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More on Contra Continuous, Irresolute Maps in Pythagorean Fuzzy Nano Topological Spaces and Application in MCDM

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Abstract: In our daily life we come across many situation which are non probabilistic and underestimation category. Since they are unpredictable, innumerable real life problems are in abundant condition for decision making from earlier 20th century. Fuzzy sets, intuitionstic fuzzy sets and Pythagorean fuzzy sets emerged one by one and many literature are adding on every day. The concept of Pythagorean fuzzy nano (resp. δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ & $\delta\beta$ or e^*)-continuity in Pythagorean fuzzy nano topological spaces and specialize some of their basic properties with examples is our main contribution to those literature. Also, we discuss about properties and characterization of Pythagorean fuzzy irresolute maps and application of Multiple Criteria Decision Making (MCDM) techniques to the real-world problem using a proposed similarity measure in Pythagorean fuzzy nano topological spaces.

Keywords: Pythagorean fuzzy nano topological spaces, $\mathcal{PFN}contra\delta Cts$, $\mathcal{PFN}contra\delta\mathcal{S}Cts$, $\mathcal{PFN}\ contra\delta Irr$, $\mathcal{PFN}contra\delta\mathcal{S}Irr$ and Zhang similarity measure.

AMS (2000) subject classification: 03E72, 54A40, 54C05, 94D05

1 Introduction

In 1965, Zadeh [40] familiarized the concept of fuzzy set which has several applications in decision theory, artificial intelligence, operations research, expert systems, computer science, data analytics, pattern recognition, management science and robotics. In 1968, Chang and Warren [14, 34] defined fuzzy topological spaces, the basic philosophies of topology such as open set, closed set, neighbourhood, interior set, closure, continuity, compactness to fuzzy topological spaces (FTS). Applications of fuzzy sets were studied [1, 13, 25, 30]. Later numerous fuzzy topological spaces raised which have unique properties. In 1997, Dogan Coker [9, 15, 19] introduced Intuitionistic fuzzy topological spaces and studied its continuity and compactness. Intuitionistic fuzzy sets have many applications [30, 27] and also flagged approach to study Pythagorean fuzzy sets. In both the sets membership and non-membership are incorporated in a different way. In Intuitionistic fuzzy set the membership μ and non-membership λ are incorporated in such a way that $\mu + \lambda \leq 1$ where as in Pythagorean fuzzy set it is $\mu^2 + \lambda^2 \leq 1$. In 2013, Yager [37] introduced the non-standard fuzzy sets called Pythagorean fuzzy sets in comparison with Intuitionistic fuzzy sets. He gave the basic definition of Pythagorean fuzzy set (PFS) and its application in decision making [3, 39, 38]. PFS has its applications in career placements based on academic performance [20], selection of mask during COVID-19 pandemic using Pythagorean TOPSIS technique [24], etc. Later Murat et.al [18] introduced the conception of Pythagorean fuzzy topological space ($PFTS$) by provoking from the conviction of FTS [16, 17, 23]. He defined Pythagorean fuzzy continuous function between $PFTS$.

Saha [26] defined δ-open sets in fuzzy topological spaces. In 2019, Acikgoz and Esenbel [2] defined neutrosophic soft δ-topology. Aranganayagi et al., Surendra et al. and Vadivel et al. [7, 8, 28, 29, 32, 33] introduced δ-open sets in neutrosophic, neutrosophic soft, neutrosophic hypersoft and neutrosophic nano topological spaces and studied its maps and separation axioms.

Similarity measure is a significant means for measuring the uncertain information. The fuzzy similarity measure is a measure that depicts the closeness (difference) among fuzzy sets. Zhang [42] proposed the Pythagorean fuzzy similarity measures for dealing the multi-attribute decision-making problems. Peng et al. [21] proposed the many new distance measures and similarity measures for dealing the issues of pattern recognition, medical diagnosis and clustering analysis, and discussed their transformation relations. Wei and Wei [35] presented some Pythagorean fuzzy cosine function for dealing with the decision-making problems. However, some existing similarity measures/distance measures cannot obey the third or fourth axiom, and also have no power to differentiate positive difference and negative difference or deal with the division by the zero problem. Due to the above counter-intuitive phenomena [35, 42, 21] of the existing similarity measures of $\mathcal{PF}s$'s, they may be hard for DM's to choose convincible or optimal alternatives. As a consequence, the goal of this paper is to deal with the above issue by proposing a novel similarity measure for Pythagorean fuzzy set, which can be without counter intuitive phenomena.

Research Gap: No investigation on some stronger and weaker forms of Pythagorean fuzzy continuous and irresolute maps such as Pythagorean fuzzy nano $contra\delta$ open map, Pythagorean fuzzy nano $contra\delta$-semi open map, Pythagorean fuzzy nano $contra\delta$-pre open map, Pythagorean fuzzy nano $contra\delta\alpha$ open map and Pythagorean fuzzy nano $contra\delta\beta$ open maps on Pythagorean fuzzy nano topological space has been reported in the Pythagorean fuzzy nano literature.

This leads to encompass the notion of $PFNts$ by introducing Pythagorean fuzzy nano $contra\delta$ (resp. $contra\delta\alpha$, $contra\delta\mathcal{S}$, $contra\delta\mathcal{P}$ & $contra\delta\beta$ or $contrae^*$)-continuous and discuss its properties. Also, we introduce the concept of Pythagorean fuzzy nano irresoluteness called Pythagorean fuzzy nano $contra\delta$ (resp. $contra\delta\mathcal{P}$, $contra\delta\mathcal{S}$, $contra\delta\alpha$ and $contra\delta\beta$) -irresolute maps by using $\mathcal{PFN}\mathcal{S}os$ (resp. $\mathcal{PFN}\delta os$, $\mathcal{PFN}\delta\mathcal{P}os$, $pf\mathcal{N}\delta\mathcal{S}os$, $pf\mathcal{N}\delta\alpha os$ and $\mathcal{PFN}\delta\beta os$)'s and study some of their basic properties.

2 Preliminaries

We recall some basic notions of fuzzy sets, IFS's and pfs's .

Definition 2.1 *[40] Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A: X \to [0,1]$. That is:*

$$\mu_A(x) = \begin{cases} 1, & \text{if } \quad x \in X \\ 0, & \text{if } \quad x \notin X \\ (0,1) & \text{if } x \text{ ispartlyin } X. \end{cases}$$

Alternatively, a fuzzy set A in X is an object having the form $A = \{< x, \mu_A(x) > | x \in X\}$ or $A = \left\{\left\langle \frac{\mu_A(x)}{x} \right\rangle | x \in X\right\}$, where the function $\mu_A(x): X \to [0,1]$ defines the degree of membership of the element, $x \in X$.

The closer the membership value $\mu_A(x)$ to 1, the more x belongs to A, where the grades 1 and 0 represent full membership and full nonmembership. Fuzzy set is a collection of objects with graded membership, that is, having degree of membership. Fuzzy set is an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in a binary terms according to a bivalent condition; an element either belongs or does not belong to the set. Classical bivalent sets are in fuzzy set theory called crisp sets. Fuzzy sets are generalized classical sets, since the indicator function of classical sets is special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. Fuzzy sets theory permits the gradual assessment of the membership of element in a set; this is described with the aid of a membership function valued in the real unit interval $[0,1]$.

Let us consider two examples:

(i) all employees of XYZ who are over $1.8m$ in height; (ii) all employees of XYZ who are tall. The first example is a classical set with a universe (all XYZ employees) and a membership rule that divides the universe into members (those over $1.8m$) and nonmembers. The second example is a fuzzy set, because some employees are definitely in the set and some are definitely not in the set, but some are borderline.

This distinction between the ins, the outs, and the borderline is made more exact by the membership function, μ. If we return to our second example and let A represent the fuzzy set of all tall employees and x represent a member of the universe X (i.e. all employees), then $\mu_A(x)$ would be $\mu_A(x) = 1$ if x is definitely tall or $\mu_A(x) = 0$ if x is definitely not tall or $0 < \mu_A(x) < 1$ for borderline cases.

Definition 2.2 *[9, 10, 11, 12] Let a nonempty set X be fixed. An IFS A in X is an object having the form: $A = \{< x, \mu_A(x), \lambda_A(x) > | x \in X\}$ or $A = \left\{\left\langle \frac{\mu_A(x), \lambda_A(x)}{x} \right\rangle | x \in X\right\}$, where the functions $\mu_A(x): X \to [0,1]$ and $\lambda_A(x): X \to [0,1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$: $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$. For each A in X: $\pi_A(x) = 1 - \mu_A(x) - \lambda_A(x)$ is the intuitionistic fuzzy set index or hesitation margin of x in X. The hesitation margin $\pi_A(x)$ is the degree of nondeterminacy of $x \in X$ to the set A and $\pi_A(x) \in [0,1]$. The hesitation margin is the function that expresses lack of knowledge of whether $x \in X$ or $x \notin X$. Thus: $\mu_A(x) + \lambda_A(x) + \pi_A(x) = 1$.*

Example 2.1 *Let* $X = \{x, y, z\}$ *be a fixed universe of discourse and* $A = \left\{\left\langle\frac{0.6,0.1}{x}\right\rangle, \left\langle\frac{0.8,0.1}{y}\right\rangle, \left\langle\frac{0.5,0.3}{z}\right\rangle\right\}$, *be the intuitionistic fuzzy set in* X. *The hesitation margins of the elements* x, y, z *to* A *are as follows:* $\pi_A(x) = 0.3, \pi_A(y) = 0.1$ *and* $\pi_A(z) = 0.2$.

Definition 2.3 *[36, 37, 39] Let a non empty set* X *be a universal set. Then, a Pythagorean fuzzy set A, which is a set of ordered pairs over* X, *is defined by the following:* $A = \{< x, \mu_A(x), \lambda_A(x) | x \in X\}$ *or* $A = \left\{\left\langle\frac{\mu_A(x),\lambda_A(x)}{x}\right\rangle | x \in X\right\}$, *where the functions* $\mu_A(x): X \to [0,1]$ *and* $\lambda_A(x): X \to [0,1]$ *define the degree of membership and the degree of nonmembership, respectively, of the element* $x \in X$ *to A, which is a subset of X, and for every* $x \in X$, $0 \leq (\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$. *Supposing* $(\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$, *then there is a degree of indeterminacy of* $x \in X$ *to* A *defined by* $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\lambda_A(x))^2]}$ *and* $\pi_A(x) \in [0,1]$. *In what follows,* $(\mu_A(x))^2 + (\lambda_A(x))^2 + (\pi_A(x))^2 = 1$. *Otherwise,* $\pi_A(x) = 0$ *whenever* $(\mu_A(x))^2 + (\lambda_A(x))^2 = 1$. *We denote the set of all PFS's over* X *by* $pfs(X)$.

