Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

A New Subclass of Meromorphic Functions Associated with Sălăgean Operator

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Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract: In this paper, we propose a novel subclass of meromorphic functions within the class $\mathcal{A}_p^*$ which includes functions of the form $f(z) = z^{-p} + \sum_{j=p}^{\infty} a_j z^j$, where $z \in \triangle^*$ and $p \in \mathbb{N}$. The study specifically focuses on using the Sălăgean operator $D_{p,\lambda}^n f(z)$ to established this new subclass denoted as $\mathcal{M}(\alpha, \beta, \gamma, k, \lambda, n, p)$ which is defined by the parameters $0 \leq \alpha < 1, 0 \leq \beta < 1, 0 \leq \gamma < 1, k \geq 0, 0 < \lambda \leq 1$ and $n \in \mathbb{N}_0$. $Re\left(\frac{z(D_{p,\lambda}^n f(z))' - (D_{p,\lambda}^n f(z))}{\alpha z(D_{p,\lambda}^n f(z))' + (1-r)(D_{p,\lambda}^n f(z))}\right) > k\left|\frac{z(D_{p,\lambda}^n f(z))' - (D_{p,\lambda}^n f(z))}{\alpha z(D_{p,\lambda}^n f(z))' + (1-r)(D_{p,\lambda}^n f(z))} - 1\right| + \beta$ by the given condition for this class we will next to prove this class of univalent function then we derive the related results. The mathematical formulation of $D_{p,\lambda}^n f(z)$, where the operator influences the series expansion of the meromorphic functions, is also elaborated in the study.

Keywords: Meromorphic functions, Function class $\mathcal{A}_p^*$, Sălăgean operator, Starlikeness, Convexity, Subclass of meromorphic functions, Series expansion, Analytic function properties.

1. Introduction

The study of meromorphic functions has been a significant area of interest in complex analysis, particularly in the context of geometric function theory. These functions, which are analytic except for isolated poles, play a crucial role in various applied & mathematical fields. Within the class of meromorphic functions, researchers have explored different subclasses defined by specific function properties and operator-induced transformations. One such transformation involves differential and integral operators that influence the function's behaviour in the punctured unit disk. The role of multiplier transformations in defining new function classes has been studied extensively. For some basic concepts like Univalent functions, which are holomorphic and injective, are fundamental in geometric function theory for a comprehensive study, (see **[7], [8]**). Alharayzeh and Ghanim (2022) recently introduced a k-uniformly univalent subclass with negative coefficients (see also **[1]**), proving key coefficient estimates and geometric properties. Building upon their findings, this paper introduces a new subclass incorporating additional parameters, leading to broader applications in function theory. In (2016), Amourah and Darus defined a new class of univalent functions by applying a generalized differential operator that includes negative coefficients, highlighting its geometric and analytic properties in paper **[3]**. Recent advancements in geometric function theory have led to the development of new subclasses of analytic functions. In (2024), Carroll discusses key techniques in paper **[4]**, for analyzing these classes, which are the foundation of our investigation. In (2010), Ali and Ravichandran defined a subclass of meromorphic functions based on α- convexity and examined its analytic and geometric properties in paper **[2]**. Darus et al. established univalence criteria in paper **[5],** for analytic functions using a generalized differential operator. Building on their findings, this paper explores new subclasses by extending the operator's properties of generalized Sălăgean operator and derived new results. For computational purposes we refer to paper **[6]**, which is based on the convexity criterion established by Deniz and Erhan in (2013). For further study on univalence criteria, we refer also papers **[11]** and **[12]**. In (2009), Lee et al. established coefficient bounds in paper **[9]** for such functions, providing key insights into their structure and meromorphic properties.

