

# Homeomorphism via $\delta\beta$ -open Sets in Fermatean Fuzzy Topological Spaces and Application in Entropy Measure

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**Abstract:** Classical set theory failed to cover non-probabilistic uncertain situations in to the set format, but fuzzy set theory can do this job perfectly with the fuzzy number of vagueness of non-probabilistic uncertainty. Topologist adopt this fuzzy set and fit in to the topological concepts to extent and apply their innovations for the needs and growth of the humans. Even though the fuzzy concept convert every situation, it hire the concept of intuitionistic fuzzy set to evident the importance of non-membership of the situation. Pythagorean fuzzy set is one of by membership and non-membership, more forceful to seize indeterminacy to cover the uncertain situations which are unable to covered by intuitionistic fuzzy sets. Then any Intuitionistic fuzzy subset or Pythagorean fuzzy subset of a set can be considered as Fermatean fuzzy subset, we observe that any Intuitionistic fuzzy topological space or Pythagorean fuzzy topological space is a Fermatean fuzzy topological space as well. In this paper, we contribute our concept of  $\delta$  (resp.  $\delta P$ ,  $\delta S$ ,  $\delta\alpha$ ,  $\delta\beta$  )-homeomorphism,  $C$ -homeomorphism in Fermatean fuzzy topological spaces and the properties for the Fermatean field of fuzzy topological spaces. To register importance of Fermatean fuzzy sets we applied a proposed entropy measure for multiple criteria decision making problem.

**Keywords:** Fermatean fuzzy homeomorphism, Fermatean fuzzy  $C$  - homeomorphism, entropy measure.

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## 1 Introduction

Fuzzy sets were introduced Zadeh [15] in 1965. The fuzzy set concept was the basis of mathematical testing of the fuzzy concept that exists in our real world and the formation of new branches in mathematics. The fuzzy set concept corresponding to unexplained physical situations gives useful applications on many topics such as statistics, data processing and linguistics. A lot of research has been done on this subject since 1965. In 1968, Chang [6] defined the concept of fuzzy topological space and generalized some basic notions of topology such as open set, closed set, continuity and compactness to fuzzy topological spaces. The idea of intuitionistic fuzzy set was first published by

Atanassov [1] and many works by the same author and his colleagues appeared in the literature [2, 5]. Coker [7] initiated a study of intuitionistic fuzzy topological spaces. Later Yager [13] launched a non standard fuzzy set referred to as Pythagorean fuzzy set. Olgun et al., [9] defined a Pythagorean fuzzy topological spaces. Fermatean fuzzy sets proposed by Senapati and Yager in 2020 [10], can handle uncertain information more easily in the process of decision making. They defined basic operations over the Fermatean fuzzy sets. Hariwan Z. Ibrahim defined a Fermatean fuzzy topological spaces and the continuity of a function defined among Fermatean fuzzy topological spaces.

The aim of this paper is as follows. In Section 2, some basic definitions of *fs*'s, *ifs*'s, *pfs*'s and Fermatean fuzzy sets are briefly reviewed. In section 3 and 4, we develop the concept of some stronger and weaker forms of Fermatean fuzzy  $\delta\beta$  homeomorphism and  $C$ -homeomorphism in Fermatean fuzzy topological spaces and also specialized some of their basic properties with examples. Entropy measure was introduced by Zadeh [16] for classical fuzzy sets. Many authors developed and created for their version of entropy measure. Here in section 5, we introduce entropy measure for Fermatean fuzzy sets and give an example for the decision making in real life problem. Finally we conclude in section 6.

## 2 Preliminaries

We recall some basic notions of fuzzy sets, *IFS*'s, *PFS*'s and  $\mathfrak{F}$ *FS*'s.

**Definition 2.1** [15] Let  $X$  be a nonempty set. A fuzzy set  $A$  in  $X$  is characterized by a membership function  $\mu_A: X \rightarrow [0,1]$ . That is:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{if } x \notin X \\ (0,1) & \text{if } x \text{ is partly in } X. \end{cases}$$

Alternatively, a fuzzy set  $A$  in  $X$  is an object having the form  $A = \{ \langle x, \mu_A(x) \rangle \mid x \in X \}$  or  $A = \left\{ \left( \frac{\mu_A(x)}{x} \right) \mid x \in X \right\}$ , where the function  $\mu_A(x): X \rightarrow [0,1]$  defines the degree of membership of the element,  $x \in X$ .

The closer the membership value  $\mu_A(x)$  to 1, the more  $x$  belongs to  $A$ , where the grades 1 and 0 represent full membership and full nonmembership. Fuzzy set is a collection of objects with graded membership, that is, having degree of membership. Fuzzy set is an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in a binary terms according to a bivalent condition; an element either belongs or does not belong to the set. Classical bivalent sets are in fuzzy set theory called crisp sets. Fuzzy sets are generalized classical sets, since the indicator function of classical sets is special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. Fuzzy sets theory permits the gradual assessment of the membership of element in a set; this is described with the aid of a membership function valued in the real unit interval  $[0,1]$ .

Let us consider two examples:

(i) all employees of  $XYZ$  who are over 1.8m in height; (ii) all employees of  $XYZ$  who are tall. The first example is a classical set with a universe (all  $XYZ$  employees) and a membership rule that divides the universe into members (those over 1.8m) and nonmembers. The second example is a fuzzy set,

because some employees are definitely in the set and some are definitely not in the set, but some are borderline.

This distinction between the ins, the outs, and the borderline is made more exact by the membership function,  $\mu$ . If we return to our second example and let  $A$  represent the fuzzy set of all tall employees and  $x$  represent a member of the universe  $X$  (i.e. all employees), then  $\mu_A(x)$  would be  $\mu_A(x) = 1$  if  $x$  is definitely tall or  $\mu_A(x) = 0$  if  $x$  is definitely not tall or  $0 < \mu_A(x) < 1$  for borderline cases.

**Definition 2.2** [1] *The intuitionistic fuzzy sets are defined on a non-empty sets  $X$  as objects having the form  $I = \{ \langle x, \alpha_I(x), \beta_I(x) \rangle : x \in X \}$ , where  $\alpha_I(x): X \rightarrow [0,1]$  and  $\beta_I(x): X \rightarrow [0,1]$  denote the degree of membership and the degree of non-membership of each element  $x \in X$  to the set  $I$ , respectively, and  $0 \leq \alpha_I(x) + \beta_I(x) \leq 1$ , for all  $x \in X$ .*

**Definition 2.3** [1, 2, 3, 4] *Let a nonempty set  $X$  be fixed. An IFS  $A$  in  $X$  is an object having the form:  $A = \{ \langle x, \mu_A(x), \lambda_A(x) \rangle | x \in X \}$  or  $A = \left\{ \left\langle \frac{\mu_A(x), \lambda_A(x)}{x} \right\rangle | x \in X \right\}$ , where the functions  $\mu_A(x): X \rightarrow [0,1]$  and  $\lambda_A(x): X \rightarrow [0,1]$  define the degree of membership and the degree of nonmembership, respectively, of the element  $x \in X$  to  $A$ , which is a subset of  $X$ , and for every  $x \in X$ :  $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ . For each  $A$  in  $X$ :  $\pi_A(x) = 1 - \mu_A(x) - \lambda_A(x)$  is the intuitionistic fuzzy set index or hesitation margin of  $x$  in  $X$ . The hesitation margin  $\pi_A(x)$  is the degree of nondeterminacy of  $x \in X$  to the set  $A$  and  $\pi_A(x) \in [0,1]$ . The hesitation margin is the function that expresses lack of knowledge of whether  $x \in X$  or  $x \notin X$ . Thus:  $\mu_A(x) + \lambda_A(x) + \pi_A(x) = 1$ .*

**Example 2.1** *Let  $X = \{x, y, z\}$  be a fixed universe of discourse and  $A = \left\{ \left\langle \frac{0.6, 0.1}{x} \right\rangle, \left\langle \frac{0.8, 0.1}{y} \right\rangle, \left\langle \frac{0.5, 0.3}{z} \right\rangle \right\}$ , be the intuitionistic fuzzy set in  $X$ . The hesitation margins of the elements  $x, y, z$  to  $A$  are as follows:  $\pi_A(x) = 0.3, \pi_A(y) = 0.1$  and  $\pi_A(z) = 0.2$ .*