Definition 2.4 *[39] Let A and B be* pfs*'s of the forms* $A = \{< a, \mu_A(a), \lambda_A(a) > | a \in X\}$ *and* $B = \{< a, \mu_B(a), \lambda_B(a) > | a \in X\}$. *Then [(i)]*

1. $A \subseteq B$ if and only if $\mu_A(a) \leq \mu_B(a)$ and $\lambda_A(a) \geq \lambda_B(a)$ for all $a \in X$.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $\bar{A} = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$.
4. $A \cap B = \{< a, \mu_A(a) \wedge \mu_B(a), \lambda_A(a) \vee \lambda_B(a) > | a \in X\}$.
5. $A \cup B = \{< a, \mu_A(a) \vee \mu_B(a), \lambda_A(a) \wedge \lambda_B(a) > | a \in X\}$.
6. $0_P = \{< a, 0,1 > | a \in X\}$ and $1_P = \{< a, 1,0 > | a \in X\}$.
7. $\bar{1}_P = 0_P$ and $\bar{0}_P = 1_P$.

Definition 2.5 *[4] Let* U *be a non-empty set and* R *be an equivalence relation on* U. *Let* A *be a Pythagorean fuzzy set in* U *with the membership function* $\mu_A(x)$ *and non membership function* $\lambda_A(x)$, $\forall\ x \in U$. *The Pythagorean fuzzy nano lower, Pythagorean fuzzy nano upper approximation and Pythagorean fuzzy nano boundary of* A *in the approximation* (U, R) *denoted by* $\underline{\mathcal{PFN}}(A), \overline{\mathcal{PFN}}(A)$ *and* $B_{\mathcal{PFN}}(A)$ *are respectively defined as follows: [(i)]*

1. $\underline{\mathcal{PFN}}(A) = \left\{\langle x, \mu_{\underline{R}(A)}(x), \lambda_{\overline{R}(A)}(x)\rangle / y \in [x]_R, x \in U\right\}$
2. $\overline{\mathcal{PFN}}(F) = \left\{\langle x, \mu_{\overline{R}(A)}(x), \lambda_{\underline{R}(A)}(x)\rangle / y \in [x]_R, x \in U\right\}$
3. $B_{\mathcal{PFN}}(F) = \overline{\mathcal{PFN}}(F) - \underline{\mathcal{PFN}}(F)$

where $\mu_{\underline{R}(A)}(x) = \wedge_{y \in [x]_R} \mu_A(y)$

$$\lambda_{\underline{R}(A)}(x) = \wedge_{y \in [x]_R} \lambda_A(y),$$

$$\mu_{\overline{R}(A)}(x) = \vee_{y \in [x]_R} \mu_A(y),$$

$$\lambda_{\overline{R}(A)}(x) = \vee_{y \in [x]_R} \lambda_A(y).$$

Definition 2.6 *[4] Let U be an universe of discourse, R be an equivalence relation on U and A be a Pythagorean fuzzy set in U and if the collection $\tau_{\mathcal{R}}(A) = \{0_{\mathcal{P}},\ 1_{\mathcal{P}},\ \underline{\mathcal{PFN}}(A),\ \overline{\mathcal{PFN}}(A),\ B_{\mathcal{PFN}}(A)\}$ forms a topology then it is said to be a Pythagorean fuzzy nano topology. We call $(U, \tau_{\mathcal{R}}(A))$ (or simply U) as the Pythagorean fuzzy nano topological space. The elements of $\tau_{\mathcal{R}}(A)$ are called Pythagorean fuzzy nano open (briefly, $\mathcal{PFN}o$) sets.*

Remark 2.1 *[4] $[\tau_{\mathcal{R}}(A)]^c$ is called the dual fuzzy nano topology of $\tau_{\mathcal{R}}(A)$. Elements of $[\tau_{\mathcal{R}}(A)]^c$ are called Pythagorean fuzzy nano closed (briefly, $\mathcal{PFN}c$) sets. Thus, we note that a Pythagorean fuzzy set G of U is Pythagorean fuzzy nano closed in $\tau_{\mathcal{R}}(A)$ if and only if $1_P - G$ is Pythagorean fuzzy nano open in $\tau_{\mathcal{R}}(A)$.*

Definition 2.7 *[4, 5] Let $(U, \tau_{\mathcal{P}}(A))$ be a $\mathcal{PFN}ts$ with respect to A where A is a Pythagorean fuzzy subset of U. Let S be a Pythagorean fuzzy subset of U. Then Pythagorean fuzzy nano [(i)]*

1. interior of S (briefly, $\mathcal{PFN}int(S)$) is defined by $\mathcal{PFN}int(S) = \cup \{I : I \leq S \ \&\ I \text{isa} \mathcal{PFN}o \text{set in} U\}$.
2. closure of S (briefly, $\mathcal{PFN}cl(S)$) is defined by $\mathcal{PFN}cl(S) = \cap \{A : S \leq A \ \&\ A \text{isa} \mathcal{PFN}c \text{set in} U\}$.
3. regular open (briefly, $\mathcal{PFN}ro$) set if $S = \mathcal{PFN}int(\mathcal{PFN}cl(S))$.
4. regular closed (briefly, $\mathcal{PFN}rc$) set if $S = \mathcal{PFN}cl(\mathcal{PFN}int(S))$.

Definition 2.8 *[6] Let $(U_1, \tau_P(A_1))$ and $(U_2, \tau_P(A_2))$ be two $pf\mathcal{N}ts$'s. Then a function $h_P: U_1 \rightarrow U_2$ is said to be a Pythagorean fuzzy nano continuous (briefly, $\mathcal{PFN}Cts$) function if $h_P^{-1}(G)$ is $\mathcal{PFN}o$ set in U_1 for all $\mathcal{PFN}o$ set G in U_2.*

Definition 2.9 *[22] Let M, N and O be three pfs's on X. A similarity measure $S(M, N)$ is mapping $S: pfs(X) \times pfs(X) \rightarrow [0,1]$, possessing the following properties: [(S1)]*

1. $0 \leq S(M, N) \leq 1$;
2. $S(M, N) = S(N, M)$;
3. $S(M, N) = 1$ iff $M = N$;
4. $S(M, M^c) = 0$ iff M is a crisp set;
5. If $M \subseteq N \subseteq O$, then $S(M, O) \leq S(M, N)$ and $S(M, O) \leq S(N, O)$.

Let $X = x_1, x_2, \ldots, x_n$ be a finite universe of discourse, and A and B be two PFS's in X, in which $A = \{< x_i, \mu_A(x_i), \lambda_A(x_i) > | x_i \in X\}$ and $B = \{< x_i, \mu_B(x_i), \lambda_B(x_i) > | x_i \in X\}$.

Using the similarity measure in section 4, we have the Zhang [42] similarity measure are defined by

$$S_Z(A, B) = \frac{1}{2} \sum_{i=1}^{n} \frac{(|\mu_A^2(x_i) - \nu_B^2(x_i)| + |\nu_A^2(x_i) - \mu_B^2(x_i)| + |\pi_A^2(x_i) - \pi_B^2(x_i)|)}{|\mu_A^2(x_i) - \mu_B^2(x_i)| + |\nu_A^2(x_i) - \nu_B^2(x_i)| + |\pi_A^2(x_i) - \pi_B^2(x_i)| + |\mu_A^2(x_i) - \nu_B^2(x_i)| + |\nu_A^2(x_i) - \mu_B^2(x_i)| + |\pi_A^2(x_i) - \pi_B^2(x_i)|}.$$

3 Pythagorean fuzzy nano contra δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$)-continuous mappings

In this section, we introduce Pythagorean fuzzy nano contra δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$)-continuous mappings and discuss some of their properties.

Definition 3.1 *Let $(U,\tau_{\mathcal{P}}(A))$ be a $\mathcal{PFN}ts$ with respect to A where A is a pfs of U. Let S be a pfs of U. Then [(i)]*

1. Pythagorean fuzzy nano δ interior of S (briefly, $\mathcal{PFN}\delta int(S)$) is defined by $\mathcal{PFN}\delta int(S) = \cup\{I: I \subseteq S \,\&\, I \text{isa} \mathcal{PFN}ro \text{ set in} U\}$.

2. Pythagorean fuzzy nano δ closure of S (briefly, $\mathcal{PFN}\delta cl(S)$) is defined by $\mathcal{PFN}\delta cl(S) = \cap\{A: S \subseteq A \,\&\, A \text{isa} \mathcal{PFN}rc \text{ set in } U\}$.

Definition 3.2 *Let $(U,\tau_{\mathcal{P}}(A))$ be a $\mathcal{PFN}ts$ with respect to A where A is a pfs of U. Then a $\mathcal{PF}s$ S in U is said to be Pythagorean: [(i)]*

1. fuzzy nano δ-open set (briefly, $\mathcal{PFN}\delta os$) if $S = \mathcal{PFN}\delta int(S)$.

2. fuzzy nano $\delta\mathcal{P}$-open set (briefly, $\mathcal{PFN}\delta\mathcal{P}os$) if $S \subseteq \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$.

3. fuzzy nano $\delta\mathcal{S}$-open set (briefly, $\mathcal{PFN}\delta\mathcal{S}os$) if $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(S))$.

4. fuzzy nano $\delta\alpha$ or a -open set (briefly, $\mathcal{PFN}\delta\alpha os$ or $\mathcal{PFN}aos$) if $S \subseteq \mathcal{PFN}int(\mathcal{PFN}cl(\mathcal{PFN}\delta int(S)))$.

5. fuzzy nano $\delta\beta$ or e^* -open set (briefly, $\mathcal{PFN}\delta\beta os$ or $\mathcal{PFN}e^*os$) if $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}int(\mathcal{PFN}\delta cl(S)))$.

The complement of a $\mathcal{PFN}\delta os$ (resp. $\mathcal{PFN}\delta\mathcal{P}os$, $\mathcal{PFN}\delta\mathcal{S}os$, $\mathcal{PFN}\delta\alpha os$ & $\mathcal{PFN}\delta\beta os$) is called a Pythagorean fuzzy nano δ (resp. $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) closed set (briefly, $\mathcal{PFN}\delta cs$ (resp. $\mathcal{PFN}\delta\mathcal{P}cs$, $\mathcal{PFN}\delta\mathcal{S}cs$, $\mathcal{PFN}\delta\alpha cs$ and $\mathcal{PFN}\delta\beta cs$)) in U.

The family of all $\mathcal{PFN}\delta os$ (resp. $\mathcal{PFN}\delta cs$, $\mathcal{PFN}\delta\mathcal{P}os$, $\mathcal{PFN}\delta\mathcal{P}cs$, $\mathcal{PFN}\delta\mathcal{S}os$, $\mathcal{PFN}\delta\mathcal{S}cs$, $\mathcal{PFN}\delta\alpha os$, $\mathcal{PFN}\delta\alpha cs$, $\mathcal{PFN}\delta\beta os$ and $\mathcal{PFN}\delta\beta cs$) of U is denoted by $\mathcal{PFN}\delta OS(U)$ (resp. $\mathcal{PFN}\delta CS(U)$, $\mathcal{PFN}\delta\mathcal{P}OS(U)$, $\mathcal{PFN}\delta\mathcal{P}CS(U)$, $\mathcal{PFN}\delta\mathcal{S}OS(U)$, $\mathcal{PFN}\delta\mathcal{S}CS(U)$, $\mathcal{PFN}\delta\alpha OS(U)$, $\mathcal{PFN}\delta\alpha CS(U)$, $\mathcal{PFN}\delta\beta OS(U)$ and $\mathcal{PFN}\delta\beta CS(U)$).

