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},C_x)}$	167.58	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},C_x)}$	399.21
$\tilde{Y}_{ER.st.sys}$	157.20	$\tilde{Y}_{E.st.sys}^{(C_x,\beta_{1x})}$	166.07	$\tilde{Y}_{G.st.sys}^{(C_x,\beta_{1x})}$	167.55	$\tilde{Y}_{AB.st.sys}^{(C_x,\beta_{1x})}$	228.72
		$\tilde{Y}_{E.st.sys}^{(\beta_{1x},\rho_{yx})}$	160.83	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},\rho_{yx})}$	167.58	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},\rho_{yx})}$	363.44
		$\tilde{Y}_{E.st.sys}^{(\rho_{yx},\beta_{1x})}$	158.55	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},\beta_{1x})}$	167.58	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},\beta_{1x})}$	386.27
		$\tilde{Y}_{E.st.sys}^{(1,\beta_{2x})}$	162.13	$\tilde{Y}_{G.st.sys}^{(1,\beta_{2x})}$	167.57	$\tilde{Y}_{AB.st.sys}^{(1,\beta_{2x})}$	349.92
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},C_x)}$	157.09	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},C_x)}$	167.58	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},C_x)}$	400.63
		$\tilde{Y}_{E.st.sys}^{(C_x,\beta_{2x})}$	128.64	$\tilde{Y}_{G.st.sys}^{(C_x,\beta_{2x})}$	167.52	$\tilde{Y}_{AB.st.sys}^{(C_x,\beta_{2x})}$	191.22
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},\rho_{yx})}$	157.75	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},\rho_{yx})}$	167.58	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},\rho_{yx})}$	394.19
		$\tilde{Y}_{E.st.sys}^{(\rho_{yx},\beta_{2x})}$	164.11	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},\beta_{2x})}$	167.57	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},\beta_{2x})}$	327.80
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},\beta_{1x})}$	157.48	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},\beta_{1x})}$	167.58	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},\beta_{1x})}$	396.83
		$\tilde{Y}_{E.st.sys}^{(\beta_{1x},\beta_{2x})}$	166.42	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},\beta_{2x})}$	167.56	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},\beta_{2x})}$	295.59

5.2 Simulation

To extend the findings of the numerical study, we conducted a simulation study using a hypothetically generated normal population. The previous section clearly demonstrated the superior effectiveness of the proposed estimator compared to competing estimators. This simulation study aims to assess the efficiency of the proposed estimator over existing ones for the stratified systematic sampling technique, utilizing auxiliary variable X. An artificial dataset was generated following the procedure outlined by Tracy et al. [30], as shown in Table 1 (Appendix 1).

A bivariate normal population was artificially generated using R software, with a population size of 1,800, divided into three strata of equal sizes: $N_1 = 600$, $N_2 = 600$ and $N_3 = 600$ respectively. The sample sizes $n_1 = 100$, $n_2 = 100$ and $n_3 = 100$ were drawn from these strata using systematic sampling, following the proportional allocation method. The correlation coefficients between the study and auxiliary variables for each stratum were set as ρ_{yx1}= 0.8, ρ_{yx2}= 0.7, ρ_{yx3}= 0.9, respectively. The standard deviations were fixed at S_{x1} = 4.7, S_{x2}= 6.2, S_{x3}= 8.4 and S_{y1}, S_{y2}, S_{y3}= 4.8 for each stratum. This process was repeated 10,000 times independently, generating 10,000 samples of size 100,100,100 units have been drawn from each stratum from given a population.

To conduct the simulation study, the procedure using R-Language software is outlined as follows:

1. Sample selection- A bivariate stratified systematic random sample of size n_i from i^{th} stratum, with $\sum n_i = n$ is selected from the above artificially generated population. the sample sizes $n_i's$ are chosen by proportional allocation method.

2. Estimation - The MSE is calculated for the repeated samples across different estimators, with the entire procedure is repeated 10,000 times and obtain 10,000 values i.e $\bar{y}$ for calculating the PREs.