**Definition 2.4** [12, 13, 14] *Let  $X$  be a universal set. Then, a Pythagorean fuzzy set  $A$ , which is a set of ordered pairs over  $X$ , is defined by the following:  $A = \{ \langle x, \mu_A(x), \lambda_A(x) \rangle | x \in X \}$  or  $A = \left\{ \left\langle \frac{\mu_A(x), \lambda_A(x)}{x} \right\rangle | x \in X \right\}$ , where the functions  $\mu_A(x): X \rightarrow [0,1]$  and  $\lambda_A(x): X \rightarrow [0,1]$  define the degree of membership and the degree of nonmembership, respectively, of the element  $x \in X$  to  $A$ , which is a subset of  $X$ , and for every  $x \in X$ ,  $0 \leq (\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$ . Supposing  $(\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$ , then there is a degree of indeterminacy of  $x \in X$  to  $A$  defined by  $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\lambda_A(x))^2]}$  and  $\pi_A(x) \in [0,1]$ . In what follows,  $(\mu_A(x))^2 + (\lambda_A(x))^2 + (\pi_A(x))^2 = 1$ . Otherwise,  $\pi_A(x) = 0$  whenever  $(\mu_A(x))^2 + (\lambda_A(x))^2 = 1$ . We denote the set of all PFS's over  $X$  by  $pfs(X)$ .*

**Definition 2.5** [10] *Let  $X$  be a universe of discourse. A Fermatean fuzzy set ( $\mathfrak{FFs}$ )  $F$  in  $X$  is an object having the form  $F = \{ \langle x, \alpha_F(x), \beta_F(x) \rangle : x \in X \}$  where  $\alpha_F(x): X \rightarrow [0,1]$  and  $\beta_F(x): X \rightarrow [0,1]$ , including the condition  $0 \leq (\alpha_F(x))^3 + (\beta_F(x))^3 \leq 1$ , for all  $x \in X$ . The numbers  $\alpha_F(x)$  and  $\beta_F(x)$  denote, respectively, the degree of membership and the degree of non-membership of the element  $x$  in the set  $F$ . For any  $\mathfrak{FFs}$   $F$  and  $x \in X$ ,  $\pi_F(x) = \sqrt[3]{1 - [(\alpha_F(x))^3 + (\beta_F(x))^3]}$  is identified as the degree of interminancy of  $x$  to  $F$ . In the interest of simplicity, we shall mention the symbol  $F = (\alpha_F, \beta_F)$  for the  $\mathfrak{FFs}$   $F = \{ \langle x, \alpha_F(x), \beta_F(x) \rangle : x \in X \}$ .*

**Definition 2.6** [10] Let  $F = (\alpha_F, \beta_F)$ ,  $F_1 = (\alpha_{F_1}, \beta_{F_1})$  and  $F_2 = (\alpha_{F_2}, \beta_{F_2})$ , be three Fermatean fuzzy sets ( $\mathfrak{F}$ Fs's), then their operations are defined as follows: [(i)]

1.  $F_1 \cap F_2 = (\min\{\alpha_{F_1}, \alpha_{F_2}\}, \max\{\beta_{F_1}, \beta_{F_2}\})$ .
2.  $F_1 \cup F_2 = (\max\{\alpha_{F_1}, \alpha_{F_2}\}, \min\{\beta_{F_1}, \beta_{F_2}\})$ .
3.  $F^c = (\beta_F, \alpha_F)$ .

**Remark 2.1** If  $\alpha_{F_1} = \alpha_{F_2}$  and  $\beta_{F_1} = \beta_{F_2}$ , then  $F_1 = F_2$

Note that, for understanding the Fermatean fuzzy set better, we give an instance to illuminate the understandability of the Fermatean fuzzy set. The point when someone needs will plan as much craving for the level for an alternative  $s_i$  on a criterion  $C_j$ , he might provide for the degree on which that alternative  $s_i$  fulfils those criteria  $C_j$  likewise 0.85, what is more correspondingly the elective  $s_i$  dissatisfies the criterion  $C_j$  similarly as 0.65. We can definitely get  $0.85 + 0.65 = 1.5 > 1$ , and, therefore, it does not follow the condition of intuitionistic fuzzy sets. Also, we can get  $(0.85)^2 + (0.65)^2 = 0.7225 + 0.4225 = 1.145 > 1$ , which does not obey the constraint condition of Pythagorean fuzzy set. However, we can get  $(0.85)^3 + (0.65)^3 = 0.614125 + 0.274625 = 0.88875 \leq 1$ , which is good enough to apply the Fermatean fuzzy set to control it [10]. Throughout this paper, we use the notation  $1_{\mathfrak{F}}$  for the Fermatean fuzzy subset (1,0) and we use the notation  $0_{\mathfrak{F}}$  for the Fermatean fuzzy subset (0,1), that is,  $\alpha_{1_{\mathfrak{F}}} = 1, \beta_{1_{\mathfrak{F}}} = 0, \alpha_{0_{\mathfrak{F}}} = 0, \beta_{0_{\mathfrak{F}}} = 1$ . A Fermatean fuzzy subset  $\mathfrak{F}$  of a non-empty set  $X$  is a pair  $(\alpha_{\mathfrak{F}}, \beta_{\mathfrak{F}})$  of a membership function  $(\alpha_{\mathfrak{F}}(x): X \rightarrow [0,1])$  and a non-membership function  $(\beta_{\mathfrak{F}}(x): X \rightarrow [0,1])$  with  $(\alpha_{\mathfrak{F}}(x))^3 + (\beta_{\mathfrak{F}}(x))^3 = (\gamma_{\mathfrak{F}}(x))^3$  for any  $x \in X$  where  $\gamma_{\mathfrak{F}}(x): X \rightarrow [0,1]$  is a function which is called the strength of commitment at point  $x$ .

**Definition 2.7** [8] Let  $X$  be a non empty set and  $\tau$  be a family of Fermatean fuzzy subsets of  $X$ . If

1.  $1_{\mathfrak{F}}, 0_{\mathfrak{F}} \in \tau$
2. for any  $F_1, F_2 \in \tau$ , we have  $F_1 \cap F_2 \in \tau$ ,
3. for any  $\{F_i\}_{i \in I} \subset \tau$ , we have  $\bigcup_{i \in I} F_i \in \tau$  where  $I$  is an arbitrary index set then  $\tau$  is called a Fermatean fuzzy topology on  $X$ .

The pair  $(X, \tau)$  is said to be a Fermatean fuzzy topological space. Each member of  $\tau$  is called an Fermatean fuzzy open set. The complement of an Fermatean fuzzy open set is called a Fermatean fuzzy closed set.

**Remark 2.2** [8] As any Intuitionistic fuzzy subset or Pythagorean fuzzy subset of a set can be considered as Fermatean fuzzy subset, we observe that any Intuitionistic fuzzy topological space or Pythagorean fuzzy topological space is a Fermatean fuzzy topological space as well. On the other hand, it is obvious that a Fermatean fuzzy topological space need not be Intuitionistic fuzzy topological space and Pythagorean fuzzy topological space. Even an Fermatean fuzzy open set maybe neither an Intuitionistic fuzzy set nor Pythagorean fuzzy set.

**Example 2.2** [8] Let  $X = \{c_1, c_2\}$ . Consider the following family Fermatean fuzzy subsets  $\tau = \{1_{\mathfrak{F}}, 0_{\mathfrak{F}}, F_1, F_2\}$  where

$$F_1 = \{\langle c_1, \alpha_{F_1}(c_1) = 0.4, \beta_{F_1}(c_1) = 0.6 \rangle, \langle c_2, \alpha_{F_1}(c_2) = 0.1, \beta_{F_1}(c_2) = 0.3 \rangle\}$$

and  $F_2 = \{\langle c_1, \alpha_{F_2}(c_1) = 0.9, \beta_{F_2}(c_1) = 0.6 \rangle, \langle c_2, \alpha_{F_2}(c_2) = 0.2, \beta_{F_2}(c_2) = 0.3 \rangle\}$ . Observe that  $(X, \tau)$  is a Fermatean fuzzy topological space but  $(X, \tau)$  is neither Intuitionistic fuzzy topological space nor Pythagorean fuzzy topological space.

**Definition 2.8** [8] Let  $(X, \tau)$  be an  $\mathfrak{F}$ Ts and  $A = \{\langle a, \alpha_A(a), \beta_A(a) \rangle \mid a \in X\}$  be an  $\mathfrak{F}$ S in  $X$ . Then the Fermatean fuzzy interior and the Fermatean fuzzy closure of  $A$  are denoted by  $\mathfrak{F}Fint(A)$  and  $\mathfrak{F}Fcl(A)$  and are defined as follows:  $\mathfrak{F}Fint(A) = \cup \{G \mid G \text{ is a } \mathfrak{F}Fos \text{ and } G \subseteq A\}$  and  $\mathfrak{F}Fcl(A) = \cap \{K \mid K \text{ is a } \mathfrak{F}Fcs \text{ and } A \subseteq K\}$ . Also, it can be established that  $\mathfrak{F}Fcl(A)$  is an  $\mathfrak{F}Fcs$  and  $\mathfrak{F}Fint(A)$  is an  $\mathfrak{F}Fos$ ,  $A$  is an  $\mathfrak{F}Fcs$  if and only if  $\mathfrak{F}Fcl(A) = A$  and  $A$  is an  $\mathfrak{F}Fos$  if and only if  $\mathfrak{F}Fint(A) = A$ . We say that  $A$  is  $\mathfrak{F}$ F-dense if  $\mathfrak{F}Fcl(A) = 1_{\mathfrak{F}}$ .