Definition 3.3 *Let $(U,\tau_{\mathcal{P}}(A))$ be a $\mathcal{PFN}ts$ with respect to A where A is a pfs of U. Let S be a pfs of U. Then Pythagorean fuzzy nano [(i)]*

1. δ pre (resp. δ semi, $\delta\alpha$ and $\delta\beta$) interior of S (briefly, $\mathcal{PFN}\delta\mathcal{P}int(S)$ (resp. $\mathcal{PFN}\delta\mathcal{S}int(S)$, $\mathcal{PFN}\delta\alpha int(S)$ and $\mathcal{PFN}\delta\beta int(S)$)) is defined by $\mathcal{PFN}\delta\mathcal{P}int(S)$ (resp. $\mathcal{PFN}\delta\mathcal{S}int(S)$, $\mathcal{PFN}\delta\alpha int(S)$ and $\mathcal{PFN}\delta\beta int(S)$) $= \cup\{I: I \subseteq S \,\&\, I \text{isa} \mathcal{PFN}\delta\mathcal{P}o$ (resp. $\mathcal{PFN}\delta\mathcal{S}o$, $\mathcal{PFN}\delta\alpha o$ & $\mathcal{PFN}\delta\beta o$) set in$U\}$.

2. δ pre (resp. δ semi, $\delta\alpha$ and $\delta\beta$) closure of S (briefly, $\mathcal{PFN}\delta\mathcal{P}cl(S)$ (resp. $\mathcal{PFN}\delta\mathcal{S}cl(S)$, $\mathcal{PFN}\delta\alpha cl(S)$ and $\mathcal{PFN}\delta\beta cl(S)$)) is defined by $\mathcal{PFN}\delta\mathcal{P}cl(S)$ (resp. $\mathcal{PFN}\delta\mathcal{S}cl(S)$, $\mathcal{PFN}\delta\alpha cl(S)$ and $\mathcal{PFN}\delta\beta cl(S)$) $= \cap\{A: S \subseteq A \,\&\, A \text{isa} \mathcal{PFN}\delta\mathcal{P}c$ (resp. $\mathcal{PFN}\delta\mathcal{S}c$, $\mathcal{PFN}\delta\alpha c$ & $\mathcal{PFN}\delta\beta c$) set in $U\}$.

Definition 3.4 *Let* $(U, \tau_{\mathcal{P}}(A))$ *be a* $\mathcal{PF}\mathfrak{N}ts$ *with respect to* A *where* A *is a* pfs *of* U. *Let* S *be a* pfs *of* U. *Then [(i)]*

1. Pythagorean fuzzy nano δ interior of S (briefly, $\mathcal{PF}\mathfrak{N}\delta int(S)$) is defined by $\mathcal{PF}\mathfrak{N}\delta int(S) = \cup \{I: I \subseteq S \;\&\; I \text{ is a } \mathcal{PF}\mathfrak{N}ro \text{ set in } U\}$.

2. Pythagorean fuzzy nano δ closure of S (briefly, $\mathcal{PF}\mathfrak{N}\delta cl(S)$) is defined by $\mathcal{PF}\mathfrak{N}\delta cl(S) = \cap \{A: S \subseteq A \;\&\; A \text{ is a } \mathcal{PF}\mathfrak{N}rc \text{ set in } U\}$.

Definition 3.5 *Let* $(U, \tau_{\mathcal{P}}(A))$ *be a* $\mathcal{PF}\mathfrak{N}ts$ *with respect to* A *where* A *is a* pfs *of* U. *Then a* $\mathcal{PF}s$ S *in* U *is said to be Pythagorean: [(i)]*

1. fuzzy nano δ-open set (briefly, $\mathcal{PF}\mathfrak{N}\delta os$) if $S = \mathcal{PF}\mathfrak{N}\delta int(S)$.
2. fuzzy nano $\delta\mathcal{P}$-open set (briefly, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}os$) if $S \subseteq \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\delta cl(S))$.
3. fuzzy nano $\delta\mathcal{S}$-open set (briefly, $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}os$) if $S \subseteq \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(S))$.
4. fuzzy nano $\delta\alpha$ or a -open set (briefly, $\mathcal{PF}\mathfrak{N}\delta\alpha os$ or $\mathcal{PF}\mathfrak{N}aos$) if $S \subseteq \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(S)))$.
5. fuzzy nano $\delta\beta$ or e^* -open set (briefly, $\mathcal{PF}\mathfrak{N}\delta\beta os$ or $\mathcal{PF}\mathfrak{N}e^*os$) if $S \subseteq \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\delta cl(S)))$.

The complement of a $\mathcal{PF}\mathfrak{N}\delta os$ (resp. $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}os$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}os$, $\mathcal{PF}\mathfrak{N}\delta\alpha os$ & $\mathcal{PF}\mathfrak{N}\delta\beta os$) is called a Pythagorean fuzzy nano δ (resp. $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) closed set (briefly, $\mathcal{PF}\mathfrak{N}\delta cs$ (resp. $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}cs$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}cs$, $\mathcal{PF}\mathfrak{N}\delta\alpha cs$ and $\mathcal{PF}\mathfrak{N}\delta\beta cs$)) in U.

The family of all $\mathcal{PF}\mathfrak{N}\delta os$ (resp. $\mathcal{PF}\mathfrak{N}\delta cs$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}os$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}cs$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}os$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}cs$, $\mathcal{PF}\mathfrak{N}\delta\alpha os$, $\mathcal{PF}\mathfrak{N}\delta\alpha cs$, $\mathcal{PF}\mathfrak{N}\delta\beta os$ and $\mathcal{PF}\mathfrak{N}\delta\beta cs$) of U is denoted by $\mathcal{PF}\mathfrak{N}\delta OS(U)$, (resp. $\mathcal{PF}\mathfrak{N}\delta CS(U)$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}OS(U)$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}CS(U)$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}OS(U)$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}CS(U)$, $\mathcal{PF}\mathfrak{N}\delta\alpha OS(U)$, $\mathcal{PF}\mathfrak{N}\delta\alpha CS(U)$, $\mathcal{PF}\mathfrak{N}\delta\beta OS(U)$ and $\mathcal{PF}\mathfrak{N}\delta\beta CS(U)$).

Definition 3.6 *Let* $(U, \tau_{\mathcal{P}}(A))$ *be a* $\mathcal{PF}\mathfrak{N}ts$ *with respect to* A *where* A *is a* pfs *of* U. *Let* S *be a* pfs *of* U. *Then Pythagorean fuzzy nano [(i)]*

1. δ pre (resp. δ semi, $\delta\alpha$ and $\delta\beta$) interior of S (briefly, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}int(S)$ (resp. $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}int(S)$, $\mathcal{PF}\mathfrak{N}\delta\alpha int(S)$ and $\mathcal{PF}\mathfrak{N}\delta\beta int(S)$)) is defined by $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}int(S)$ (resp. $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}int(S)$, $\mathcal{PF}\mathfrak{N}\delta\alpha int(S)$ and $\mathcal{PF}\mathfrak{N}\delta\beta int(S)) = \cup \{I: I \subseteq S \;\&\; I \text{ is a } \mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ (resp. $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}o$, $\mathcal{PF}\mathfrak{N}\delta\alpha o \;\&\; \mathcal{PF}\mathfrak{N}\delta\beta o$) set in$U\}$.

2. δ pre (resp. δ semi, $\delta\alpha$ and $\delta\beta$) closure of S (briefly, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(S)$ (resp. $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}cl(S)$, $\mathcal{PF}\mathfrak{N}\delta\alpha cl(S)$ and $\mathcal{PF}\mathfrak{N}\delta\beta cl(S)$)) is defined by $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(S)$ (resp. $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}cl(S)$, $\mathcal{PF}\mathfrak{N}\delta\alpha cl(S)$ and $\mathcal{PF}\mathfrak{N}\delta\beta cl(S)) = \cap \{A: S \subseteq A \;\&\; A \text{ is a } \mathcal{PF}\mathfrak{N}\delta\mathcal{P}c$ (resp. $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}c$, $\mathcal{PF}\mathfrak{N}\delta\alpha c \;\&\; \mathcal{PF}\mathfrak{N}\delta\beta c$) set in $U\}$.

Definition 3.7 *Let* $(U_1, \tau_P(A_1))$ *and* $(U_2, \tau_P(A_2))$ *be two* $PFNts$ *'s. Then a function* $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_p(A_2))$ *is said to be a Pythagorean fuzzy nano* δ *(resp.* δ *pre,* δ *semi,* $\delta\alpha$ *and* $\delta\beta$*) continuous (briefly,* $\mathcal{PFN}\delta Cts$ *(resp.* $\mathcal{PFN}\delta\mathcal{P}Cts$, $\mathcal{PFN}\delta\mathcal{S}Cts$, $\mathcal{PFN}\delta\alpha Cts$ *and* $\mathcal{PFN}\delta\beta Cts$*)) function if* $h_P^{-1}(G)$ *is* $\mathcal{PFN}\delta o$ *(resp.* $\mathcal{PFN}\delta\mathcal{P}o$, $\mathcal{PFN}\delta\mathcal{S}o$, $\mathcal{PFN}\delta\alpha o$ &

$\mathcal{PFN}\delta\beta o$) set in U_1 for all $\mathcal{PFN}o$ set G in U_2.

Definition 3.8 *Let $(U_1, \tau_P(A_1))$ and $(U_2, \tau_P(A_2))$ be two $PF\mathcal{N}ts$'s. Then a function $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_p(A_2))$ is said to be a Pythagorean fuzzy nano contra δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) continuous (briefly, $\mathcal{PFN}contra\delta Cts$ (resp. $\mathcal{PFN}contra\delta\mathcal{P}Cts$, $\mathcal{PFN}contra\delta\mathcal{S}Cts$, $\mathcal{PFN}contra\delta\alpha Cts$ and $\mathcal{PFN}contra\delta\beta Cts$)) function if $h_P^{-1}(G)$ is $\mathcal{PFN}\delta c$ (resp. $\mathcal{PFN}\delta\mathcal{P}c$, $\mathcal{PFN}\delta\mathcal{S}c$, $\mathcal{PFN}\delta\alpha c$ & $\mathcal{PFN}\delta\beta c$) set in U_1 for all $\mathcal{PFN}o$ set G in U_2.*

Lemma 3.1 *Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a function. Then the following statements hold.*

1. If S and T are pfs's of U_1 such that $S \subseteq T$, then $h_P(S) \subseteq h_P(T)$.
2. If S and T are pfs's of U_2 such that $S \subseteq T$, then $h_P^{-1}(S) \subseteq h_P^{-1}(T)$.

Lemma 3.2 *Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a function. If S is a pfs of U_1 and T is a pfs of U_2. Then*

1. $h_P(h_P^{-1}(S)) \subseteq S$
2. $h_P(h_P^{-1}(S)) = S \Leftrightarrow h_P$ is surjective.
3. $h_P^{-1}(h_P(S)) \supseteq S$
4. $h_P^{-1}(h_P(S)) = S$ whenever h_P is injective.