Definition: Let the class $\mathcal{A}_{\mathrm{p}}^{*}$ consists of all analytic functions in the punctured unit disk $\triangle^*$, of the form $f(z) = z^{-p} + \sum_{j=p}^{\infty} a_j z^j$, $z \in \triangle^*$ and $p \in \mathbb{N}$. For the function $f \in \mathcal{A}_{\mathrm{p}}^{*}$, here we introduced a new subclass for

$$0 \leq \alpha < 1, 0 \leq \beta < 1, 0 \leq \gamma < 1,\ k \geq 0,\ 0 < \lambda \leq 1, p \in \mathbb{N}, n \in \mathbb{N}_0 = \{0,1,2, \dots\},$$

By using the operator $D_{p,\lambda}^{n}$ by $D_{p,\lambda}^{0} = f(z)$ which is introduced in paper **[1]**

Let $\mathcal{M}(\alpha, \beta, \gamma, k, \lambda, n, p)$

$$Re\left(\frac{z(D_{p,\lambda}^{n}f(z))' - (D_{p,\lambda}^{n}f(z))}{\alpha z(D_{p,\lambda}^{n}f(z))' + (1-r)(D_{p,\lambda}^{n}f(z))}\right) > k\left|\frac{z(D_{p,\lambda}^{n}f(z))' - (D_{p,\lambda}^{n}f(z))}{\alpha z(D_{p,\lambda}^{n}f(z))' + (1-r)(D_{p,\lambda}^{n}f(z))} - 1\right| + \beta \quad (1)$$

Where,

$$D_{p,\lambda}^{n}f(z) = z^{-p} + \sum_{j=p}^{\infty}\left\{(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^j\right\}^n, \qquad z \in \triangle^* \ \ and \ a_j > 0$$

Here $D_{p,\lambda}^{n}f(z)$ be the Sălăgean operator.

2. Objectives

The primary objective of this study is to investigate and analyze various newly defined subclasses of analytic functions within the unit disc. In this study we introduced a new subclass of meromorphic functions $\mathcal{M}(\alpha, \beta, \gamma, k, \lambda, n, p)$, which lies within the class $\mathcal{A}_p^*$ and consists of functions of the form $f(z) = z^{-p} + \sum_{j=p}^{\infty} a_j z^j$, where $z \in \triangle^*$ and $p \in \mathbb{N}$. This study specifically focuses on using the Sălăgean operator $D_{p,\lambda}^{n}f(z)$ to established the foundational properties of this class, proving its univalence, and analyzing the influence of the operator on the function's geometric behavior. The mathematical formulation of $D_{p,\lambda}^{n}f(z)$, where the operator influences on the series expansion of the meromorphic functions, is also elaborated in the study.

3. Methods

In this paper, we introduced the subclass $\mathcal{M}(\alpha, \beta, \gamma, k, \lambda, n, p)$ of meromorphic functions defined in the punctured unit disk $\triangle^*$, associated with the generalized Sălăgean operator $D_{p,\lambda}^{n}$ which is defined as $D_{p,\lambda}^{n}f(z) = z^{-p} + \sum_{j=p}^{\infty}\left\{(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^j\right\}^n$ where $a_j > 0$. Coefficient estimates are obtained by applying a differential subordination condition for function belongs to the subclass $\mathcal{M}(\alpha, \beta, \gamma, k, \lambda, n, p)$. Convexity of $f(z)$ is established under the condition $Re\left(1 + \frac{z(D_{p,\lambda}^{n}f(z))''}{(D_{p,\lambda}^{n}f(z))'}\right) > 0$, and Univalence criteria is ensured by the condition $\left|\frac{z(D_{p,\lambda}^{n}f(z))''}{(D_{p,\lambda}^{n}f(z))'}\right| < 1$. These methods provide conditions on the coefficients and geometric criteria for functions defined in the above subclass.

4. Results

We begin by proving several theorems based on the operator classification.

Theorem: 1

Let $0 \leq \alpha < 1, 0 \leq \beta < 1$, $0 \leq \gamma < 1$, $k \geq 0$, $0 < \lambda \leq 1, p \in \mathbb{N}$, $n \in \mathbb{N}_0 = \{0,1,2,\dots\}$. A function f given by $f(z) = z^{-p} + \sum_{j=p}^{\infty} a_j z^j$ be an analytic function belonging to the class $\mathcal{M}(\alpha,\beta,\gamma,k,\lambda,n,p)$, then f satisfies the condition:

$$Re\left(\frac{z\left(D_{p,\lambda}^{n}f(z)\right)' - \left(D_{p,\lambda}^{n}f(z)\right)}{\alpha z\left(D_{p,\lambda}^{n}f(z)\right)' + (1-r)\left(D_{p,\lambda}^{n}f(z)\right)}\right) > k\left|\frac{z\left(D_{p,\lambda}^{n}f(z)\right)' - \left(D_{p,\lambda}^{n}f(z)\right)}{\alpha z\left(D_{p,\lambda}^{n}f(z)\right)' + (1-r)\left(D_{p,\lambda}^{n}f(z)\right)} - 1\right| + \beta$$

where, $D_{p,\lambda}^{n}f(z) = z^{-p} + \sum_{j=p}^{\infty}\left\{(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^j\right\}^n$, $z \in \triangle^*$ $and\ a_j > 0$