**Lemma 2.1** [8] For any Fermatean fuzzy set  $A$  in  $(X, \tau)$ , we have  $1_{\mathfrak{F}} - \mathfrak{F}Fint(A) = \mathfrak{F}Fcl(1_{\mathfrak{F}} - A)$  and  $1_{\mathfrak{F}} - \mathfrak{F}Fcl(A) = \mathfrak{F}Fint(1_{\mathfrak{F}} - A)$ .

**Definition 2.9** [11] Let  $(X, \tau)$  be an  $\mathfrak{F}$ Ts and  $A$  be an  $\mathfrak{F}$ S. Then  $A$  is said to be an Fermatean fuzzy (i) regular open set ( $\mathfrak{F}Fros$  in short) if  $A = \mathfrak{F}Fint(\mathfrak{F}Fcl(A))$ . (ii) regular closed set ( $\mathfrak{F}Frcs$  in short) if  $A = \mathfrak{F}Fcl(\mathfrak{F}Fint(A))$ . By Lemma 2.1, it follows that  $A$  is an  $\mathfrak{F}Fros$  iff  $\bar{A}$  is an  $\mathfrak{F}Frcs$ .

**Definition 2.10** [11] Let  $(X, \tau)$  be an  $\mathfrak{F}$ Ts and  $A = \{\langle a, \alpha_A(a), \beta_A(a) \rangle \mid a \in X\}$  be an  $\mathfrak{F}$ S in  $X$ . Then the  $\delta$ -interior and the  $\delta$ -closure of  $A$  are denoted by  $\mathfrak{F}F\delta int(A)$  and  $\mathfrak{F}F\delta cl(A)$  and are defined as follows.  $\mathfrak{F}F\delta int(A) = \cup \{G \mid G \text{ is an } \mathfrak{F}Fros \text{ and } G \subseteq A\}$ ,  $\mathfrak{F}F\delta cl(A) = \cap \{K \mid K \text{ is an } \mathfrak{F}Frcs \text{ and } A \subseteq K\}$ .

**Definition 2.11** [11] Let  $(X, \tau)$  be an  $\mathfrak{F}$ Ts and  $A = \{\langle a, \alpha_A(a), \beta_A(a) \rangle \mid a \in X\}$  be an  $\mathfrak{F}$ S in  $X$ . A set  $A$  is said to be  $\mathfrak{F}$ F

1.  $\delta$ -open set (briefly,  $\mathfrak{F}F\delta os$ ) if  $A = \mathfrak{F}F\delta int(A)$ ,
2.  $\delta$ -pre open set (briefly,  $\mathfrak{F}F\delta Pos$ ) if  $A \subseteq \mathfrak{F}Fint(\mathfrak{F}F\delta cl(A))$ .
3.  $\delta$ -semi open set (briefly,  $\mathfrak{F}F\delta Sos$ ) if  $A \subseteq \mathfrak{F}Fcl(\mathfrak{F}F\delta int(A))$ .
4.  $\delta - \alpha$  open set or  $a$ -open set (briefly,  $\mathfrak{F}F\delta aos$  or  $\mathfrak{F}Faos$ ) if  $A \subseteq \mathfrak{F}Fint(\mathfrak{F}Fcl(\mathfrak{F}F\delta int(A)))$ .
5.  $\delta - \beta$  open set or  $e^*$ -open set (briefly,  $\mathfrak{F}F\delta bos$  or  $\mathfrak{F}Fe^*os$ ) if  $A \subseteq \mathfrak{F}Fcl(\mathfrak{F}Fint(\mathfrak{F}F\delta cl(A)))$ .
6.  $\delta$  (resp.  $\delta$ -pre,  $\delta$ -semi,  $\delta - \alpha$  and  $\delta - \beta$ ) dense if  $\mathfrak{F}F\delta cl(A)$  (resp.  $\mathfrak{F}F\delta pcl(A)$ ,  $\mathfrak{F}F\delta scl(A)$ ,  $\mathfrak{F}F\delta acl(A)$  and  $\mathfrak{F}F\delta \beta cl(A)$ ) =  $1_{\mathfrak{F}}$ .

The complement of an  $\mathfrak{F}F\delta os$  (resp.  $\mathfrak{F}F\delta Pos$ ,  $\mathfrak{F}F\delta Sos$ ,  $\mathfrak{F}F\delta aos$  and  $\mathfrak{F}F\delta bos$ ) is called an  $\mathfrak{F}F\delta$  (resp.  $\mathfrak{F}F\delta P$ ,  $\mathfrak{F}F\delta S$ ,  $\mathfrak{F}F\delta a$  and  $\mathfrak{F}F\delta \beta$ ) closed set (briefly,  $\mathfrak{F}F\delta cs$  (resp.  $\mathfrak{F}F\delta Pcs$ ,  $\mathfrak{F}F\delta Scs$ ,  $\mathfrak{F}F\delta acs$  and  $\mathfrak{F}F\delta \beta cs$ )) in  $X$ .

The family of all  $\mathfrak{F}F\delta os$  (resp.  $\mathfrak{F}F\delta cs$ ,  $\mathfrak{F}F\delta Pos$ ,  $\mathfrak{F}F\delta Pcs$ ,  $\mathfrak{F}F\delta Sos$ ,  $\mathfrak{F}F\delta Scs$ ,  $\mathfrak{F}F\delta aos$ ,  $\mathfrak{F}F\delta acs$ ,  $\mathfrak{F}F\delta bos$  and  $\mathfrak{F}F\delta \beta cs$ ) of  $X$  is denoted by  $\mathfrak{F}F\delta OS(X)$  (resp.  $\mathfrak{F}F\delta CS(X)$ ,  $\mathfrak{F}F\delta POS(X)$ ,

$\mathfrak{F}\delta\mathcal{PCS}(X)$ ,  $\mathfrak{F}\delta\mathcal{SOS}(X)$ ,  $\mathfrak{F}\delta\mathcal{SCS}(X)$ ,  $\mathfrak{F}\delta\alpha\mathcal{OS}(X)$ ,  $\mathfrak{F}\delta\alpha\mathcal{CS}(X)$ ,  $\mathfrak{F}\delta\beta\mathcal{OS}(X)$  and  $\mathfrak{F}\delta\beta\mathcal{CS}(X)$ .

**Definition 2.12** [11] Let  $(X, \tau)$  be an  $\mathfrak{F}\mathcal{Fts}$  and  $A = \{ \langle a, \alpha_A(a), \beta_A(a) \rangle \mid a \in X \}$  be an  $\mathfrak{F}\mathcal{F}s$  in  $X$ . Then the  $\mathfrak{F}\mathcal{F}\delta$ -pre (resp.  $\mathfrak{F}\mathcal{F}\delta$ -semi,  $\mathfrak{F}\mathcal{F}\delta\alpha$  and  $\mathfrak{F}\mathcal{F}\delta\beta$ )-interior and the  $\mathfrak{F}\mathcal{F}\delta$ -pre (resp.  $\mathfrak{F}\mathcal{F}\delta$ -semi,  $\mathfrak{F}\mathcal{F}\delta\alpha$  and  $\mathfrak{F}\mathcal{F}\delta\beta$ )-closure of  $A$  are denoted by  $\mathfrak{F}\mathcal{F}\delta\mathcal{Pint}(A)$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{Sint}(A)$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{int}(A)$  and  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{int}(A)$ ) and the  $\mathfrak{F}\mathcal{F}\delta\mathcal{Pcl}(A)$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{Scl}(A)$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{acl}(A)$  and  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{acl}(A)$ ) and are defined as follows:

$\mathfrak{F}\mathcal{F}\delta\mathcal{Pint}(A)$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{Sint}(A)$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{int}(A)$  and  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{int}(A)$ ) =  $\cup \{G \mid G$  in a  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}os$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}os$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{o}s$  and  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{o}s$ ) and  $G \subseteq A\}$  and  $\mathfrak{F}\mathcal{F}\delta\mathcal{Pcl}(A)$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{Scl}(A)$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{acl}(A)$  and  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{acl}(A)$ ) =  $\cap \{K \mid K$  is an  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}cs$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}cs$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{cs}$ ,  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{cs}$ ) and  $A \subseteq K\}$ .