3. Mean Square Error - The MSE of the estimators is calculated using as $MSE = \frac{\sum_{i=1}^{10,000}(\tilde{Y}_i - \bar{Y})^2}{10,000}$.

Where,
$\tilde{Y}_i = \tilde{Y}_{0.st.sys}$, $\tilde{Y}_{R.st.sys}$, $\tilde{Y}_{P.st.sys}$….., $\tilde{Y}^{(1,0)}_{E.st.sys}$,….., $\tilde{Y}^{(\beta_{1x},\beta_{2x})}_{E.st.sys}$, $\tilde{Y}^{(1,0)}_{G.st.sys}$,…., $\tilde{Y}^{(\beta_{1x},\beta_{2x})}_{G.st.sys}$, $\tilde{Y}^{(1,0)}_{AB.st.sys}$,….., $\tilde{Y}^{(\beta_{1x},\beta_{2x})}_{AB.st.sys}$.

4. Percent Relative Efficiency - The PRE of the estimators is calculated as $PRE(\tilde{Y}_{i_i}, \tilde{Y}_{0.st.sys}) = \frac{var(\tilde{Y}_{0.st.sys})}{MSE(\tilde{Y}_i)} \times 100$

$\tilde{Y}_i = \tilde{Y}_{0.st.sys}$, $\tilde{Y}_{R.st.sys}$, $\tilde{Y}_{P.st.sys}$….., $\tilde{Y}^{(1,0)}_{E.st.sys}$,….., $\tilde{Y}^{(\beta_{1x},\beta_{2x})}_{E.st.sys}$, $\tilde{Y}^{(1,0)}_{G.st.sys}$,…., $\tilde{Y}^{(\beta_{1x},\beta_{2x})}_{G.st.sys}$, $\tilde{Y}^{(1,0)}_{AB.st.sys}$,….., $\tilde{Y}^{(\beta_{1x},\beta_{2x})}_{AB.st.sys}$

Simulation results for the PREs of the proposed estimator w.r.t existing estimators across different strata, are provided in **Table (8)**.

Estimators	PRE	Estimator	PRE	Estimator	PRE	Estimator	PRE
$\tilde{Y}_{0.st.sys}$	100.0	$\tilde{Y}^{(1,0)}_{E.st.sys}$	177.33	$\tilde{Y}^{(1,0)}_{G.st.sys}$	1124.77	$\tilde{Y}^{(1,0)}_{AB.st.sys}$	5112.90