Theorem 3.1 *Let $(U_1, \tau_P(A_1))$ and $(U_2, \tau_P(A_2))$ be two $P\mathcal{N}ts$'s and let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$, then*

(i) Every $\mathcal{PFN}contra\delta Cts$ is a $\mathcal{PFN}contraCts$.

(ii) Every $\mathcal{PFN}contra\delta Cts$ is a $\mathcal{PFN}contra\delta\mathcal{P}Cts$.

(iii) Every $\mathcal{PFN}contra\delta Cts$ is a $\mathcal{PFN}contra\delta\mathcal{S}Cts$.

(iV) Every $\mathcal{PFN}contra\delta\mathcal{S}Cts$ is a $\mathcal{PFN}contra\delta\beta Cts$.

(V) Every $\mathcal{PFN}contra\delta\mathcal{P}Cts$ is a $\mathcal{PFN}contra\delta\beta Cts$.

(Vi) Every $\mathcal{PFN}contra\delta\alpha Cts$ is a $\mathcal{PFN}contra\delta\mathcal{P}Cts$.

(Vii) Every $\mathcal{PFN}contra\delta\alpha Cts$ is a $\mathcal{PFN}contra\delta\mathcal{S}Cts$.

But not converse.

Proof. (i) Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a $\mathcal{PFN}contra\delta Cts$. Let S be a $\mathcal{PFN}o$ set in $(U_2, \tau_P(A_2))$. Then $h_P^{-1}(S)$ is $\mathcal{PFN}\delta c$ set in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta c$ set is $\mathcal{PFN}cs$, $h_P^{-1}(S)$ is $pf\mathcal{N}c$ set in $(U_1, \tau_P(A_1))$. Hence h_P is $\mathcal{PFN}contraCts$ function.

(ii) Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a $\mathcal{PFN}contra\delta Cts$. Let S be a $\mathcal{PFN}o$ set in $(U_2, \tau_P(A_2))$. Then $h_P^{-1}(S)$ is $\mathcal{PFN}\delta c$ set in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta c$ set is $\mathcal{PFN}\delta\mathcal{P}cs$, $h_P^{-1}(S)$ is $pf\mathcal{N}\delta\mathcal{P}c$ set in $(U_1, \tau_P(A_1))$. Hence h_P is $\mathcal{PFN}contra\delta\mathcal{P}Cts$ function.

(iii) Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a $\mathcal{PFN}contra\delta Cts$. Let S be a $\mathcal{PFN}o$ set in

$(U_2, \tau_P(A_2))$. Then $h_P^{-1}(S)$ is $\mathcal{PFN}\delta c$ set in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta c$ set is $\mathcal{PFN}\delta\mathcal{S}cs$, $h_P^{-1}(S)$ is $pf\mathcal{N}\delta\mathcal{S}c$ set in $(U_1, \tau_P(A_1))$. Hence h_P is $\mathcal{PFN}contra\delta\mathcal{S}Cts$ function.

(iv) Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a $\mathcal{PFN}contra\delta\mathcal{S}Cts$. Let S be a $\mathcal{PFN}o$ set in $(U_2, \tau_P(A_2))$. Then $h_P^{-1}(S)$ is $\mathcal{PFN}\delta\mathcal{S}c$ set in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta\mathcal{S}c$ set is $\mathcal{PFN}\delta\beta cs$, $h_P^{-1}(S)$ is $pf\mathcal{N}\delta\beta c$ set in $(U_1, \tau_P(A_1))$. Hence h_P is $\mathcal{PFN}contra\delta\beta Cts$ function.

(v) Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a $\mathcal{PFN}contra\delta\mathcal{P}Cts$. Let S be a $\mathcal{PFN}c$ set in $(U_2, \tau_P(A_2))$. Then $h_P^{-1}(S)$ is $\mathcal{PFN}\delta\mathcal{P}c$ set in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta\mathcal{P}c$ set is $\mathcal{PFN}\delta\beta cs$, $h_P^{-1}(S)$ is $pf\mathcal{N}\delta\beta c$ set in $(U_1, \tau_P(A_1))$. Hence h_P is $\mathcal{PFN}contra\delta\beta Cts$ function.

(vi) Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a $\mathcal{PFN}contra\delta\alpha Cts$. Let S be a $\mathcal{PFN}o$ set in $(U_2, \tau_P(A_2))$. Then $h_P^{-1}(S)$ is $\mathcal{PFN}\delta\alpha c$ set in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta\alpha c$ set is $\mathcal{PFN}\delta\mathcal{P}cs$, $h_P^{-1}(S)$ is $pf\mathcal{N}\delta\mathcal{P}c$ set in $(U_1, \tau_P(A_1))$. Hence h_P is $\mathcal{PFN}contra\delta\mathcal{P}Cts$ function.

(vii) Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a $\mathcal{PFN}contra\delta\alpha Cts$. Let S be a $\mathcal{PFN}o$ set in $(U_2, \tau_P(A_2))$. Then $h_P^{-1}(S)$ is $\mathcal{PFN}\delta\alpha c$ set in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta\alpha c$ set is $\mathcal{PFN}\delta\mathcal{S}cs$, $h_P^{-1}(S)$ is $pf\mathcal{N}\delta\mathcal{S}c$ set in $(U_1, \tau_P(A_1))$. Hence h_P is $\mathcal{PFN}contra\delta\mathcal{S}Cts$ function.

Remark 3.1 *The following Figure shows the relations among the different types of Pythagorean fuzzy δ continuous mappings that were studied in this section.*

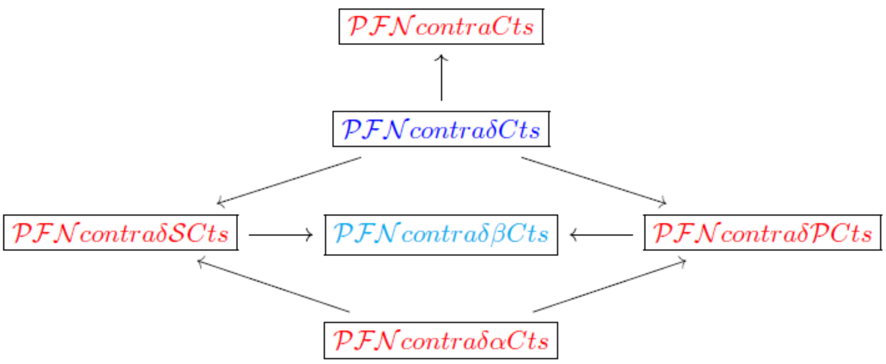

Figure : $\mathcal{PFN}contra\delta Cts$ mappings in $\mathcal{PFN}contraCts$

Example 3.1 *Assume $U_1 = U_2 = U = \{s_1, s_2, s_3, s_4\}$ be the universe set and the equivalence relation is $U/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$.*

Let $A = \left\{\left\langle\frac{s_1}{0.3,0.1}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle, \left\langle\frac{s_4}{0.4,0.25}\right\rangle\right\}$ be a Pythagorean fuzzy subset of U.

$$\underline{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1,s_4}{0.3,0.25}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1,s_4}{0.4,0.1}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1,s_4}{0.25,0.3}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\}.$$

Now $\tau_P(A_1) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PF}\mathfrak{N}}(A), \overline{\mathcal{PF}\mathfrak{N}}(A), B_{\mathcal{PF}\mathfrak{N}}(A)\}$ and $\tau_P(A_2) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, (\underline{\mathcal{PF}\mathfrak{N}}(A))^c,$

$(\overline{\mathcal{PFN}}(A))^c$, $(B_{\mathcal{PFN}}(A))^c\}$. Let $h_P: (U, \tau_P(A_1)) \to (U, \tau_P(A_2))$ be an identity function, Then h_P is $\mathcal{PFN}contraCts$ but not $\mathcal{PFN}contra\delta Cts$. Since, $\overline{\mathcal{PFN}}(A)$ is a $\mathcal{PFN}o$ set in U_2 but $h_P^{-1}(\overline{\mathcal{PFN}}(A)) = \overline{\mathcal{PFN}}(A)$ is not $\mathcal{PFN}\delta c$ set in U_1.

Example 3.2 *Let $U_1 = \{s_1, s_2, s_3, s_4\}$, $U_2 = \{t_1, t_2, t_3, t_4\}$ are the universe sets and the equivalence relations are $U_1/R = \{\{s_1, s_3\}, \{s_2, s_4\}\}$ and $U_2/R = \{\{t_1, t_3\}, \{t_2, t_4\}\}$. Let $A_1 = \left\{\left\langle\frac{s_1}{0.4,0.8}\right\rangle, \left\langle\frac{s_2}{0.5,0.4}\right\rangle, \left\langle\frac{s_3}{0.6,0.6}\right\rangle, \left\langle\frac{s_4}{0.7,0.6}\right\rangle\right\}$ and $A_2 = \left\{\left\langle\frac{t_1}{0.4,0.8}\right\rangle, \left\langle\frac{t_2}{0.5,0.7}\right\rangle, \left\langle\frac{t_3}{0.4,0.6}\right\rangle, \left\langle\frac{t_4}{0.3,0.7}\right\rangle\right\}$ be a Pythagorean fuzzy subsets of U_1 and U_2 respectively.*

$$\underline{\mathcal{PFN}}(A_1) = \left\{\left\langle\frac{s_1,s_3}{0.4,0.8}\right\rangle, \left\langle\frac{s_2,s_4}{0.5,0.6}\right\rangle\right\}$$

$$\overline{\mathcal{PFN}}(A_1) = \left\{\left\langle\frac{s_1,s_3}{0.6,0.6}\right\rangle, \left\langle\frac{s_2,s_4}{0.7,0.4}\right\rangle\right\}$$

$$B_{\mathcal{PFN}}(A_1) = \left\{\left\langle\frac{s_1,s_3}{0.6,0.6}\right\rangle, \left\langle\frac{s_2,s_4}{0.6,0.5}\right\rangle\right\}$$

$$\underline{\mathcal{PFN}}(A_2) = \left\{\left\langle\frac{t_1,t_3}{0.4,0.7}\right\rangle, \left\langle\frac{t_2,t_4}{0.3,0.7}\right\rangle\right\}$$

$$B_{\mathcal{PFN}}(A_2) = \overline{\mathcal{PFN}}(A_2) = \left\{\left\langle\frac{t_1,t_3}{0.4,0.6}\right\rangle, \left\langle\frac{t_2,t_4}{0.5,0.7}\right\rangle\right\}.$$

Now $\tau_P(A_1) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFN}}(A_1), \overline{\mathcal{PFN}}(A_1), B_{\mathcal{PFN}}(A_1)\}$, $\tau_P(A_2) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFN}}(A_2), \overline{\mathcal{PFN}}(A_2) = B_{\mathcal{PFN}}(A_2)\}$. Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be an identity function, then h_P is $\mathcal{PFN}contra\delta\mathcal{P}Cts$ (resp. $\mathcal{PFN}contra\delta\beta Cts$ and $\mathcal{PFN}contra\delta\mathcal{P}Cts$) but not $\mathcal{PFN}contra\delta Cts$ (resp. $\mathcal{PFN}contra\delta\mathcal{S}Cts$ and $\mathcal{PFN}contra\delta\alpha Cts$). Since, $\underline{\mathcal{PFN}}(A_2)$ is a $\mathcal{PFN}o$ set in U_2 but $h_P^{-1}(\underline{\mathcal{PFN}}(A_2)) = \underline{\mathcal{PFN}}(A_2)$ is not $\mathcal{PFN}\delta c$ (resp. $\mathcal{PFN}\delta\mathcal{S}c$ and $\mathcal{PFN}\delta\alpha c$) set in U_1.