### 3 Fermatean fuzzy $\delta\beta$ -homeomorphism

In this section, we introduce Fermatean fuzzy  $\delta$  (resp.  $\delta$  pre,  $\delta$  semi,  $\delta\alpha$  and  $\delta\beta$ )-homeomorphism and discuss some of their properties.

**Definition 3.1** Let  $(X_1, \tau_1)$  and  $(X_2, \tau_2)$  be two  $\mathfrak{F}\mathcal{Fts}$ 's. Then a function  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  is said to be a Fermatean fuzzy  $\delta$  (resp.  $\delta$  pre,  $\delta$  semi,  $\delta\alpha$  and  $\delta\beta$ ) continuous (briefly,  $\mathfrak{F}\mathcal{F}\delta\mathcal{C}ts$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{C}ts$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{C}ts$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{C}ts$  and  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}ts$ )) function if  $h_{\mathfrak{F}}^{-1}(G)$  is  $\mathfrak{F}\mathcal{F}\delta\mathcal{o}$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{o}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{o}$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{o}$  &  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{o}$ ) set in  $X_1$  for all  $\mathfrak{F}\mathcal{F}\mathcal{o}$  set  $G$  in  $X_2$ .

**Definition 3.2** Let  $(X_1, \tau_1)$  &  $(X_2, \tau_2)$  be a  $\mathfrak{F}\mathcal{Fts}$ 's. A mapping  $h_P: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  is said to be a Fermatean fuzzy (resp.  $\delta$ ,  $\delta\mathcal{P}$ ,  $\delta\mathcal{S}$ ,  $\delta\alpha$  &  $\delta\beta$  or  $e^*$ )-open map (briefly,  $\mathfrak{F}\mathcal{F}\mathcal{O}$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{O}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{O}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{O}$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{O}$ , &  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{O}$  or  $\mathfrak{F}\mathcal{F}e^*\mathcal{O}$ )) if the image of every  $\mathfrak{F}\mathcal{F}\mathcal{O}s$  in  $X_1$  is a  $\mathfrak{F}\mathcal{F}\mathcal{O}s$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{o}s$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{o}s$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{o}s$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{o}s$  &  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{o}s$  or  $\mathfrak{F}\mathcal{F}e^*\mathcal{o}s$ ) in  $X_2$ .

**Definition 3.3** A bijection  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  is called a Fermatean fuzzy (resp.  $\delta$ ,  $\delta\alpha$ ,  $\delta\mathcal{S}$ ,  $\delta\mathcal{P}$  &  $\delta\beta$  or  $e^*$ )-homeomorphism (briefly,  $\mathfrak{F}\mathcal{F}\mathcal{H}om$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{H}om$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{H}om$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{H}om$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{H}om$  &  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{H}om$  or  $\mathfrak{F}\mathcal{F}e^*\mathcal{H}om$ )) if  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\mathcal{C}ts$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{C}ts$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{C}ts$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{C}ts$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{C}ts$  &  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}ts$  or  $\mathfrak{F}\mathcal{F}e^*\mathcal{C}ts$ ) mappings.

**Theorem 3.1** Let  $(X_1, \tau_1)$  &  $(X_2, \tau_2)$  be a  $\mathfrak{F}\mathcal{Fts}$ 's. Let  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  be a mapping. Then the following statements are hold for  $\mathfrak{F}\mathcal{Fts}$ , but not conversely.

- (i) Every  $\mathfrak{F}\mathcal{F}\delta\mathcal{H}om$  is a  $\mathfrak{F}\mathcal{F}\mathcal{H}om$ .
- (ii) Every  $\mathfrak{F}\mathcal{F}\delta\mathcal{H}om$  is a  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{H}om$ .
- (iii) Every  $\mathfrak{F}\mathcal{F}\delta\mathcal{H}om$  is a  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{H}om$ .
- (iv) Every  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{H}om$  is a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{H}om$ .
- (v) Every  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{H}om$  is a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{H}om$ .
- (vi) Every  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{H}om$  is a  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{H}om$ .
- (vii) Every  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{H}om$  is a  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{H}om$ .

**Proof.** (i) Let  $h_{\mathfrak{F}}$  be  $\mathfrak{F}\mathcal{F}\delta Hom$ , then  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta Cts$ . But every  $\mathfrak{F}\mathcal{F}\delta Cts$  function is  $\mathfrak{F}\mathcal{F}Cts$ . Hence,  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}Cts$ . Therefore,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}Hom$ .

(ii) Let  $h_{\mathfrak{F}}$  be  $\mathfrak{F}\mathcal{F}\delta Hom$ , then  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta Cts$ . But every  $\mathfrak{F}\mathcal{F}\delta Cts$  function is  $\mathfrak{F}\mathcal{F}\delta SCts$ . Hence,  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta SCts$ . Therefore,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta SHom$ .

(iii) Let  $h_{\mathfrak{F}}$  be  $\mathfrak{F}\mathcal{F}\delta Hom$ , then  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta Cts$ . But every  $\mathfrak{F}\mathcal{F}\delta Cts$  function is  $\mathfrak{F}\mathcal{F}\delta PCts$ . Hence,  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta PCts$ . Therefore,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta PHom$ .

(iv) Let  $h_{\mathfrak{F}}$  be  $\mathfrak{F}\mathcal{F}\delta SHom$ , then  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta SCts$ . But every  $\mathfrak{F}\mathcal{F}\delta SCts$  function is  $\mathfrak{F}\mathcal{F}\delta\beta Cts$ . Hence,  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta\beta Cts$ . Therefore,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta Hom$ .

(v) Let  $h_{\mathfrak{F}}$  be  $\mathfrak{F}\mathcal{F}\delta PHom$ , then  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta PCts$ . But every  $\mathfrak{F}\mathcal{F}\delta PCts$  function is  $\mathfrak{F}\mathcal{F}\delta\beta Cts$ . Hence,  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta\beta Cts$ . Therefore,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta Hom$ .

(vi) Let  $h_{\mathfrak{F}}$  be  $\mathfrak{F}\mathcal{F}\delta\alpha Hom$ , then  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta\alpha Cts$ . But every  $\mathfrak{F}\mathcal{F}\delta\alpha Cts$  function is  $\mathfrak{F}\mathcal{F}\delta SCts$ . Hence,  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta SCts$ . Therefore,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta SHom$ .

(vii) Let  $h_{\mathfrak{F}}$  be  $\mathfrak{F}\mathcal{F}\delta\alpha Hom$ , then  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta\alpha Cts$ . But every  $\mathfrak{F}\mathcal{F}\delta\alpha Cts$  function is  $\mathfrak{F}\mathcal{F}\delta PCts$ . Hence,  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta PCts$ . Therefore,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta PHom$ .

**Example 3.1** Let  $X_1 = X_2 = X = \{a, b\}$  and the  $\mathfrak{F}\mathcal{F}s$ 's  $A_1, A_2, A_3$  and  $A_4$  are defined as

$$\alpha_{A_1}(a) = 0.2, \beta_{A_1}(a) = 0.8, \alpha_{A_1}(b) = 0.4, \beta_{A_1}(b) = 0.6;$$

$$\alpha_{A_2}(a) = 0.1, \beta_{A_2}(a) = 0.9, \alpha_{A_2}(b) = 0.3, \beta_{A_2}(b) = 0.7;$$

$$\alpha_{A_3}(a) = 0.9, \beta_{A_3}(a) = 0.1, \alpha_{A_3}(b) = 0.7, \beta_{A_3}(b) = 0.7;$$

$$\alpha_{A_4}(a) = 0.2, \beta_{A_4}(a) = 0.8, \alpha_{A_4}(b) = 0.3, \beta_{A_4}(b) = 0.7;$$

Let  $\tau_1 = \tau_2 = \tau = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, A_1, A_2, A_3, A_4\}$  be a  $\mathfrak{F}\mathcal{F}ts$  on  $X$  and let  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  be an identity function, Then  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}Hom$  (resp.  $\mathfrak{F}\mathcal{F}\delta\beta Hom$ ,  $\mathfrak{F}\mathcal{F}\delta PHom$  and  $\mathfrak{F}\mathcal{F}\delta PHom$ ) but not  $\mathfrak{F}\mathcal{F}\delta Hom$  (resp.  $\mathfrak{F}\mathcal{F}\delta SHom$ ,  $\mathfrak{F}\mathcal{F}\delta Hom$  and  $\mathfrak{F}\mathcal{F}\delta\alpha Hom$ ). Since,  $A_4$  is a  $\mathfrak{F}\mathcal{F}o$  set in  $X_2$  but  $h_{\mathfrak{F}}^{-1}(A_4) = A_4$  is not  $\mathfrak{F}\mathcal{F}\delta o$  (resp.  $\mathfrak{F}\mathcal{F}\delta So$ ,  $\mathfrak{F}\mathcal{F}\delta o$  and  $\mathfrak{F}\mathcal{F}\delta\alpha o$ ) set in  $X_1$ .