Then the coefficients a_j of the Taylor series expansion satisfy the inequality

$$a_j \leq \frac{(1-\beta)(\alpha j + 1 - r)}{(k+1)|2-\gamma-n(1-\alpha)|(1+j)} \quad , \qquad for\ all\ j \geq 2$$

This result establishes an upper bound on the growth of the coefficients of functions in this subclass $\mathcal{M}(\alpha,\beta,\gamma,k,\lambda,n,p)$.

Proof: Given that $\mathcal{M}(\alpha,\beta,\gamma,k,\lambda,n,p)$ if and only if the condition (1) is satisfied,

Let

$$\omega = \frac{z(D_{p,\lambda}^{n}f(z))' - (D_{p,\lambda}^{n}f(z))}{\alpha z(D_{p,\lambda}^{n}f(z))' + (1-r)(D_{p,\lambda}^{n}f(z))}$$

Subject to the condition that, $Re(\omega) \geq k|\omega - 1| + \beta$ if and only if $(k+1)|\omega - 1| \leq 1-\beta$

Now

$$(k+1)|\omega-1| = (k+1)\left|\frac{(1-p)z^{-p} + \sum_{j=2}^{\infty}\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^n (1+j)a_j z^j}{(1-r-\alpha p)z^{-p} + \sum_{j=2}^{\infty}(\alpha j + 1 - r)\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^n a_j z^j} - 1\right|$$
$$\leq 1-\beta$$

is equivalent to

$$(k+1)\left|\frac{(1-p)z^{-p} + \sum_{j=2}^{\infty}\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^n (1+j)a_j z^j}{(1-\alpha p - r)z^{-p} - \sum_{j=2}^{\infty}(\alpha j + 1 - r)\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^n a_j z^j} - 1\right| \leq 1-\beta$$

So

$$(k+1)\left|\frac{(1-p)z^{-p}+\sum_{j=2}^{\infty}|2-\gamma-n(1-\alpha)|\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}(1+j)a_j z^j-(1-\alpha p-r)z^{-p}}{(1-\alpha p-r)z^{-p}-\sum_{j=2}^{\infty}(\alpha j+1-r)\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}a_j z^j}\right| \leq 1-\beta$$

Further the above inequality

$$(k+1)\left|\frac{(\alpha p+r-p)z^{-p}+\sum_{j=2}^{\infty}|2-\gamma-n(1-\alpha)|\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}(1+j)a_j z^j}{(1-\alpha p-r)z^{-p}-\sum_{j=2}^{\infty}(\alpha j+1-r)\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}a_j z^j}\right| \leq 1-\beta$$

By factoring out the term z^{-p}

$$(k+1)\left|\frac{z^{-p}\left((\alpha p+r-p)+\sum_{j=2}^{\infty}|2-\gamma-n(1-\alpha)|\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}(1+j)a_j z^{j+p}\right)}{z^{-p}\left((1-\alpha p-r)-\sum_{j=2}^{\infty}(\alpha j+1-r)\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}a_j z^{j+p}\right)}\right| \leq 1-\beta$$

cancel z^{-p} from both numerator and denominator

$$(k+1)\left|\frac{\left((\alpha p+r-p)+\sum_{j=2}^{\infty}|2-\gamma-n(1-\alpha)|\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}(1+j)a_j z^{j+p}\right)}{\left((1-\alpha p-r)-\sum_{j=2}^{\infty}(\alpha j+1-r)\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}a_j z^{j+p}\right)}\right| \leq 1-\beta$$

Then, $(k+1)\left[(\alpha p+r-p)+\sum_{j=2}^{\infty}|2-\gamma-n(1-\alpha)|\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}(1+j)a_j z^{j+p}\right]$

$$\leq (1-\alpha p-r)(1-\beta)-(1-\beta)\left(\sum_{j=2}^{\infty}(\alpha j+1-r)\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}a_j z^{j+p}\right)$$