Example 3.3 *Let $U_1 = \{s_1, s_2, s_3, s_4\}$, $U_2 = \{t_1, t_2, t_3, t_4\}$ are the universe sets and the equivalence relations are $U_1/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$ and $U_2/R = \{\{t_1, t_4\}, \{t_2\}, \{t_3\}\}$. Let $A_1 = \left\{\left\langle\frac{s_1}{0.3,0.7}\right\rangle, \left\langle\frac{s_2}{0.1,0.6}\right\rangle, \left\langle\frac{s_3}{0.4,0.6}\right\rangle, \left\langle\frac{s_4}{0.4,0.6}\right\rangle\right\}$ and $A_2 = \left\{\left\langle\frac{t_1}{0.6,0.4}\right\rangle, \left\langle\frac{t_2}{0.6,0.2}\right\rangle, \left\langle\frac{t_3}{0.6,0.4}\right\rangle, \left\langle\frac{t_4}{0.6,0.4}\right\rangle\right\}$ be a Pythagorean fuzzy subsets of U_1 and U_2 respectively.*

$$\underline{\mathcal{PFN}}(A_1) = \left\{\left\langle\frac{s_1,s_4}{0.3,0.7}\right\rangle, \left\langle\frac{s_2}{0.1,0.6}\right\rangle, \left\langle\frac{s_3}{0.4,0.6}\right\rangle\right\},$$

$$B_{\mathcal{PFN}}(A_1) = \overline{\mathcal{PFN}}(A_1) = \left\{\left\langle\frac{s_1,s_4}{0.4,0.6}\right\rangle, \left\langle\frac{s_2}{0.1,0.6}\right\rangle, \left\langle\frac{s_3}{0.4,0.6}\right\rangle\right\},$$

$$\underline{\mathcal{PFN}}(A_2) = \left\{\left\langle\frac{t_1,t_4}{0.6,0.4}\right\rangle, \left\langle\frac{t_2}{0.6,0.2}\right\rangle, \left\langle\frac{t_3}{0.6,0.4}\right\rangle\right\},$$

$$\overline{\mathcal{PFN}}(A_2) = \left\{\left\langle\frac{t_1,t_4}{0.6,0.4}\right\rangle, \left\langle\frac{t_2}{0.6,0.2}\right\rangle, \left\langle\frac{t_3}{0.6,0.4}\right\rangle\right\},$$

$$B_{\mathcal{PFN}}(A_2) = \left\{\left\langle\frac{t_1,t_4}{0.4,0.6}\right\rangle, \left\langle\frac{t_2}{0.2,0.6}\right\rangle, \left\langle\frac{t_3}{0.4,0.6}\right\rangle\right\}.$$

Now $\tau_P(A_1) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFR}}(A_1), \overline{\mathcal{PFR}}(A_1) = B_{\mathcal{PFR}}(A_1)\}$, $\tau_P(A_2) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFR}}(A_2), \overline{\mathcal{PFR}}(A_2), B_{\mathcal{PFR}}(A_2)\}$. Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be an identity function, then h_P is $\mathcal{PFN}contra\delta\mathcal{S}Cts$ (resp. $\mathcal{PFN}contra\delta\mathcal{S}Cts$) but not $\mathcal{PFN}contra\delta Cts$ (resp. $\mathcal{PFN}contra\delta\alpha Cts$). Since, $\underline{\mathcal{PFR}}(A_2)$ is a $\mathcal{PFN}o$ set in U_2 but $h_P^{-1}(\underline{\mathcal{PFR}}(A_2)) = \underline{\mathcal{PFR}}(A_2)$ is not $\mathcal{PFN}\delta c$ (resp. $\mathcal{PFN}\delta\alpha c$) set in U_1.

Example 3.4 *Let* $U_1 = \{s_1, s_2, s_3, s_4\}$, $U_2 = \{t_1, t_2, t_3, t_4\}$ *are the universe sets and the equivalence relations are* $U_1/R = \{\{s_1, s_3\}, \{s_2, s_4\}\}$ *and* $U_2/R = \{\{t_1, t_3\}, \{t_2, t_4\}\}$. *Let* $A_1 = \left\{\left\langle\frac{s_1}{0.4,0.7}\right\rangle, \left\langle\frac{s_2}{0.5,0.7}\right\rangle, \left\langle\frac{s_3}{0.4,0.6}\right\rangle, \left\langle\frac{s_4}{0.3,0.7}\right\rangle\right\}$ *and* $A_2 = \left\{\left\langle\frac{t_1}{0.4,0.8}\right\rangle, \left\langle\frac{t_2}{0.5,0.4}\right\rangle, \left\langle\frac{t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_4}{0.7,0.6}\right\rangle\right\}$ *be a Pythagorean fuzzy subsets of* U_1 *and* U_2 *respectively.*

$$\underline{\mathcal{PFR}}(A_1) = \left\{\left\langle\frac{s_1, s_3}{0.4,0.7}\right\rangle, \left\langle\frac{s_2, s_4}{0.3,0.7}\right\rangle\right\}$$

$$\overline{\mathcal{PFR}}(A_1) = \left\{\left\langle\frac{s_1, s_3}{0.4,0.6}\right\rangle, \left\langle\frac{s_2, s_4}{0.5,0.7}\right\rangle\right\}$$

$$B_{\mathcal{PFR}}(A_1) = \left\{\left\langle\frac{s_1, s_3}{0.4,0.6}\right\rangle, \left\langle\frac{s_2, s_4}{0.5,0.7}\right\rangle\right\}$$

$$\underline{\mathcal{PFR}}(A_2) = \left\{\left\langle\frac{t_1, t_3}{0.4,0.8}\right\rangle, \left\langle\frac{t_2, t_4}{0.5,0.6}\right\rangle\right\}$$

$$\overline{\mathcal{PFR}}(A_2) = \left\{\left\langle\frac{t_1, t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_2, t_4}{0.7,0.4}\right\rangle\right\}$$

$$B_{\mathcal{PFR}}(A_2) = \left\{\left\langle\frac{t_1, t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_2, t_4}{0.6,0.5}\right\rangle\right\}.$$

Now $\tau_P(A_1) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFR}}(A_1), \overline{\mathcal{PFR}}(A_1), B_{\mathcal{PFR}}(A_1)\}$, $\tau_P(A_2) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFR}}(A_2), \overline{\mathcal{PFR}}(A_2), B_{\mathcal{PFR}}(A_2)\}$. Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be an identity function, then h_P is $\mathcal{PFN}contra\delta\beta Cts$ but not $\mathcal{PFN}contra\delta\mathcal{P}Cts$. Since, $\overline{\mathcal{PFR}}(A_2)$ is a $\mathcal{PFN}o$ set in U_2 but $h_P^{-1}(\overline{\mathcal{PFR}}(A_2)) = \overline{\mathcal{PFR}}(A_2)$ is not $\mathcal{PFN}\delta\mathcal{P}c$ set in U_1.

Theorem 3.2 *Let* $(U_1, \tau_P(A_1))$ & $(U_2, \tau_P(A_2))$ *be a* $\mathcal{PFN}ts$*'s. A mapping* $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ *satisfies the following conditions are equivalent.*

(i) h_P is $\mathcal{PFN}contra\delta\beta Cts$;

(ii) The inverse $h_P^{-1}(K)$ of all $\mathcal{PFN}cs$ set K in U_2 is $\mathcal{PFN}\delta\beta os$ in U_1.

Proof. (i) → (ii): Consider a $\mathcal{PFN}cs$ K in U_2. Then K^c is $\mathcal{PFN}os$ in U_2. As h_P is $\mathcal{PFN}contra\delta\beta Cts$, $h_P^{-1}(K^c)$ is $\mathcal{PFN}\delta\beta cs$ in U_1. As $h_P^{-1}(K^c) = (h_P^{-1}(K))^c$, $h_P^{-1}(K)$ is a $\mathcal{PFN}\delta\beta os$ in U_1.

(ii) → (i): Consider a $\mathcal{PFN}cs$ K in U_2. So K^c is a $\mathcal{PFN}os$ in U_2. By presumption, $h_P^{-1}(K^c)$ is $\mathcal{PFN}\delta\beta cs$ in U_1. As $h_P^{-1}(K^c) = (h_P^{-1}K)^c$, $(h_P^{-1}(K))^c$ is a $\mathcal{PFN}\delta\beta cs$ in U_1. Hence $h_P^{-1}(K)$ is a $\mathcal{PFN}\delta\beta os$ in U_1. Thus h_P is $\mathcal{PFN}contra\delta\beta Cts$.

Theorem 3.3 *Let* $(U_1, \tau_P(A_1))$ & $(U_2, \tau_P(A_2))$ *be a* $\mathcal{PFN}ts$*'s. A mapping* $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ *is* $\mathcal{PFN}contra\delta\beta Cts$ *satisfies the following conditions are hold.*

(i) $h_P(\mathcal{PFN}\delta\beta cl(L)) \supseteq \mathcal{PFN}\delta int(h_P(L))$, for all $pfs\ L$ in U_1.

(ii) $\mathcal{PFN}\delta\beta cl(h_P^{-1}(K)) \supseteq h_P^{-1}(\mathcal{PFN}\delta int(K))$, for all $pfs\ K$ in U_2.

Proof. (i) Since $\mathcal{PFN}\delta cl(h_P(L))$ is a $\mathcal{PFN}\delta cs$ in U_2 and h_P is $\mathcal{PFN}contra\delta\beta Cts$, then $h_P^{-1}(\mathcal{PFN}\delta int(\ h_P(L)))$ is $\mathcal{PFN}\delta\beta os$ in U_1. Now, since $L \supseteq h_P^{-1}(\mathcal{PFN}\delta int(h_P(L)))$, $\mathcal{PFN}\delta\beta cl(L)\ \supseteq h_P^{-1}(\mathcal{PFN}\delta int(h_P(L)))$. Therefore, $h_P\big(\mathcal{PFN}\delta\beta cl(L)\big) \supseteq\ \mathcal{PFN}\delta int(h_P(L))$.

(ii) By replacing L with K in (i),

we obtain $h_P\left(\mathcal{PFN}\delta\beta cl\left(h_P^{-1}(K)\right)\right) \supseteq \mathcal{PFN}\delta int(h_P(h_P^{-1}(K))) \supseteq \mathcal{PFN}\delta in(K)$. Hence, $\mathcal{PFN}\delta\beta cl(h_P^{-1}(K)) \supseteq h_P^{-1}(\mathcal{PFN}\delta int(K))$.