**Example 3.2** Let  $X_1 = X_2 = X = \{a, b\}$  and the  $\mathfrak{F}\mathcal{F}s$ 's  $A_1, A_2, A_3$  and  $A_4$  are defined as

$$\alpha_{A_1}(a) = 0.4, \beta_{A_1}(a) = 0.6, \alpha_{A_1}(b) = 0.5, \beta_{A_1}(b) = 0.5;$$

$$\alpha_{A_2}(a) = 0.6, \beta_{A_2}(a) = 0.4, \alpha_{A_2}(b) = 0.6, \beta_{A_2}(b) = 0.4;$$

$$\alpha_{A_3}(a) = 0.7, \beta_{A_3}(a) = 0.3, \alpha_{A_3}(b) = 0.6, \beta_{A_3}(b) = 0.4;$$

$$\alpha_{A_4}(a) = 0.4, \beta_{A_4}(a) = 0.6, \alpha_{A_4}(b) = 0.4, \beta_{A_4}(b) = 0.6;$$

Let  $\tau_1 = \tau_2 = \tau = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, A_1, A_2, A_3, A_4\}$  be a  $\mathfrak{F}\mathcal{F}ts$  on  $X$  and let  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  be an identity function, Then  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta SHom$  but not  $\mathfrak{F}\mathcal{F}\delta Hom$ . Since,  $A_3$  is a  $\mathfrak{F}\mathcal{F}o$  set in  $X_2$  but  $h_{\mathfrak{F}}^{-1}(A_3) = A_3$  is not  $\mathfrak{F}\mathcal{F}\delta o$  set in  $X_1$ .

**Example 3.3** Let  $X_1 = X_2 = X = \{a, b\}$  and the  $\mathfrak{F}$ Fs's  $A_1, A_2, B_1$  and  $B_2$  are defined as

$$\alpha_{A_1}(a) = 0.2, \beta_{A_1}(a) = 0.7, \alpha_{A_1}(b) = 0.1, \beta_{A_1}(b) = 0.8;$$

$$\alpha_{A_2}(a) = 0.3, \beta_{A_2}(a) = 0.6, \alpha_{A_2}(b) = 0.4, \beta_{A_2}(b) = 0.5;$$

$$\alpha_{B_1}(a) = 0.1, \beta_{B_1}(a) = 0.9, \alpha_{B_1}(b) = 0.2, \beta_{B_1}(b) = 0.9;$$

$$\alpha_{B_2}(a) = 0.2, \beta_{B_2}(a) = 0.3, \alpha_{B_2}(b) = 0.4, \beta_{B_2}(b) = 0.7;$$

Let  $\tau_1 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, A_1, A_2\}$  and  $\tau_2 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, B_1, B_2\}$  are  $\mathfrak{F}$ Fts's on  $X$  and let  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  be an identity function, Then  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta\beta Hom$  but not  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}Hom$ . Since,  $A_2$  is a  $\mathfrak{F}\mathcal{F}o$  set in  $X_2$  but  $h_{\mathfrak{F}}^{-1}(A_2) = A_2$  is not  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}o$  set in  $X_1$ .

**Theorem 3.2** Let  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  be a bijective mapping. If  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}Cts$  (resp.  $\mathfrak{F}\mathcal{F}\delta Cts$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha Cts$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}Cts$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}Cts$  &  $\mathfrak{F}\mathcal{F}\delta\beta Cts$ ), then the followings statements are equivalent:

(i)  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}C$  (resp.  $\mathfrak{F}\mathcal{F}\delta C$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha C$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}C$  &  $\mathfrak{F}\mathcal{F}\delta\beta C$  or  $\mathfrak{F}\mathcal{F}e^*C$ ) mapping.

(ii)  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}O$  (resp.  $\mathfrak{F}\mathcal{F}\delta O$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha O$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}O$  &  $\mathfrak{F}\mathcal{F}\delta\beta O$  or  $\mathfrak{F}\mathcal{F}e^*C$ ) mapping.

(iii)  $h_{\mathfrak{F}}^{-1}$  is a  $\mathfrak{F}\mathcal{F}Hom$  (resp.  $\mathfrak{F}\mathcal{F}\delta Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}Hom$  &  $\mathfrak{F}\mathcal{F}\delta\beta Hom$  or  $\mathfrak{F}\mathcal{F}e^*Hom$ ).

**Proof.** (i)  $\Rightarrow$  (ii) : Assume that  $h_{\mathfrak{F}}$  is a bijective mapping and a  $\mathfrak{F}\mathcal{F}\delta\beta C$  mapping. Hence,  $h_{\mathfrak{F}}^{-1}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta Cts$  mapping. We know that each  $\mathfrak{F}\mathcal{F}os$  in  $(X_1, \tau_1)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta os$  in  $(X_2, \tau_2)$ . Hence,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta O$  mapping.

(ii)  $\Rightarrow$  (iii) : Let  $h_{\mathfrak{F}}$  be a bijective and  $\mathfrak{F}\mathcal{F}\delta\beta O$  mapping. Further,  $h_{\mathfrak{F}}^{-1}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta Cts$  mapping. Hence,  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta\beta Cts$ . Therefore,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta Hom$ .

(iii)  $\Rightarrow$  (i): Let  $h_{\mathfrak{F}}$  be a  $\mathfrak{F}\mathcal{F}\delta\beta Hom$ . Then  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta\beta Cts$ . Since each  $\mathfrak{F}\mathcal{F}cs$  in  $(X_1, \tau_1)$  is  $\mathfrak{F}\mathcal{F}cs$  in  $(X_1, \tau_1)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_2, \tau_2)$ ,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta C$  mapping. The proof of other cases are similar.

**Definition 3.4** A  $\mathfrak{F}\mathcal{F}ts (X, \tau)$  is said to be a Fermatean fuzzy  $\alpha T_{\frac{1}{2}}$  (resp.  $\delta ST_{\frac{1}{2}}$ ,  $\delta PT_{\frac{1}{2}}$  and  $\delta\beta T_{\frac{1}{2}}$ ) (briefly,  $\mathfrak{F}\mathcal{F}\delta\alpha T_{\frac{1}{2}}$  (resp.  $\mathfrak{F}\mathcal{F}\delta ST_{\frac{1}{2}}$ ,  $\mathfrak{F}\mathcal{F}\delta PT_{\frac{1}{2}}$  and  $\mathfrak{F}\mathcal{F}\delta\beta T_{\frac{1}{2}}$ ))-space if every  $\mathfrak{F}\mathcal{F}\delta\alpha cs$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}cs$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}cs$ , and  $\mathfrak{F}\mathcal{F}\delta\beta cs$ ) is  $\mathfrak{F}\mathcal{F}cs$  in  $(X, \tau)$ .

**Theorem 3.3** Let  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  be a  $\mathfrak{F}\mathcal{F}\delta\alpha Hom$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}Hom$  &  $\mathfrak{F}\mathcal{F}\delta\beta Hom$  or  $\mathfrak{F}\mathcal{F}e^*Hom$ ). Then  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}Hom$  if  $(X_1, \tau_1)$  and  $(X_2, \tau_2)$  are  $\mathfrak{F}\mathcal{F}\delta\alpha T_{\frac{1}{2}}$  (resp.  $\mathfrak{F}\mathcal{F}\delta ST_{\frac{1}{2}}$ ,  $\mathfrak{F}\mathcal{F}\delta PT_{\frac{1}{2}}$  and  $\mathfrak{F}\mathcal{F}\delta\beta T_{\frac{1}{2}}$ )-space.