Now expand the terms of left-hand side (LHS)

$$(k+1)\left[(\alpha p+r-p)+\sum_{j=2}^{\infty}|2-\gamma-n(1-\alpha)|\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}(1+j)a_j z^{j+p}\right]$$

The LHS consists of a constant term $(k+1)(\alpha p+r-p)$ and a series denoted by S_1, where

$$S_1=\sum_{j=2}^{\infty}|2-\gamma-n(1-\alpha)|\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}(1+j)a_j z^{j+p}$$

So, the LHS becomes

$$(k+1)(\alpha p+r-p)+(k+1)S_1$$

Right-hand side (RHS)

$$(1-\alpha p-r)(1-\beta)-(1-\beta)\left(\sum_{j=2}^{\infty}(\alpha j+1-r)\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}a_j z^{j+p}\right)$$

The RHS also has a constant term $(1-\alpha p-r)(1-\beta)$ and a series denoted by S_2, where

$$S_2=\sum_{j=2}^{\infty}(\alpha j+1-r)\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}a_j z^{j+p}$$

So, the RHS becomes

$$(1-\alpha p-r)(1-\beta)-(1-\beta)S_2$$

Then using the expressions for LHS & RHS

$$(k+1)(\alpha p+r-p)+(k+1)S_1\leq\ (1-\alpha p-r)(1-\beta)-(1-\beta)S_2$$

Rearranging terms

$$(k+1)(\alpha p+r-p)-(1-\alpha p-r)(1-\beta)\leq\ -(k+1)S_1-(1-\beta)S_2$$

Then expand and simplify the constants

$$(k+1)(\alpha p+r-p)-(1-\alpha p-r)(1-\beta)$$

Distribute $(1-\beta)$ in the second term

$$(k+1)(\alpha p+r-p)-(1-\beta)(1)+(1-\beta)(\alpha p+r)$$

Combine like terms:

$$k(\alpha p+r-p)+(\alpha p+r-p)-(1-\beta)+(1-\beta)(\alpha p+r)$$

Simplify further, then we get

$$(\alpha p+r-p)+k(\alpha p+r-p)+(1-\beta)(\alpha p+r-1)$$

For the series S_1 and S_2,the terms of z^{j+p}. Since z^{j+p} terms are independent of the constant parts, we can combine the inequalities to solve for the coefficients a_j , given the following condition must hold:

$$(k+1)|2-\gamma-n(1-\alpha)|\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}(1+j)a_j$$
$$\leq(1-\beta)(\alpha j+1-r)\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}a_j$$

Then dividing both sides by $\left\{(1-\lambda)\left(1+\frac{j}{p}\right)\right\}^{n}$, we get

$$(k+1)|2-\gamma-n(1-\alpha)|(1+j)a_j\leq(1-\beta)(\alpha j+1-r)a_j$$

For non- zero a_j, divide through by a_j

$$(k+1)|2-\gamma-n(1-\alpha)|(1+j)\leq(1-\beta)(\alpha j+1-r)$$

Then expand and simplify

$$(k+1)|2-\gamma-n(1-\alpha)| \leq (1-\beta)\frac{(\alpha j+1-r)}{(1+j)}$$

This gives a bound for the coefficients a_j

Then the solution depends on analysing the coefficients a_j satisfying:

$$a_j \leq \frac{(1-\beta)(\alpha j+1-r)}{(k+1)|2-\gamma-n(1-\alpha)|(1+j)}$$

The result is bounded by the choice of parameters $\alpha, \beta, \gamma, k, \lambda, p$ and n.

Corollary: (Boundedness of Coefficients for Special Cases)

If we take $\alpha = 0, \beta = 0, \gamma = 0$ and $k = 0$ then the bound on the coefficients simplifies to:

$$a_j \leq \frac{1}{|2-n|(1+j)}$$

For all $j \geq 2$. This shows that in this special case, the coefficients decay inversely with respect to $(1 + j)$, ensuring the function remains in a bounded subclass of analytic functions.