Remark 3.2 *Let* $(U_1, \tau_P(A_1))$ & $(U_2, \tau_P(A_2))$ *be a* $\mathcal{PFN}ts$ *'s. Let* $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ *be a mapping. If* h_P *is* $\mathcal{PFN}contra\delta\beta Cts$*, then*

1. $h_P(\mathcal{PFN}\delta\beta cl(L))$ is not necessarily equal to $\mathcal{PFN}\delta int(h_P(L))$ where $L \in U_1$.
2. $\mathcal{PFN}\delta\beta cl(h_P^{-1}(K))$ is not necessarily equal to $h_P^{-1}(\mathcal{PFN}\delta int(K))$ where $K \in U_2$.

Example 3.5 Assume $U_1 = U_2 = U = \{s_1, s_2, s_3, s_4\}$ be the universe set and the equivalence relation is $U/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$.

Let $A = \left\{\left\langle\frac{s_1}{0.3,0.1}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle, \left\langle\frac{s_4}{0.4,0.25}\right\rangle\right\}$ be a Pythagorean fuzzy subset of U.

$$\underline{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1, s_4}{0.3,0.25}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1, s_4}{0.4,0.1}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1, s_4}{0.25,0.3}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\}.$$

Now $\tau_P(A_1) = \tau_P(A_2) = \tau_P(A) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PF}\mathfrak{N}}(A), \overline{\mathcal{PF}\mathfrak{N}}(A), B_{\mathcal{PF}\mathfrak{N}}(A)\}$ is a $\mathcal{PFN}ts$. Let $h_P: (U, \tau_P(A_1)) \to (U, \tau_P(A_2))$ be an identity function, then h_P is $\mathcal{PFN}\delta\beta Cts$.

1. $h_P(\mathcal{PFN}\delta\beta cl(\underline{\mathcal{PF}\mathfrak{N}}(A))) = B_{\mathcal{PF}\mathfrak{N}}(A)$. But $\mathcal{PFN}\delta cl\left(h_P\left(\underline{\mathcal{PF}\mathfrak{N}}(A)\right)\right) = B_{\mathcal{PF}\mathfrak{N}}(A)^c$. Thus $h_P(\mathcal{PFN}\delta\beta cl(\underline{\mathcal{PF}\mathfrak{N}}(A))) \neq \mathcal{PFN}\delta cl(h_P(\underline{\mathcal{PF}\mathfrak{N}}(A)))$.
2. $\mathcal{PFN}\delta\beta cl(h_P^{-1}(\underline{\mathcal{PF}\mathfrak{N}}(A))) = B_{\mathcal{PF}\mathfrak{N}}(A)$. But $h_P^{-1}(\mathcal{PFN}\delta cl(\underline{\mathcal{PF}\mathfrak{N}}(A))) = B_{\mathcal{PF}\mathfrak{N}}(A)^c$. Thus $\mathcal{PFN}\delta\beta cl(h_P^{-1}(\underline{\mathcal{PF}\mathfrak{N}}(A))) \neq h_P^{-1}(\mathcal{PFN}\delta cl(\underline{\mathcal{PF}\mathfrak{N}}(A)))$.

4 Pythagorean fuzzy nano contra δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$)-irresolute maps

In this section, we introduce the concept of Pythagorean fuzzy nano contra irresoluteness called Pythagorean fuzzy nano contra (resp. δ, $\delta\mathcal{P}, \delta\mathcal{S}, \delta\alpha$ and $\delta\beta$)-irresolute maps by using $\mathcal{PFNS}os$ (resp. $\mathcal{PFN}contra\delta os$, $\mathcal{PFN}contra\delta\mathcal{P}os$, $\mathcal{PFN}\ \ contra\delta\mathcal{S}os$, $\mathcal{PFN}contra\delta\alpha os$ and

$\mathcal{PFN}contra\delta\beta os$)'s and study some of their basic properties. This definition enables us to obtain conditions under which maps and inverse maps preserve respective open sets.

Definition 4.1 *A map $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ is said to be Pythagorean fuzzy nano contra (resp. δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$)-irresolute (in short, $\mathcal{PFN}contraIrr$ (resp. $\mathcal{PFN}contra\delta Irr$, $\mathcal{PFN}contra\delta\mathcal{P}Irr$, $\mathcal{PFN}contra\delta\mathcal{S}Irr$, $\mathcal{PFN}contra\delta\alpha Irr$ and $\mathcal{PFN}contra\delta\beta Irr$)) map if $h_P^{-1}(K)$ is a $\mathcal{PFNS}cs$ (resp. $\mathcal{PFN}\delta cs$, $\mathcal{PFN}\delta\mathcal{P}cs$, $\mathcal{PFN}\delta\mathcal{S}cs$, $\mathcal{PFN}\delta\alpha cs$ and $\mathcal{PFN}\delta\beta cs$) in $(U_1, \tau_P(A_1))$ for each $\mathcal{PFNS}os$ (resp. $\mathcal{PFN}\delta os$, $\mathcal{PFN}\delta\mathcal{P}os$, $\mathcal{PFN}\delta\mathcal{S}os$, $\mathcal{PFN}\delta\alpha os$ and $\mathcal{PFN}\delta\beta os$) K of $(U_2, \tau_P(A_2))$.*

Theorem 4.1 *Let $(U_1, \tau_P(A_1))$ & $(U_2, \tau_P(A_2))$ be a $\mathcal{PFN}ts$'s. Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a mapping. Then the following statements are hold for $\mathcal{PFN}ts$, but not conversely.*

(i) Every $\mathcal{PFN}contraIrr$ map is a $\mathcal{PFN}contra\mathcal{SC}ts$.

(ii) Every $\mathcal{PFN}contra\delta\mathcal{S}Irr$ map is a $\mathcal{PFN}contra\delta\mathcal{SC}ts$.

(iii) Every $\mathcal{PFN}contra\delta\mathcal{P}Irr$ map is a $\mathcal{PFN}contra\delta\mathcal{PC}ts$.

(iv) Every $\mathcal{PFN}contra\delta\alpha Irr$ map is a $\mathcal{PFN}contra\delta\alpha Cts$.

(V) Every $\mathcal{PFN}contra\delta\beta Irr$ map is a $\mathcal{PFN}contra\delta\beta Cts$.

Proof. (i) Consider a $\mathcal{PFN}contraIrr$ map h_P and a $\mathcal{PFN}os$ K in U_2. As each $\mathcal{PFN}os$ is a $\mathcal{PFNS}os$, K is a $\mathcal{PFNS}os$ in U_2. By presumption, $h_P^{-1}(K)$ is a $\mathcal{PFNS}cs$ in U_1. Thus h_P is a $\mathcal{PFN}contra\mathcal{SC}ts$ map.

(ii) Consider a $\mathcal{PFN}contra\delta\mathcal{S}Irr$ map h_P and a $\mathcal{PFN}\delta os$ K in U_2. As each $\mathcal{PFN}\delta os$ is a $\mathcal{PFN}os$ and $\mathcal{PFN}\delta\mathcal{S}os$, K is a $\mathcal{PFN}\delta os$ and $\mathcal{PFN}\delta\mathcal{S}os$ in U_2. By presumption, $h_P^{-1}(K)$ is a $\mathcal{PFN}\delta\mathcal{S}cs$ in U_1. Thus h_P is a $\mathcal{PFN}contra\delta\mathcal{SC}ts$ map.

(iii) Consider a $\mathcal{PFN}contra\delta\mathcal{P}Irr$ map h_P and a $\mathcal{PFN}\delta os$ K in U_2. As each $\mathcal{PFN}\delta os$ is a $\mathcal{PFN}os$ and $\mathcal{PFN}\delta\mathcal{P}os$, K is a $\mathcal{PFN}\delta os$ and $\mathcal{PFN}\delta\mathcal{P}os$ in U_2. By presumption, $h_P^{-1}(K)$ is a $\mathcal{PFN}\delta\mathcal{P}cs$ in U_1. Thus h_P is a $\mathcal{PFN}contra\delta\mathcal{PC}ts$ map.

(iv) Consider a $\mathcal{PFN}contra\delta\alpha Irr$ map h_P and a $\mathcal{PFN}\delta os$ K in U_2. As each $\mathcal{PFN}\delta os$ is a $\mathcal{PFN}os$ and $\mathcal{PFN}\delta\alpha os$, K is a $\mathcal{PFN}\delta os$ and $\mathcal{PFN}\delta\alpha os$ in U_2. By presumption, $h_P^{-1}(K)$ is a $\mathcal{PFN}\delta\alpha cs$ in U_1. Thus h_P is a $\mathcal{PFN}contra\delta\alpha Cts$ map.

(v) Consider a $\mathcal{PFN}contra\delta\beta Irr$ map h_P and a $\mathcal{PFN}\delta os$ K in U_2. As each $\mathcal{PFN}\delta os$ is a $\mathcal{PFN}os$ and $\mathcal{PFN}\delta\beta os$, K is a $\mathcal{PFN}\delta os$ and $\mathcal{PFN}\delta\beta os$ in U_2. By presumption, $h_P^{-1}(K)$ is a $\mathcal{PFN}\delta\beta cs$ in U_1. Thus h_P is a $\mathcal{PFN}contra\delta\beta Cts$ map.

Example 4.1 *Let $U_1 = \{s_1, s_2, s_3, s_4\}$, $U_2 = \{t_1, t_2, t_3, t_4\}$ are the universe sets and the equivalence relations are $U_1/R = \{\{s_1, s_3\}, \{s_2, s_4\}\}$ and $U_2/R = \{\{t_1, t_3\}, \{t_2, t_4\}\}$. Let $A_1 = \left\{\left\langle\frac{s_1}{0.1,0.8}\right\rangle, \left\langle\frac{s_2}{0.3,0.7}\right\rangle, \left\langle\frac{s_3}{0.2,0.9}\right\rangle, \left\langle\frac{s_4}{0.4,0.6}\right\rangle\right\}$ and $A_2 = \left\{\left\langle\frac{t_1}{0.4,0.8}\right\rangle, \left\langle\frac{t_2}{0.5,0.4}\right\rangle, \left\langle\frac{t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_4}{0.7,0.6}\right\rangle\right\}$ be a Pythagorean fuzzy subsets of U_1 and U_2 respectively.*

$$\underline{\mathcal{PF}\mathfrak{N}}(A_1) = \left\{\left\langle\frac{s_1, s_3}{0.1,0.9}\right\rangle, \left\langle\frac{s_2, s_4}{0.3,0.7}\right\rangle\right\},$$

$B_{\mathcal{PF}\mathfrak{R}}(A_1) = \overline{\mathcal{PF}\mathfrak{R}}(A_1) = \left\{\left\langle\frac{s_1,s_3}{0.2,0.8}\right\rangle, \left\langle\frac{s_2,s_4}{0.4,0.6}\right\rangle\right\},$

$\underline{\mathcal{PF}\mathfrak{R}}(A_2) = \left\{\left\langle\frac{t_1,t_3}{0.4,0.8}\right\rangle, \left\langle\frac{t_2,t_4}{0.5,0.6}\right\rangle\right\},$

$\overline{\mathcal{PF}\mathfrak{R}}(A_2) = \left\{\left\langle\frac{t_1,t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_2,t_4}{0.7,0.4}\right\rangle\right\},$

$B_{\mathcal{PF}\mathfrak{R}}(A_2) = \left\{\left\langle\frac{t_1,t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_2,t_4}{0.6,0.5}\right\rangle\right\}.$

Here $\tau_p(A_1) = \{0_P,\ 1_P,\ \underline{\mathcal{PF}\mathfrak{R}}(A_1),\ \overline{\mathcal{PF}\mathfrak{R}}(A_1) = B_{\mathcal{PF}\mathfrak{R}}(A_1)\}$ and $\tau_p(A_2) = \{0_P, 1_P,\ \underline{\mathcal{PF}\mathfrak{R}}(A_2),$ $\overline{\mathcal{PF}\mathfrak{R}}(A_2),\ B_{\mathcal{PF}\mathfrak{R}}(A_2)\}$ are the $\mathcal{PFN}ts's$ on U_1 and U_2 respectively.

Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be an identity function, then h_P is $\mathcal{PFN}contra\mathcal{SC}ts$ but not $\mathcal{PFN}contraIrr$, because the set $(\overline{\mathcal{PF}\mathfrak{R}}(A_2))^c$ is a $\mathcal{PFNS}os$ in U_2 but $h_P^{-1}((\overline{\mathcal{PF}\mathfrak{R}}(A_2))^c) = (\overline{\mathcal{PF}\mathfrak{R}}(A_2))^c$ is not $\mathcal{PFNS}cs$ in U_1.

Example 4.2 *Let $U_1 = \{s_1, s_2, s_3, s_4\}$, $U_2 = \{t_1, t_2, t_3, t_4\}$ are the universe sets and the equivalence relations are $U_1/R = \{\{s_1, s_3\}, \{s_2, s_4\}\}$ and $U_2/R = \{\{t_1, t_3\}, \{t_2, t_4\}\}$. Let $A_1 = \left\{\left\langle\frac{s_1}{0.4,0.7}\right\rangle, \left\langle\frac{s_2}{0.5,0.7}\right\rangle, \left\langle\frac{s_3}{0.4,0.6}\right\rangle, \left\langle\frac{s_4}{0.3,0.7}\right\rangle\right\}$ and $A_2 = \left\{\left\langle\frac{t_1}{0.4,0.8}\right\rangle, \left\langle\frac{t_2}{0.5,0.4}\right\rangle, \left\langle\frac{t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_4}{0.7,0.6}\right\rangle\right\}$ be a Pythagorean fuzzy subsets of U_1 and U_2 respectively.*

$\underline{\mathcal{PF}\mathfrak{R}}(A_1) = \left\{\left\langle\frac{s_1,s_3}{0.4,0.7}\right\rangle, \left\langle\frac{s_2,s_4}{0.3,0.7}\right\rangle\right\}$

$\overline{\mathcal{PF}\mathfrak{R}}(A_1) = \left\{\left\langle\frac{s_1,s_3}{0.4,0.6}\right\rangle, \left\langle\frac{s_2,s_4}{0.5,0.7}\right\rangle\right\}$

$B_{\mathcal{PF}\mathfrak{R}}(A_1) = \left\{\left\langle\frac{s_1,s_3}{0.4,0.6}\right\rangle, \left\langle\frac{s_2,s_4}{0.5,0.7}\right\rangle\right\}$

$\underline{\mathcal{PF}\mathfrak{R}}(A_2) = \left\{\left\langle\frac{t_1,t_3}{0.4,0.8}\right\rangle, \left\langle\frac{t_2,t_4}{0.5,0.6}\right\rangle\right\}$

$\overline{\mathcal{PF}\mathfrak{R}}(A_2) = \left\{\left\langle\frac{t_1,t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_2,t_4}{0.7,0.4}\right\rangle\right\}$

$B_{\mathcal{PF}\mathfrak{R}}(A_2) = \left\{\left\langle\frac{t_1,t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_2,t_4}{0.6,0.5}\right\rangle\right\}$

Now $\tau_P(A_1) = \{0_{\mathcal{P}},\ 1_{\mathcal{P}},\ \underline{\mathcal{PF}\mathfrak{R}}(A_1),\ \overline{\mathcal{PF}\mathfrak{R}}(A_1),\ B_{\mathcal{PF}\mathfrak{R}}(A_1)\}$, $\tau_P(A_2) = \{0_{\mathcal{P}},\ 1_{\mathcal{P}},\ \underline{\mathcal{PF}\mathfrak{R}}(A_2),\ \overline{\mathcal{PF}\mathfrak{R}}(A_2),\ B_{\mathcal{PF}\mathfrak{R}}(A_2)\}$. Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be an identity function, then h_P is $\mathcal{PFN}contra\delta\mathcal{P}Cts$ but not $\mathcal{PFN}contra\delta\mathcal{P}Irr$. Since, $(\underline{\mathcal{PF}\mathfrak{R}}(A_2))^c$ is a $\mathcal{PFN}\delta\mathcal{P}o$ set in U_2 but $h_P^{-1}((\underline{\mathcal{PF}\mathfrak{R}}(A_2))^c) = (\underline{\mathcal{PF}\mathfrak{R}}(A_2))^c$ is not $\mathcal{PFN}\delta\mathcal{P}c$ set in U_1.

Example 4.3 *Let $U_1 = \{s_1, s_2, s_3, s_4\}$, $U_2 = \{t_1, t_2, t_3, t_4\}$ are the universe sets and the equivalence relations are $U_1/R = \{\{s_1, s_4\},\ \{s_2\}, \{s_3\}\}$ and $U_2/R = \{\{t_1, t_4\}, \{t_2\},\ \{t_3\}\}$. Let $A_1 = \left\{\left\langle\frac{s_1}{0.4,0.3}\right\rangle, \left\langle\frac{s_2}{0.4,0.2}\right\rangle, \left\langle\frac{s_3}{0.5,0.5}\right\rangle, \left\langle\frac{s_4}{0.5,0.2}\right\rangle\right\}$ and $A_2 = \left\{\left\langle\frac{t_1}{0.3,0.1}\right\rangle, \left\langle\frac{t_2}{0.1,0.5}\right\rangle, \left\langle\frac{t_3}{0.2,0.45}\right\rangle, \left\langle\frac{t_4}{0.4,0.25}\right\rangle\right\}$ be a Pythagorean fuzzy subsets of U_1 and U_2 respectively.*

$\underline{\mathcal{PFN}}(A_1) = \left\{\left\langle\frac{s_1,s_4}{0.4,0.3}\right\rangle, \left\langle\frac{s_2}{0.4,0.2}\right\rangle, \left\langle\frac{s_3}{0.5,0.3}\right\rangle\right\}$,

$\overline{\mathcal{PFN}}(A_1) = \left\{\left\langle\frac{s_1,s_4}{0.5,0.2}\right\rangle, \left\langle\frac{s_2}{0.4,0.2}\right\rangle, \left\langle\frac{s_3}{0.5,0.3}\right\rangle\right\}$,

$B_{\mathcal{PFN}}(A_1) = \left\{\left\langle\frac{s_1,s_4}{0.3,0.4}\right\rangle, \left\langle\frac{s_2}{0.2,0.4}\right\rangle, \left\langle\frac{s_3}{0.3,0.5}\right\rangle\right\}$,

$\underline{\mathcal{PFN}}(A_2) = \left\{\left\langle\frac{t_1,t_4}{0.3,0.25}\right\rangle, \left\langle\frac{t_2}{0.1,0.5}\right\rangle, \left\langle\frac{t_3}{0.2,0.45}\right\rangle\right\}$,

$\overline{\mathcal{PFN}}(A_2) = \left\{\left\langle\frac{t_1,t_4}{0.4,0.1}\right\rangle, \left\langle\frac{t_2}{0.1,0.5}\right\rangle, \left\langle\frac{t_3}{0.2,0.45}\right\rangle\right\}$,

$B_{\mathcal{PFN}}(A_2) = \left\{\left\langle\frac{t_1,t_4}{0.25,0.3}\right\rangle, \left\langle\frac{t_2}{0.1,0.5}\right\rangle, \left\langle\frac{t_3}{0.2,0.45}\right\rangle\right\}$.

Here $\tau_p(A_1) = \{0_P,\ 1_P,\ \underline{\mathcal{PFN}}(A_1),\ \overline{\mathcal{PFN}}(A_1),\ B_{\mathcal{PFN}}(A_1)\}$ and $\tau_p(A_2) = \{0_P,\ 1_P,\ \underline{\mathcal{PFN}}(A_2),\ \overline{\mathcal{PFN}}(A_2),\ B_{\mathcal{PFN}}(A_2)\}$ are the $\mathcal{PFN}ts's$ on U_1 and U_2 respectively.

Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be an identity function, then h_P is $\mathcal{PFN}contra\delta\beta Cts$ but not $\mathcal{PFN}contra\delta\beta Irr$, because the set $B = \left\{\left\langle\frac{t_1,t_4}{0.4,0.1}\right\rangle, \left\langle\frac{t_2}{0.4,0.2}\right\rangle, \left\langle\frac{t_3}{0.2,0.3}\right\rangle\right\}$ is $\mathcal{PFN}\delta\beta os$ in U_2 but $h_P^{-1}(B) = B$ is not $\mathcal{PFN}\delta\beta cs$ in U_1.

Definition 4.2 *A $\mathcal{PFN}ts$ $(U_1, \tau_P(A_1))$ is known as a Pythagorean fuzzy nano $\delta\mathcal{S}U_{\frac{1}{2}}$ (resp. $\delta\mathcal{P}U_{\frac{1}{2}}$, $\delta\alpha U_{\frac{1}{2}}$ and $\delta\beta U_{\frac{1}{2}}$) (in short, $\mathcal{PFN}\delta\mathcal{S}U_{\frac{1}{2}}$ (resp. $\mathcal{PFN}\delta\mathcal{P}U_{\frac{1}{2}}$, $\mathcal{PFN}\delta\alpha U_{\frac{1}{2}}$ and $\mathcal{PFN}\delta\beta U_{\frac{1}{2}}$))-space, if each $\mathcal{PFN}\delta\mathcal{S}os$ (resp. $\mathcal{PFN}\delta\mathcal{P}os$, $\mathcal{PFN}\delta\alpha os$ and $\mathcal{PFN}\delta\beta os$) in X is $\mathcal{PFN}os$ in U_1.*

Theorem 4.2 *Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a $\mathcal{PFN}contra\delta\mathcal{S}Irr$ (resp. $\mathcal{PFN}contra\delta\mathcal{P}Irr$, $\mathcal{PFN}contra\delta\alpha Irr$ and $\mathcal{PFN}contra\delta\beta Irr$) map. Then h_P is a $\mathcal{PFN}contraCts$ map if X is a $\mathcal{PFN}\delta\mathcal{S}U_{\frac{1}{2}}$ (resp. $\mathcal{PFN}\delta\mathcal{P}U_{\frac{1}{2}}$, $\mathcal{PFN}\delta\alpha U_{\frac{1}{2}}$ and $\mathcal{PFN}\delta\beta U_{\frac{1}{2}}$)-space.*

Proof. (i) Consider a $\mathcal{PFN}os$ K in U_2. Then K is a $\mathcal{PFN}\delta\mathcal{S}os$ in U_2. Therefore $h_P^{-1}(K)$ is a $\mathcal{PFN}\delta\mathcal{S}cs$ in U_1. Since U_1 is a $\mathcal{PFN}\delta\mathcal{S}U_{\frac{1}{2}}$-space, $h_P^{-1}(K)$ is a $\mathcal{PFN}cs$ in U_1. Hence h_P is a $\mathcal{PFN}contraCts$ map.