$\tilde{Y}_{R.st.sys}$	374.03	$\tilde{Y}_{E.st.sys}^{(1,1)}$	176.82	$\tilde{Y}_{G.st.sys}^{(1,1)}$	1216.96	$\tilde{Y}_{AB.st.sys}^{(1,1)}$	4857.57
$\tilde{Y}_{P.st.sys}$	43.78	$\tilde{Y}_{E.st.sys}^{(1,C_x)}$	177.32	$\tilde{Y}_{G.st.sys}^{(1,C_x)}$	1216.60	$\tilde{Y}_{AB.st.sys}^{(1,C_x)}$	4890.82
$\tilde{Y}_{Reg.st.sys}$	76.11	$\tilde{Y}_{E.st.sys}^{(1,\rho_{yx})}$	176.83	$\tilde{Y}_{G.st.sys}^{(1,\rho_{yx})}$	1216.96	$\tilde{Y}_{AB.st.sys}^{(1,\rho_{yx})}$	4854.97
$\tilde{Y}_{RE.st.sys}$	177.05	$\tilde{Y}_{E.st.sys}^{(C_x,\rho_{yx})}$	136.42	$\tilde{Y}_{G.st.sys}^{(C_x,\rho_{yx})}$	1247.22	$\tilde{Y}_{AB.St.sys}^{(C_x,\rho_{yx})}$	2436.44
$\tilde{Y}_{Pe.st.sys}$	63.57	$\tilde{Y}_{E.st.sys}^{(\rho_{yx},C_x)}$	177.32	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},C_x)}$	1216.60	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},C_x)}$	4890.82
$\tilde{Y}_{D.st.sys}$	1187.49	$\tilde{Y}_{E.st.sys}^{(1,\beta_{1x})}$	176.58	$\tilde{Y}_{G.st.sys}^{(1,\beta_{1x})}$	1216.96	$\tilde{Y}_{AB.st.sys}^{(1,\beta_{1x})}$	4843.78
$\tilde{Y}_{RP.st.sys}$	1287.53	$\tilde{Y}_{E.st.sys}^{(\beta_{1x},C_x)}$	177.32	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},C_x)}$	1216.60	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},C_x)}$	4903.78
$\tilde{Y}_{ER.st.sys}$	155.01	$\tilde{Y}_{E.st.sys}^{(C_x,\beta_{1x})}$	128.75	$\tilde{Y}_{G.st.sys}^{(C_x,\beta_{1x})}$	1251.34	$\tilde{Y}_{AB.st.sys}^{(C_x,\beta_{1x})}$	2124.20
		$\tilde{Y}_{E.st.sys}^{(\beta_{1x},\rho_{yx})}$	176.99	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},\rho_{yx})}$	1216.60	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},\rho_{yx})}$	4874.21
		$\tilde{Y}_{E.st.sys}^{(\rho_{yx},\beta_{1x})}$	176.58	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},\beta_{1x})}$	1216.96	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},\beta_{1x})}$	4842.74
		$\tilde{Y}_{E.st.sys}^{(1,\beta_{2x})}$	174.35	$\tilde{Y}_{G.st.sys}^{(1,\beta_{2x})}$	1219.09	$\tilde{Y}_{AB.st.sys}^{(1,\beta_{2x})}$	4661.24
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},C_x)}$	177.33	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},C_x)}$	1216.60	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},C_x)}$	4898.57
		$\tilde{Y}_{E.st.sys}^{(C_x,\beta_{2x})}$	109.70	$\tilde{Y}_{G.st.sys}^{(C_x,\beta_{2x})}$	809.20	$\tilde{Y}_{AB.st.sys}^{(C_x,\beta_{2x})}$	1828.42
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},\rho_{yx})}$	177.24	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},\rho_{yx})}$	1216.60	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},\rho_{yx})}$	4898.01
		$\tilde{Y}_{E.st.sys}^{(\rho_{yx},\beta_{2x})}$	174.34	$\tilde{Y}_{G.st.sys}^{(\rho_{yx},\beta_{2x})}$	1219.09	$\tilde{Y}_{AB.st.sys}^{(\rho_{yx},\beta_{2x})}$	4658.87
		$\tilde{Y}_{E.st.sys}^{(\beta_{2x},\beta_{1x})}$	177.46	$\tilde{Y}_{G.st.sys}^{(\beta_{2x},\beta_{1x})}$	1216.60	$\tilde{Y}_{AB.st.sys}^{(\beta_{2x},\beta_{1x})}$	4883.17
		$\tilde{Y}_{E.st.sys}^{(\beta_{1x},\beta_{2x})}$	175.31	$\tilde{Y}_{G.st.sys}^{(\beta_{1x},\beta_{2x})}$	1218.38	$\tilde{Y}_{AB.st.sys}^{(\beta_{1x},\beta_{2x})}$	4736.37

6. Results and Discussion

As discussed earlier, we utilized three real datasets and conducted a simulation study to evaluate the efficiency of the proposed generalized class of exponential-type estimator, along with other existing estimators, within the stratified systematic sampling scheme using information from a single auxiliary variable X. The proposed estimator and its adapted version in stratified systematic sampling were compared with respect to their percent relative efficiencies ($PREs$). Tables 2-4 provide the data descriptions, while Tables 5-7 present the $PREs$ results for the real datasets. It was observed that the $PREs$ of the proposed estimator vary with different choices of α and β. Additionally, the proposed estimator proved to be more precise than other existing estimators in both the real datasets and the simulation study, based on $PREs$.