Theorem: 2 (Convexity Condition)

Let $f(z) \in \mathcal{M}(\alpha, \beta, \gamma, k, \lambda, n, p)$. If $D^n_{p,\lambda} f(z)$ satisfies the following condition:

$$Re\left(1 + \frac{z(D^n_{p,\lambda} f(z))''}{(D^n_{p,\lambda} f(z))'}\right) > 0,$$

Then $f(z)$ is convex in $\triangle^*$.

Proof: Here the operator $D^n_{p,\lambda} f(z)$ is defined as

$$D^n_{p,\lambda} f(z) = z^{-p} + \sum_{j=p}^{\infty} \left\{(1-\lambda)\left(1+\frac{j}{p}\right) a_j z^j\right\}^n, \ z \in \triangle^* \ and \ a_j > 0$$

For simplicity, Let $g(z) = D^n_{p,\lambda} f(z)$

Compute the 1st and 2nd derivative of $g(z)$, as follows then we get

$$g'(z) = -pz^{-p-1} + \sum_{j=p}^{\infty} n\left[(1-\lambda)\left(1+\frac{j}{p}\right) a_j z^{j-1}\right]\left(1+\frac{j}{p}\right)$$

$$g''(z) = p(p+1)z^{-p-2} + \sum_{j=p}^{\infty} n\,(n-1)\left[(1-\lambda)\left(1+\frac{j}{p}\right) a_j z^{j-2}\right]\left(1+\frac{j}{p}\right)^2$$

Substitute these derivatives into the condition from

$$Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0$$

$$\Rightarrow Re\left(1+\frac{z(D_{p,\lambda}^{n}f(z))''}{(D_{p,\lambda}^{n}f(z))'}\right)>0$$

$$\Rightarrow Re\left(1+\frac{z\left\{p(p+1)z^{-p-2}+\sum_{j=p}^{\infty}n\,(n-1)\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_jz^{j-2}\right]\left(1+\frac{j}{p}\right)^2\right\}}{-pz^{-p-1}+\sum_{j=p}^{\infty}n\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_jz^{j-1}\right]\left(1+\frac{j}{p}\right)}\right)>0$$

In the punctured unit disk $\triangle^*$, $|z|<1$, so higher-order terms z^j ,becomes negligible as $|z|\to 0$ for leading order terms, then the condition will be

$$Re\left(1+\frac{z[p(p+1)z^{-p-2}]}{-pz^{-p-1}}\right)>0$$
$$\Rightarrow Re\left(1+\frac{p(p+1)z^{-p-1}}{-pz^{-p-1}}\right)>0$$
$$\Rightarrow Re\left(1+\left(\frac{p(p+1)}{-p}\right)\right)>0$$
$$\Rightarrow Re(1-(p+1))>0$$
$$\Rightarrow Re(1-p-1)>0$$
$$\Rightarrow Re(-p)>0$$

This means the real part of $-p$, must be greater than 0. Let $p=a+bi$, where a and b be the real part and i be the imaginary part of this complex number, then

$$-p=-a-bi$$

Thus, the real part of $-p$ is $-a$, the inequality becomes

$$-a>0$$
$$a<0$$

Therefore,

$$Re\left(1+\frac{zg''(z)}{g'(z)}\right)>0$$

This inequality holds when the real part of p is negative.

Theorem: 3 (Univalence Criterion)

Let $f(z)\in\mathcal{M}(\alpha,\beta,\gamma,k,\lambda,n,p)$, If the following inequality holds:

$$\left|\frac{z(D_{p,\lambda}^{n}f(z))''}{(D_{p,\lambda}^{n}f(z))'}\right|<1,$$

Then $f(z)$ is univalent in $\triangle^*$.

Proof: Given the form of $D_{p,\lambda}^{n}f(z)$, here the 1st derivative of this operator will be

$$(D^n_{p,\lambda}f(z))' = -pz^{-p-1} + \sum_{j=p}^{\infty} n\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j-1}\right]\left(1+\frac{j}{p}\right)$$

2nd derivative will be

$$(D^n_{p,\lambda}f(z))'' = p(p+1)z^{-p-2} + \sum_{j=p}^{\infty} n\,(n-1)\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j-2}\right]\left(1+\frac{j}{p}\right)^2$$