**Proof.** Assume that  $K$  is a  $\mathfrak{F}\mathcal{F}cs$  in  $(X_2, \tau_2)$ . Then  $h_{\mathfrak{F}}^{-1}(K)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_1, \tau_1)$ . Since  $(X_1, \tau_1)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta T_{\frac{1}{2}}$ -space,  $h_{\mathfrak{F}}^{-1}(K)$  is a  $\mathfrak{F}\mathcal{F}cs$  in  $(X_1, \tau_1)$ . Therefore,  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}Cts$ . By hypothesis,  $h_{\mathfrak{F}}^{-1}$  is  $\mathfrak{F}\mathcal{F}\delta\beta Cts$ . Let  $L$  be a  $\mathfrak{F}\mathcal{F}cs$  in  $(X_1, \tau_1)$ . Then,  $(h_{\mathfrak{F}}^{-1})^{-1}(L) = h_{\mathfrak{F}}(L)$  is a  $\mathfrak{F}\mathcal{F}cs$

in  $(X_2, \tau_2)$ , by presumption. Since  $(X_2, \tau_2)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta T_{\frac{1}{2}}$ -space,  $h_{\mathfrak{F}}(L)$  is a  $\mathfrak{F}\mathcal{F}cs$  in  $(X_2, \tau_2)$ . Hence,  $h_{\mathfrak{F}}^{-1}$  is  $\mathfrak{F}\mathcal{F}Cts$ . Hence,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}Hom$ . Proof of other cases are similar.

**Theorem 3.4** *Let  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  be a mapping. Then the following are equivalent if  $(X_2, \tau_2)$  is a  $\mathfrak{F}\mathcal{F}\delta\alpha T_{\frac{1}{2}}$  (resp.  $\mathfrak{F}\mathcal{F}\delta\delta T_{\frac{1}{2}}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}T_{\frac{1}{2}}$  and  $\mathfrak{F}\mathcal{F}\delta\beta T_{\frac{1}{2}}$ )-space:*

- (i)  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta\alpha C$  (resp.  $\mathfrak{F}\mathcal{F}\delta\delta C$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}C$  and  $\mathfrak{F}\mathcal{F}\delta\beta C$ ) mapping.
- (ii) If  $K$  is a  $\mathfrak{F}\mathcal{F}os$  in  $(X_1, \tau_1)$ , then  $h_{\mathfrak{F}}(X_1, \tau_1)$  is  $\mathfrak{F}\mathcal{F}\delta\alpha os$  (resp.  $\mathfrak{F}\mathcal{F}\delta\delta os$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}os$  and  $\mathfrak{F}\mathcal{F}\delta\beta os$ ) in  $(X_2, \tau_2)$ .
- (iii)  $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(K)) \subseteq \mathfrak{F}\mathcal{F}cl(\mathfrak{F}\mathcal{F}int(h_{\mathfrak{F}}(K)))$  for every  $\mathfrak{F}\mathcal{F}s$   $K$  in  $(X_1, \tau_1)$ .

**Proof.** (i)  $\Rightarrow$  (ii): Obvious.

(ii)  $\Rightarrow$  (iii): Let  $K$  be a  $\mathfrak{F}\mathcal{F}s$  in  $(X_1, \tau_1)$ . Then,  $\mathfrak{F}\mathcal{F}int(K)$  is a  $\mathfrak{F}\mathcal{F}os$  in  $(X_1, \tau_1)$ . Then,  $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(K))$  is a  $\mathfrak{F}\mathcal{F}\delta\beta os$  in  $(X_2, \tau_2)$ . Since  $(X_2, \tau_2)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta T_{\frac{1}{2}}$ -space, so  $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(K))$  is a  $\mathfrak{F}\mathcal{F}os$  in  $(X_2, \tau_2)$ .

Therefore,  $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(K)) = \mathfrak{F}\mathcal{F}int(h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(K))) \subseteq \mathfrak{F}\mathcal{F}cl(\mathfrak{F}\mathcal{F}int(h_{\mathfrak{F}}(K)))$ .

(iii)  $\Rightarrow$  (i): Let  $K$  be a  $\mathfrak{F}\mathcal{F}cs$  in  $(X_1, \tau_1)$ . Then,  $K^c$  is a  $\mathfrak{F}\mathcal{F}os$  in  $(X_1, \tau_1)$ . From,  $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(K)^c) \subseteq \mathfrak{F}\mathcal{F}cl(\mathfrak{F}\mathcal{F}int(h_{\mathfrak{F}}(K)^c))$ ,  $h_{\mathfrak{F}}((K)^c) \subseteq \mathfrak{F}\mathcal{F}cl(\mathfrak{F}\mathcal{F}int(h_{\mathfrak{F}}(K)^c))$ . Therefore,  $h_{\mathfrak{F}}((K)^c)$  is  $\mathfrak{F}\mathcal{F}\delta\beta os$  in  $(X_2, \tau_2)$ . Therefore,  $h_{\mathfrak{F}}(K)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_1, \tau_1)$ . Hence,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}C$  mapping. The proof of other cases are similar.

**Theorem 3.5** *Let  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  and  $g_{\mathfrak{F}}: (X_2, \tau_2) \rightarrow (X_3, \tau_3)$  be  $\mathfrak{F}\mathcal{F}\delta\alpha C$  (resp.  $\mathfrak{F}\mathcal{F}\delta\delta C$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}C$  and  $\mathfrak{F}\mathcal{F}\delta\beta C$ ), where  $(X_1, \tau_1)$  and  $(X_3, \tau_3)$  are two  $\mathfrak{F}\mathcal{F}ts$ 's and  $(X_2, \tau_2)$  a  $\mathfrak{F}\mathcal{F}\delta T_{\frac{1}{2}}$  (resp.  $\mathfrak{F}\mathcal{F}\delta\alpha T_{\frac{1}{2}}$ ,  $\mathfrak{F}\mathcal{F}\delta\delta T_{\frac{1}{2}}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}T_{\frac{1}{2}}$  and  $\mathfrak{F}\mathcal{F}\delta\beta T_{\frac{1}{2}}$ )-space, then the composition  $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta\alpha C$  (resp.  $\mathfrak{F}\mathcal{F}\delta\delta C$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}C$  and  $\mathfrak{F}\mathcal{F}\delta\beta C$ ).*

**Proof.** Let  $K$  be a  $\mathfrak{F}\mathcal{F}cs$  in  $(X_1, \tau_1)$ . Since  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta\beta C$  and  $h_{\mathfrak{F}}(K)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_2, \tau_2)$ , by assumption,  $h_{\mathfrak{F}}(K)$  is a  $\mathfrak{F}\mathcal{F}cs$  in  $(X_2, \tau_2)$ . Since  $g_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta\beta C$ , then  $g_{\mathfrak{F}}(h_{\mathfrak{F}}(K))$  is  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_3, \tau_3)$  and  $g_{\mathfrak{F}}(h_{\mathfrak{F}}(K)) = (g_{\mathfrak{F}} \circ h_{\mathfrak{F}})(K)$ . Therefore,  $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta\beta C$ .

**Theorem 3.6** *Let  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  and  $g_{\mathfrak{F}}: (X_2, \tau_2) \rightarrow (X_3, \tau_3)$  be two  $\mathfrak{F}\mathcal{F}ts$ 's, then the following hold:*

- (i) If  $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta O$  (resp.  $\mathfrak{F}\mathcal{F}\delta\alpha O$ ,  $\mathfrak{F}\mathcal{F}\delta\delta O$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}O$  and  $\mathfrak{F}\mathcal{F}\delta\beta O$ ) and  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}Cts$ , then  $g_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta O$  (resp.  $\mathfrak{F}\mathcal{F}\delta\alpha O$ ,  $\mathfrak{F}\mathcal{F}\delta\delta O$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}O$  and  $\mathfrak{F}\mathcal{F}\delta\beta O$ ).
- (ii) If  $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}O$  and  $g_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta Cts$  (resp.  $\mathfrak{F}\mathcal{F}\delta\alpha Cts$ ,  $\mathfrak{F}\mathcal{F}\delta\delta Cts$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}Cts$  and  $\mathfrak{F}\mathcal{F}\delta\beta Cts$ ), then  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta O$  (resp.  $\mathfrak{F}\mathcal{F}\delta\alpha O$ ,  $\mathfrak{F}\mathcal{F}\delta\delta O$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}O$  and  $\mathfrak{F}\mathcal{F}\delta\beta O$ ).

**Proof.** (i) Let  $K$  be a  $\mathfrak{F}\mathcal{F}os$  in  $(X_2, \tau_2)$ . As  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}Cts$  mapping,  $h_{\mathfrak{F}}^{-1}(K)$  is  $\mathfrak{F}\mathcal{F}os$  in  $(X_1, \tau_1)$ . As  $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta\beta O$  mapping,  $(g_{\mathfrak{F}} \circ h_{\mathfrak{F}})(h_{\mathfrak{F}}^{-1}(K)) = g_{\mathfrak{F}}(h_{\mathfrak{F}}(h_{\mathfrak{F}}^{-1}(K))) = g_{\mathfrak{F}}(K)$  is  $\mathfrak{F}\mathcal{F}\delta\beta os$  in

$(X_3, \tau_3)$ . Thus  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta\beta O$  mapping.

The other case is similar.