Conclusion

In this paper, we introduced an improved generalized class of exponential-type estimator for estimating the finite population mean under the stratified systematic sampling scheme. Auxiliary variable information is used to enhance precision. The mean squared error (MSE) of the proposed exponential estimator was derived using the Taylor series. An empirical study, based on both simulated and real datasets, was conducted to assess the efficiency of the proposed estimator. The MSE and percent relative efficiency (PRE) were employed to compare the precision of the proposed estimator with that of the usual ratio, product, regression, and other existing estimators. The study demonstrated that the proposed improved class of exponential-type estimator yields better results compared to other existing estimators, making it a reliable and more precise option for practical use.

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Appendix -1

Table 1. Parameters and distributions of study and auxiliary variables

Population	Parameters and distributions of the study Variable	Parameters and distributions of the auxiliary variable
Population h = 1,2,3	$f(y^*_{hi}) = \frac{1}{\Gamma(1.5)} y^{*1.5-1}_{hi} e^{-y^*_{hi}}$	$f(x^*_{hi}) = \frac{1}{\Gamma(0.3)} x^{*0.3-1}_{hi} e^{-x^*_{hi}}$

Table 2. Properties of strata

Strata	Study variable	Auxiliary variable
Stratum 1	$y_{1i} = 50 + y^*_{1i}$	$x_{1i} = 15 + \sqrt{(1-\rho^2_{xy1})} x^*_{1i} + \rho_{xy1}\left(\frac{S_{1x}}{S_{1y}}\right) y^*_{1i}$
Stratum 2	$y_{2i} = 150 + y^*_{2i}$	$x_{2i} = 100 + \sqrt{(1-\rho^2_{xy2})} x^*_{2i} + \rho_{xy2}\left(\frac{S_{2x}}{S_{2y}}\right) y^*_{2i}$
Stratum 3	$y_{2i} = 100 + y^*_{3i}$	$x_{3i} = 200 + \sqrt{(1-\rho^2_{xy3})} x^*_{3i} + \rho_{xy3}\left(\frac{S_{3x}}{S_{3y}}\right) y^*_{3i}$

Appendix- 2

```
N1<-600;N2<-600;N3<-600
n1<-100;n2<-100;n3<-100
N<-1800
R1<-0.8;R2<-0.7;R3<-0.9
Sx1<-4.7;Sx2<-6.2; Sx3<-8.4
Sy1<-4.8;Sy2<-4.8;Sy3<-4.8
set.seed(12345)
Y_1<-rgamma(N1,1.5,1)+50
y1<-Y_1[order(Y_1)]
Y_2<-rgamma(N2,1.5,1)+100
y2<-Y_2[order(Y_2)]
Y_3<-rgamma(N3,1.5,1)+150
```

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
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```
y3<-Y_3[order(Y_3)]
X_1<-15+sqrt(1-R1^2)*rgamma(N1,0.3,1)+R1*(Sx1/Sy1)*Y_1
x1<-X_1[order(X_1)]
X_2<-100+sqrt(1-R2^2)*rgamma(N2,0.3,1)+R2*(Sx2/Sy2)*Y_2
x2<-X_2[order(X_2)]
X_3<-200+sqrt(1-R3^2)*rgamma(N3,0.3,1)+R3*(Sx3/Sy3)*Y_3
x3<-X_3[order(X_3)]
Df1<-cbind(x1,y1);Df2<-cbind(x2,y2);Df3<-cbind(x3,y3)
#====mean
W<-c(0.3333333,0.3333333,0.3333333)
W1<-(N1/N);W2<-(N2/N);W3<-(N3/N)
Xbar<-228.716;Ybar<-101.4751
d<-0.8126446
d1<-0.9999706
d2<-0.8126207
d3<-1.415815
d4<-0.9998337
d5<-0.9999706
d6<-0.8124975
d7<-(-0.8689623)
d8<-189.65
a<-1
b<-(N*Xbar)
#======
set.seed(12345)
mst<-NA;yst<-NA;xst<-NA;yr<-NA          ;myr<-NA;yre<-NA;yp<-NA;myp<-NA;myre<-NA;yrt<-
NA;myrt<-NA;ypt<-NA;mypt<-NA;yd<-NA;myd<-NA;yrp<-NA;myrp<-NA;yer<-NA;myer<-
NA;ye<-NA;mye<-NA;yg<-NA;myg<-NA;yA<-NA;myA<-NA
for (i in 1:10000) {
  m1<-c(sample(1:600,100,replace=F))
  m2<-c(sample(1:600,100,replace=F))
```

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ISSN: 1074-133X
Vol 32 No. 10s (2025)