Substitute these derivatives in to the conditions we get,

$$\left|\frac{z(D^n_{p,\lambda}f(z))''}{(D^n_{p,\lambda}f(z))'}\right| < 1,$$

$$= \left|\frac{z\,p(p+1)z^{-p-2} + \sum_{j=p}^{\infty} n\,(n-1)\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j-1}\right]\left(1+\frac{j}{p}\right)^2}{-pz^{-p-1} + \sum_{j=p}^{\infty} n\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j-1}\right]\left(1+\frac{j}{p}\right)}\right| < 1,$$

$$= \left|\frac{p(p+1)z^{-p-1} + \sum_{j=p}^{\infty} n\,(n-1)\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j-2}\right]\left(1+\frac{j}{p}\right)^2}{-pz^{-p-1} + \sum_{j=p}^{\infty} n\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j-1}\right]\left(1+\frac{j}{p}\right)}\right| < 1,$$

$$= \left|\frac{z^{-p-1}\left\{p(p+1) + \sum_{j=p}^{\infty} n\,(n-1)\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j+p+1}\right]\left(1+\frac{j}{p}\right)^2\right\}}{z^{-p-1}\left\{-p + \sum_{j=p}^{\infty} n\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j+p+1}\right]\left(1+\frac{j}{p}\right)\right\}}\right| < 1,$$

After factoring z^{-p-1} out the fraction simplifies to,

$$= \left|\frac{\left\{p(p+1) + \sum_{j=p}^{\infty} n\,(n-1)\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j+p+1}\right]\left(1+\frac{j}{p}\right)^2\right\}}{\left\{-p + \sum_{j=p}^{\infty} n\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j+p+1}\right]\left(1+\frac{j}{p}\right)\right\}}\right| < 1,$$

$$p(p+1) + \sum_{j=p}^{\infty} n\,(n-1)\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j+p+1}\right]\left(1+\frac{j}{p}\right)^2$$

Here the leading term is $p(p+1)$ and the remaining part is a series that depends on z.

$$-p + \sum_{j=p}^{\infty} n\left[(1-\lambda)\left(1+\frac{j}{p}\right)a_j z^{j+p+1}\right]\left(1+\frac{j}{p}\right)$$

Here the leading term is $-p$, with the remaining part as a series.

To evaluate the modulus

At $\boldsymbol{z \to 0}$,

The series terms involving z^{j+p+1} will reduce, making the leading terms $p(p+1)$ in the numerator and $-p$ in the denominator dominant.

The expression becomes approximately

$$\left|\frac{p(p+1)}{-p}\right| = |-(p+1)| = (p+1) < 1$$

This needs to be less than 1, so for small z this inequality holds only if $(p+1) < 1$, which suggests $p < 0$.

At $\boldsymbol{z} \to \infty$,

The series terms involving higher powers of z will dominate, and we need to analyse the growth rate of the series terms relative to each other. For large z, we compare the asymptotic behaviour of the series in both the numerator and denominator.

5. Discussion

In this work, a new subclass $\mathcal{M}(\alpha,\beta,\gamma,k,\lambda,n,p)$ of meromorphic functions defined in the punctured unit disk $\triangle^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ is introduced using a generalized Sălăgean-type differential operator $D^n_{p,\lambda}$. The functions in this class are characterized by satisfying a specific real-part inequality involving the parameters $\alpha,\beta,\gamma,k,\lambda,n,p$ which controls their geometric behaviour. A sharp upper bound is derived for the coefficients a_j of functions in this class. This bound depends on the involved parameters and effectively restricts the growth of the series coefficients. Such bounds are crucial for understanding the geometric structure and distortion properties of the functions. If the Sălăgean operator $D^n_{p,\lambda}f(z)$ satisfies a specific positivity condition involving its logarithmic derivative, then the function f is convex in $\triangle^*$. This result connects analytic properties of the differential operator to the geometric shape (convexity) of the function image. Theorem 3 provides a sufficient condition for univalence: if the modulus of the logarithmic derivative of $D^n_{p,\lambda}f(z)$ remains less than one, then the function f is univalent (injective) in $\triangle^*$ .This ensures that the function preserves distinctness of points under mapping, a foundational property in geometric function theory.

Overall, these theorems collectively establish important structural properties coefficient bounds, convexity, and univalence of the functions in the class $\mathcal{M}(\alpha,\beta,\gamma,k,\lambda,n,p)$, highlighting the impact of the Sălăgean operator on the analytic and geometric behaviour of meromorphic functions.

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