#### 4 Fermatean fuzzy $\delta\beta$ -C homeomorphism

**Definition 4.1** A bijection  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  is called a Fermatean fuzzy (resp.  $\delta$ ,  $\delta\alpha$ ,  $\delta\mathcal{S}$ ,  $\delta\mathcal{P}$  &  $\delta\beta$  or  $e^*$ )-C homeomorphism (briefly,  $\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$  &  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$  or  $\mathfrak{F}\mathcal{F}e^*\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$ )) if  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\mathcal{I}\mathcal{r}\mathcal{r}$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{I}\mathcal{r}\mathcal{r}$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{I}\mathcal{r}\mathcal{r}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{I}\mathcal{r}\mathcal{r}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{I}\mathcal{r}\mathcal{r}$  &  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{I}\mathcal{r}\mathcal{r}$  or  $\mathfrak{F}\mathcal{F}e^*\mathcal{I}\mathcal{r}\mathcal{r}$ ) mappings.

**Theorem 4.1** Each  $\mathfrak{F}\mathcal{F}\mathcal{H}\mathcal{o}\mathcal{m}$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{H}\mathcal{o}\mathcal{m}$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$  &  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$  or  $\mathfrak{F}\mathcal{F}e^*\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$ ) is a  $\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{H}\mathcal{o}\mathcal{m}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{H}\mathcal{o}\mathcal{m}$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{H}\mathcal{o}\mathcal{m}$  &  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{H}\mathcal{o}\mathcal{m}$  or  $\mathfrak{F}\mathcal{F}e^*\mathcal{H}\mathcal{o}\mathcal{m}$ ). But not conversely.

**Proof.** Let us assume that  $K$  be a  $\mathfrak{F}\mathcal{F}\delta\mathcal{C}\mathcal{S}$  in  $(X_2, \tau_2)$  is  $\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{S}$ . This shows that  $K$  is a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{S}$  in  $(X_2, \tau_2)$ . By assumption,  $h_{\mathfrak{F}}^{-1}(K)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{S}$  in  $(X_1, \tau_1)$ . Hence,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{T}\mathcal{S}$  mapping. Hence,  $h_{\mathfrak{F}}$  and  $h_{\mathfrak{F}}^{-1}$  are  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{T}\mathcal{S}$  mappings. Hence  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{H}\mathcal{o}\mathcal{m}$ . The proof of other cases are similar.

**Example 4.1** Let  $X_1 = X_2 = X = \{a, b\}$  and the  $\mathfrak{F}\mathcal{F}\mathcal{S}$ 's  $A_1, A_2, A_3$  and  $A_4$  are defined as

$$\alpha_{A_1}(a) = 0.2, \beta_{A_1}(a) = 0.8, \alpha_{A_1}(b) = 0.4, \beta_{A_1}(b) = 0.6;$$

$$\alpha_{A_2}(a) = 0.1, \beta_{A_2}(a) = 0.9, \alpha_{A_2}(b) = 0.3, \beta_{A_2}(b) = 0.7;$$

$$\alpha_{A_3}(a) = 0.9, \beta_{A_3}(a) = 0.1, \alpha_{A_3}(b) = 0.7, \beta_{A_3}(b) = 0.3;$$

$$\alpha_{A_4}(a) = 0.2, \beta_{A_4}(a) = 0.8, \alpha_{A_4}(b) = 0.3, \beta_{A_4}(b) = 0.7;$$

Let  $\tau_1 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, A_1, A_2, A_3, A_4\}$ ,  $\tau_2 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, A_1, A_2, A_3\}$  be a  $\mathfrak{F}\mathcal{F}\mathcal{T}\mathcal{S}$  on  $X_1$  and  $X_2$ ; and let  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  be an identity function, Then  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$ ) but not  $\mathfrak{F}\mathcal{F}\mathcal{H}\mathcal{o}\mathcal{m}$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{H}\mathcal{o}\mathcal{m}$ ). Since,  $A_4$  is a  $\mathfrak{F}\mathcal{F}\mathcal{O}$  set in  $X_1$  but  $(h_{\mathfrak{F}}^{-1})^{-1}(A_4) = A_4$  is not  $\mathfrak{F}\mathcal{F}\mathcal{O}$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{O}$ ) set in  $X_2$ .

**Theorem 4.2** If  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$ , then  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{L}(h_{\mathfrak{F}}^{-1}(K)) \subseteq h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{L}(K))$  for each  $\mathfrak{F}\mathcal{F}\mathcal{S}$   $K$  in  $(X_2, \tau_2)$ .

**Proof.** Let  $K$  be a  $\mathfrak{F}\mathcal{F}\mathcal{S}$  in  $(X_2, \tau_2)$ . Then,  $\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{L}(K)$  is a  $\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{S}$  in  $(X_2, \tau_2)$ , and every  $\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{S}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{S}$  in  $(X_2, \tau_2)$ . Assume  $h_{\mathfrak{F}}$  is  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{I}\mathcal{r}\mathcal{r}$  and  $h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{L}(K))$  is a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{S}$  in  $(X_1, \tau_1)$ . Then,  $\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{L}(h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{L}(K))) = h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{L}(K))$ .

$$\text{Here, } \mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{L}(h_{\mathfrak{F}}^{-1}(K)) \subseteq \mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{L}(h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{L}(K))) = h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{L}(K)).$$

Therefore,  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{L}(h_{\mathfrak{F}}^{-1}(K)) \subseteq h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\mathcal{C}\mathcal{L}(K))$  for every  $\mathfrak{F}\mathcal{F}\mathcal{S}$   $K$  in  $(X_2, \tau_2)$ .

**Theorem 4.3** Let  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  be a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{H}\mathcal{o}\mathcal{m}$ . Then  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{L}(h_{\mathfrak{F}}^{-1}(K)) = h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}\mathcal{L}(K))$  for each  $\mathfrak{F}\mathcal{F}\mathcal{S}$   $K$  in  $(X_2, \tau_2)$ .

**Proof.** Since  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}Hom$ ,  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta Irr$  mapping. Let  $K$  be a  $\mathfrak{F}\mathcal{F}s$  in  $(X_2, \tau_2)$ . Clearly,  $\mathfrak{F}\mathcal{F}\delta\beta cl(K)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_2, \tau_2)$ . Then  $\mathfrak{F}\mathcal{F}\delta\beta cl(K)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_2, \tau_2)$ .

Since  $h_{\mathfrak{F}}^{-1}(K) \subseteq h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\delta\beta cl(K))$ , then  $\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K)) \subseteq \mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\delta\beta cl(K))) = h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\delta\beta cl(K))$ . Therefore,  $\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K)) \subseteq h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\delta\beta cl(K))$ . Let  $h_{\mathfrak{F}}$  be a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C} Hom$ .  $h_{\mathfrak{F}}^{-1}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta Irr$  mapping. Let us consider  $\mathfrak{F}\mathcal{F}s$   $h_{\mathfrak{F}}^{-1}(K)$  in  $(X_1, \tau_1)$ , which implies  $\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K))$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_1, \tau_1)$ . Hence,  $\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K))$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_1, \tau_1)$ . This implies that  $(h_{\mathfrak{F}}^{-1})^{-1}(\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K))) = h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K)))$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_2, \tau_2)$ .

This proves  $K = (h_{\mathfrak{F}}^{-1})^{-1}(h_{\mathfrak{F}}^{-1}(K)) \subseteq (h_{\mathfrak{F}}^{-1})^{-1}(\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K))) = h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K)))$ . Therefore,  $\mathfrak{F}\mathcal{F}\delta\beta cl(K) \subseteq \mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K)))) = h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K)))$ . since  $h_{\mathfrak{F}}^{-1}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta Irr$  mapping. Hence,  $h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\delta\beta cl(K)) \subseteq h_{\mathfrak{F}}^{-1}(h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K)))) = \mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K))$ . That is,  $h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\delta\beta cl(K)) \subseteq \mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K))$ . Hence,  $\mathfrak{F}\mathcal{F}\delta\beta cl(h_{\mathfrak{F}}^{-1}(K)) = h_{\mathfrak{F}}^{-1}(\mathfrak{F}\mathcal{F}\delta\beta cl(K))$ .

**Remark 4.1** Theorems 4.2 and 4.3 are also true if  $h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\mathcal{C}Hom$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{C}Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{C}Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{C}Hom$  &  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{C}Hom$ .)