```
m3<-c(sample(1:600,100,replace=F))
ma1<-Df1[m1,]
head(ma1)
ma2<-Df2[m2,]
head(ma2)
ma3<-Df3[m3,]
head(ma3)
maa1<-as.data.frame(ma1)
maa2<-as.data.frame(ma2)
maa3<-as.data.frame(ma3)
y11<-mean(maa1[,2])
y22<-mean(maa2[,2])
y33<-mean(maa3[,2])
x11<-mean(maa1[,1])
x22<-mean(maa2[,1])
x33<-mean(maa3[,1])
yst[i]<-W[1]*y11+W[2]*y22+W[3]*y33
mst[i]<-(yst[i]-Ybar)^2
xst[i]<-W[1]*x11+W[2]*x22+W[3]*x33
yr[i]<-yst[i]*(Xbar/xst[i])
myr[i]<-(yr[i]-Ybar)^2
yp[i]<-yst[i]*(xst[i]/Xbar)
myp[i]<-(yp[i]-Ybar)^2
yre[i]<-yst[i]+(d*(Xbar-xst[i]))
myre[i]<-(yre[i]-Ybar)^2
yrt[i]<-yst[i]*(exp((Xbar-xst[i])/(Xbar+xst[i])))
myrt[i]<-(yrt[i]-Ybar)^2
ypt[i]<-yst[i]*(exp((xst[i]-Xbar)/(xst[i]+Xbar)))
mypt[i]<-(ypt[i]-Ybar)^2
yd[i]<-(d1*yst[i])+(d2*(Xbar-xst[i]))
myd[i]<-(yd[i]-Ybar)^2
```

```
 yrp[i]<-yst[i]*((d3*(Xbar/xst[i]))+((1-d3)*(xst[i]/Xbar)))
 myrp[i]<-(yrp[i]-Ybar)^2
 yer[i]<-d4*yst[i]*(exp((Xbar-xst[i])/(Xbar+xst[i])))
 myer[i]<-(yer[i]-Ybar)^2
 ye[i]<-yst[i]*(exp((a*(Xbar-xst[i]))/((a*(Xbar+xst[i]))+2*b)))
 mye[i]<-(ye[i]-Ybar)^2
 yg[i]<-((d5*yst[i])+(d6*(Xbar-xst[i])))*(exp((a*(Xbar-xst[i]))/((a*(Xbar+xst[i]))+2*b)))
 myg[i]<-(yg[i]-Ybar)^2
 yA[i]<-((d7*yst[i])+(d8*((xst[i])/Xbar)))*(exp((a*(Xbar-xst[i]))/((a*(Xbar+xst[i]))+2*b)))
 myA[i]<-(yA[i]-Ybar)^2
}
mse1<-mean(mst)
MSE1<-1/10000*sum(myr);MSE2<-1/10000*sum(myp);MSE3-1/10000*sum(myre);MSE4<-
1/10000*sum(myrt);MSE5<-1/10000*sum(mypt);MSE6<-1/10000*sum(myd);MSE7<-
1/10000*sum(myrp);MSE8<-1/10000*sum(myer);MSE9<-1/10000*sum(myg);MSE10<-
1/10000*sum(myA);MSE11<1/10000*sum(mye)
    pre1<-(mse1/MSE1) *100; pre2<-(mse1/MSE2)*100;………….. pre11<-(mse1/MSE11)*100
```