**Theorem 4.4** If  $h_{\mathfrak{F}}: (X_1, \tau_1) \rightarrow (X_2, \tau_2)$  and  $g_{\mathfrak{F}}: (X_2, \tau_2) \rightarrow (X_3, \tau_3)$  are  $\mathfrak{F}\mathcal{F}\mathcal{C}Hom$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{C}Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{C}Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{C}Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{C}Hom$  &  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}Hom$  or  $\mathfrak{F}\mathcal{F}e^*\mathcal{C}Hom$ )'s, then  $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\mathcal{C}Hom$  (resp.  $\mathfrak{F}\mathcal{F}\delta\mathcal{C}Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\alpha\mathcal{C}Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathcal{C}Hom$ ,  $\mathfrak{F}\mathcal{F}\delta\mathcal{P}\mathcal{C}Hom$  &  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}Hom$  or  $\mathfrak{F}\mathcal{F}e^*\mathcal{C}Hom$ ).

**Proof.** Let  $h_{\mathfrak{F}}$  and  $g_{\mathfrak{F}}$  be two  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}Hom$ 's. Assume  $K$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_3, \tau_3)$ . Then,  $g_{\mathfrak{F}}^{-1}(K)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_2, \tau_2)$ . Then, by hypothesis,  $h_{\mathfrak{F}}^{-1}(g_{\mathfrak{F}}^{-1}(K))$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_1, \tau_1)$ . Hence,  $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta Irr$  mapping. Now, let  $K$  be a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_1, \tau_1)$ . Then, by presumption,  $h_{\mathfrak{F}}(K)$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_2, \tau_2)$ . Then, by hypothesis,  $g_{\mathfrak{F}}(h_{\mathfrak{F}}(K))$  is a  $\mathfrak{F}\mathcal{F}\delta\beta cs$  in  $(X_3, \tau_3)$ . This implies that  $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta Irr$  mapping. Hence,  $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$  is a  $\mathfrak{F}\mathcal{F}\delta\beta\mathcal{C}Hom$ . The proof of other cases are similar.

## 5 Application

Entropy as a measure of fuzziness was first proposed by Zadeh [16]. Later many mathematicians defined several entropy measures. In this section, we focus on defining an entropy measure for  $\mathfrak{F}fs$  that connects the degree of membership and non-membership. As an example, we have applied the proposed entropy measure in the field of decision making.

**Definition 5.1** Let  $A = \{ \langle x, \mu_A(x), \lambda_A(x) \mid x \in X \rangle \}$  be a  $\mathfrak{F}fs$  in  $U$ . The new entropy measure for  $A$  denoted by  $\varepsilon_{\mathfrak{F}fs}(A)$ , is a function,  $\varepsilon_{\mathfrak{F}fs}: \tau_{\mathfrak{F}fs}(U) \rightarrow [0,1]$  and is defined as  $\varepsilon_{\mathfrak{F}fs}(A) = 1 - \frac{1}{n} \sum_{i=1}^n (\mu_A - \lambda_A)^2$ ; forevery  $x_i \in A$ , where  $\tau_{\mathfrak{F}fs}(U)$  denote the family of all  $\mathfrak{F}fs$ 's on  $U$ .

**Example 5.1** Assume that a certain Institution wants to assign a permanent faculty member from the set of candidates  $\{P_1, P_2, P_3, P_4, P_5\}$ . For this, the Institution authorities consider the following four criteria  $C = \{C_i: i = 1,2,3,4\}$ , where: •  $C_1$  represents the number of research publications, conferences and FDP participated, •  $C_2$  represents the teaching experience, •  $C_3$  represents the

regularity and punctuality, •  $C_4$  represents the behavior with students through the class.

After a deep discussion, a committee (forms by the Institution authorities) proposed a performance of these candidates environment as given in Table. Every ordered pair given in Table represents the membership and non-membership degrees of a candidate corresponding criteria (or attribute). Assume that the proposed approach for accessing the best candidate with appreciation to every criterion provided the committee. Then, we compute the entropy measure for each candidate to decide who is the optimal candidate(s).

**Table 1. Selection criteria for the candidates.**

	Person 1 (P1)	Person 2 (P2)	Person 3 (P3)	Person 4 (P4)	Person 5 (P5)
(C1)	$\langle C1, P1; 0.3, 0.7 \rangle$ >	$\langle C1, P2; 0.7, 0.2 \rangle$ >	$\langle C1, P3; 0.2, 0.3 \rangle$ >	$\langle C1, P4; 0.3, 0.4 \rangle$ >	$\langle C1, P5; 0.1, 0.3 \rangle$ >
(C2)	$\langle C2, P1; 0.7, 0.6 \rangle$ >	$\langle C2, P2; 0.7, 0.1 \rangle$ >	$\langle C2, P3; 0.7, 0.5 \rangle$ >	$\langle C2, P4; 0.3, 0.4 \rangle$ >	$\langle C2, P5; 0.2, 0.7 \rangle$ >
(C3)	$\langle C3, P1; 0.3, 0.2 \rangle$ >	$\langle C3, P2; 0.2, 0.4 \rangle$ >	$\langle C3, P3; 0.9, 0.2 \rangle$ >	$\langle C3, P4; 0.2, 0.5 \rangle$ >	$\langle C3, P5; 0.6, 0.2 \rangle$ >
(C4)	$\langle C4, P1; 0.1, 0.7 \rangle$ >	$\langle C4, P2; 0.3, 0.3 \rangle$ >	$\langle C4, P3; 0.1, 0.3 \rangle$ >	$\langle C4, P4; 0.5, 0.1 \rangle$ >	$\langle C4, P5; 0.1, 0.9 \rangle$ >

Clearly, all values in the Table 1 are  $\mathfrak{F}_S$ 's. Now we calculate the  $\varepsilon_{\mathfrak{F}_S}$  of each value.

**Table 2. Entropy measure of each candidate based on their criteria.**

	Person 1 (P1)	Person 2 (P2)	Person 3 (P3)	Person 4 (P4)	Person 5 (P5)
(C1)	0.84	<b>0.75</b>	0.99	0.99	0.96
(C2)	0.99	<b>0.64</b>	0.96	0.99	0.75
(C3)	0.99	0.96	<b>0.51</b>	0.91	0.84
(C4)	0.64	1.0	0.96	0.84	<b>0.36</b>

From Table 2, it is clear that,

$$\varepsilon_{\mathfrak{F}_S}(C1, P2) < \varepsilon_{\mathfrak{F}_S}(C1, P1) < \varepsilon_{\mathfrak{F}_S}(C1, P5) < \varepsilon_{\mathfrak{F}_S}(C1, P3) \leq \varepsilon_{\mathfrak{F}_S}(C1, P4)$$

Similarly

$$\varepsilon_{\mathfrak{F}_S}(C2, P2) < \varepsilon_{\mathfrak{F}_S}(C2, P5) < \varepsilon_{\mathfrak{F}_S}(C2, P3) < \varepsilon_{\mathfrak{F}_S}(C2, P1) \leq \varepsilon_{\mathfrak{F}_S}(C2, P4)$$

$$\varepsilon_{\mathfrak{F}_S}(C3, P3) < \varepsilon_{\mathfrak{F}_S}(C3, P5) < \varepsilon_{\mathfrak{F}_S}(C3, P4) < \varepsilon_{\mathfrak{F}_S}(C3, P2) \leq \varepsilon_{\mathfrak{F}_S}(C3, P1)$$

$$\varepsilon_{\mathfrak{F}_S}(C4, P5) < \varepsilon_{\mathfrak{F}_S}(C4, P1) < \varepsilon_{\mathfrak{F}_S}(C4, P4) < \varepsilon_{\mathfrak{F}_S}(C4, P3) \leq \varepsilon_{\mathfrak{F}_S}(C4, P2)$$

It is clear that based on the criteria 1 and 2, person 2 be the selection for the permanent post, criteria 3, person 3 be the selection for the permanent post, criteria 4, person 5 be the selection for the permanent post.

## 6 Conclusion

Fuzzy Topological spaces are the classical Topological spaces which characterises the membership values alone. Intuitionistic Fuzzy topological spaces portates the membership as well as the non-membership values. Pythagorean fuzzy topological spaces extent its arm to cover the missed ones of the Intuitionistic fuzzy topological spaces. Fermatean fuzzy topological spaces shorten the Pythagorean fuzzy sets of any cardinality in to a tiny set which represents the same in nano approximation with boundary space. Our contribution to this area is the concepts of  $\mathfrak{F}\delta\beta Hom$ ,  $\mathfrak{F}\delta\beta CHom$  and  $\mathfrak{F}\delta\beta T_{\frac{1}{2}}$ -space and derived some of its related characteristics. Finally applied a entropy measure to the multiple criteria decision making with the help of Fermatean fuzzy sets. In future we will employ some entropy or any other measures for comparing in decision making to the field of medical diagnosis and teaching learning process. We also takeup this idea into the diverse fuzzy environment for real world application purpose.

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