(ii) Consider a $\mathcal{PFN}os$ K in U_2. Then K is a $\mathcal{PFN}\delta\mathcal{P}os$ in U_2. Therefore $h_P^{-1}(K)$ is a $\mathcal{PFN}\delta\mathcal{P}cs$ in U_1. Since X is a $\mathcal{PFN}\delta\mathcal{P}U_{\frac{1}{2}}$-space, $h_P^{-1}(K)$ is a $\mathcal{PFN}cs$ in U_1. Hence h_P is a $\mathcal{PFN}contraCts$ map.

(iii) Consider a $\mathcal{PFN}os$ K in U_2. Then K is a $\mathcal{PFN}\delta\alpha os$ in U_2. Therefore $h_P^{-1}(K)$ is a $\mathcal{PFN}\delta\alpha cs$ in U_1. Since U_1 is a $\mathcal{PFN}\delta\alpha U_{\frac{1}{2}}$-space, $h_P^{-1}(K)$ is a $\mathcal{PFN}cs$ in U_1. Hence h_P is a $\mathcal{PFN}contraCts$ map.

(iv) Consider a $\mathcal{PFN}os$ K in U_2. Then K is a $\mathcal{PFN}\delta\beta os$ in U_2. Therefore $h_P^{-1}(K)$ is a

$\mathcal{PFN}\delta\beta cs$ in U_1. Since U_1 is a $\mathcal{PFN}\delta\beta U_{\frac{1}{2}}$-space, $h_P^{-1}(K)$ is a $\mathcal{PFN}cs$ in U_1. Hence h_P is a $\mathcal{PFN}contraCts$ map.

Theorem 4.3 *Let* $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ *and* $g_P: (U_2, \tau_P(A_2)) \to (U_3, \tau_P(A_3))$ *be mappings. Then* $g_P \circ h_P: (U_1, \tau_P(A_1)) \to (U_3, \tau_P(A_3))$ *is*

(i) $\mathcal{PFN}contra\delta Cts$ (resp. $\mathcal{PFN}contra\delta\mathcal{S}Cts$, $\mathcal{PFN}contra\delta\mathcal{P}Cts$, $\mathcal{PFN}contra\delta\alpha Cts$ and $\mathcal{PFN}contra\delta\beta Cts$) if h_P is $\mathcal{PFN}\delta Irr$ (resp. $\mathcal{PFN}\delta\mathcal{S}Irr$, $\mathcal{PFN}\delta\mathcal{P}Irr$, $\mathcal{PFN}\delta\alpha Irr$ and $\mathcal{PFN}\delta\beta Irr$) maps and g_P is $\mathcal{PFN}contra\delta Cts$ (resp. $\mathcal{PFN}contra\delta\mathcal{S}Cts$, $\mathcal{PFN}contra\delta\mathcal{P}Cts$, $\mathcal{PFN}contra\delta\alpha Cts$ and $\mathcal{PFN}contra\delta\beta Cts$).

(ii) $\mathcal{PFN}contraIrr$ (resp. $\mathcal{PFN}contra\delta Irr$, $\mathcal{PFN}contra\delta\mathcal{P}Irr$, $\mathcal{PFN}contra\delta\mathcal{S}Irr$, $\mathcal{PFN}contra\delta\alpha Irr$ and $\mathcal{PFN}$ $contra\delta\beta Irr$) if h_P is $\mathcal{PFN}contraIrr$ (resp. $\mathcal{PFN}contra\delta Irr$, $\mathcal{PFN}contra\delta\mathcal{P}Irr$, $\mathcal{PFN}contra\delta\mathcal{S}Irr$, $\mathcal{PFN}contra\delta\alpha Irr$ and $\mathcal{PFN}contra\delta\beta Irr$) and g_P is $\mathcal{PFN}\delta Irr$ (resp. $\mathcal{PFN}\delta\mathcal{S}Irr$, $\mathcal{PFN}\delta\mathcal{P}Irr$, $\mathcal{PFN}\delta\alpha Irr$ and $\mathcal{PFN}\delta\beta Irr$).

(iii) $\mathcal{PFN}contraIrr$ (resp. $\mathcal{PFN}contra\delta Irr$, $\mathcal{PFN}contra\delta\mathcal{P}Irr$, $\mathcal{PFN}contra\delta\mathcal{S}Irr$, $\mathcal{PFN}contra\delta\alpha Irr$ and $\mathcal{PFN}$ $contra\delta\beta Irr$) if h_P is $\mathcal{PFN}\delta Irr$ (resp. $\mathcal{PFN}\delta\mathcal{S}Irr$, $\mathcal{PFN}\delta\mathcal{P}Irr$, $\mathcal{PFN}\delta\alpha Irr$ and $\mathcal{PFN}\delta\beta Irr$) and $\mathcal{PFN}contraIrr$ (resp. $\mathcal{PFN}contra\delta Irr$, $\mathcal{PFN}contra\delta\mathcal{P}Irr$, $\mathcal{PFN}contra\delta\mathcal{S}Irr$, $\mathcal{PFN}contra\delta\alpha Irr$ & $\mathcal{PFN}contra\delta\beta Irr$).

Proof. (i) Let K be a $\mathcal{PFN}os$ in U_3. Then $g_P^{-1}(K)$ is a $\mathcal{PFN}\delta\beta cs$ in U_2. As g_P is a $\mathcal{PFN}\delta\beta Irr$, $h_P^{-1}(g_P^{-1}(K))$ is a $\mathcal{PFN}\delta\beta cs$ in U_1. Thus $g_P \circ h_P$ is a $\mathcal{PFN}contraCts$ map. The other cases are similar.

Theorem 4.4 *Let* $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ *be a mapping.*

(i) If $(U_1, \tau_P(A_1))$ is $\mathcal{PFN}\delta\mathcal{S}U_{\frac{1}{2}}$ (resp. $\mathcal{PFN}\delta\mathcal{P}U_{\frac{1}{2}}$, $\mathcal{PFN}\delta\alpha U_{\frac{1}{2}}$ and $\mathcal{PFN}\delta\beta U_{\frac{1}{2}}$)-space, then the concepts of $\mathcal{PFN}$ $contraCts$ and $\mathcal{PFN}\delta\mathcal{S}Cts$ (resp. $\mathcal{PFN}\delta\mathcal{P}Cts$, $\mathcal{PFN}\delta\alpha Cts$ and $\mathcal{PFN}\delta\beta Cts$) are equivalent.

(ii) If g_P is $\mathcal{PFN}\delta\mathcal{S}U_{\frac{1}{2}}$ (resp. $\mathcal{PFN}\delta\mathcal{P}U_{\frac{1}{2}}$, $\mathcal{PFN}\delta\alpha U_{\frac{1}{2}}$ and $\mathcal{PFN}\delta\beta U_{\frac{1}{2}}$)-space, then the concepts of $\mathcal{PFN}contra\delta\mathcal{S}Cts$ (resp. $\mathcal{PFN}contra\delta\mathcal{P}Cts$, $\mathcal{PFN}contra\delta\alpha Cts$ and $\mathcal{PFN}contra\delta\beta Cts$) and $\mathcal{PFN}contra\delta\mathcal{S}Irr$ (resp. $\mathcal{PFN}contra\delta\mathcal{P}Irr$, $\mathcal{PFN}contra\delta\alpha Irr$ and $\mathcal{PFN}contra\delta\beta Irr$) are equivalent.

(iii) If h_P and g_P are $\mathcal{PFN}\delta\mathcal{S}U_{\frac{1}{2}}$ (resp. $\mathcal{PFN}\delta\mathcal{P}U_{\frac{1}{2}}$, $\mathcal{PFN}\delta\alpha U_{\frac{1}{2}}$ and $\mathcal{PFN}\delta\beta U_{\frac{1}{2}}$)-spaces, then the concepts of $N_S ScontraCts$, $\mathcal{PFN}contra\delta\mathcal{S}Cts$ (resp. $\mathcal{PFN}contra\delta\mathcal{P}Cts$, $\mathcal{PFN}contra\delta\alpha Cts$ and $\mathcal{PFN}contra\delta\beta Cts$) and $\mathcal{PFN}contra\delta\mathcal{S}Irr$ (resp. $\mathcal{PFN}contra\delta\mathcal{P}Irr$, $\mathcal{PFN}contra\delta\alpha Irr$ and $\mathcal{PFN}contra\delta\beta Irr$) are equivalent.

Proof.

(i) Let K be a $\mathcal{PFN}cs$ in U_2. Then $h_P^{-1}(K)$ is a $\mathcal{PFN}\delta\beta os$ in U_1 if h_P is $\mathcal{PFN}contra\delta\beta Cts$. As U_1 is a $\mathcal{PFN}\delta\beta U_{\frac{1}{2}}$-space, $h_P^{-1}(K)$ is a $\mathcal{PFNS}os$ in U_1. Hence h_P is also $\mathcal{PFN}contraCts$

map. The other cases are similar.

Theorem 4.5 *Let* $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ *and* $g_P: (U_2, \tau_P(A_2)) \to (U_3, \tau_P(A_3))$ *be* $\mathcal{PFN}\delta\mathcal{S}Cts$ *(resp.* $\mathcal{PFN}\delta\mathcal{P}Cts$ *,* $\mathcal{PFN}\delta\alpha Cts$ *and* $\mathcal{PFN}\delta\beta Cts$ *) mappings and* U_2 *be a* $\mathcal{PFN}\delta\mathcal{S}U_{\frac{1}{2}}$ *(resp.* $\mathcal{PFN}\delta\mathcal{P}U_{\frac{1}{2}}$ *,* $\mathcal{PFN}\delta\alpha U_{\frac{1}{2}}$ *and* $\mathcal{PFN}\delta\beta U_{\frac{1}{2}}$ *)-spaces. Then* $g_P \circ h_P: (U_1, \tau_P(A_1)) \to (U_3, \tau_P(A_3))$ *is* $\mathcal{PFN}\delta\mathcal{S}Cts$ *(resp.* $\mathcal{PFN}\delta\mathcal{P}Cts$ *,* $\mathcal{PFN}\delta\alpha Cts$ *and* $\mathcal{PFN}\delta\beta Cts$*).*

Proof. Let K be a $\mathcal{PFN}cs$ in U_3. Then $g_P^{-1}(K)$ is a $\mathcal{PFN}\delta\beta os$ in U_2 since g_P is $\mathcal{PFN}contra\delta\beta Cts$. As U_2 is a $\mathcal{PFN}\delta\beta U_{\frac{1}{2}}$-space, $g_P^{-1}(K)$ is a $\mathcal{PFN}os$ in U_2. Then, $g_P(h_P^{-1}(K))$ is $\mathcal{PFN}\delta\beta cs$ in U_1 because h_P is $\mathcal{PFN}contra\delta\beta Cts$. Hence, $g_P \circ h_P$ is a $\mathcal{PFN}\delta\beta Cts$ map.

Theorem 4.6 *Let a map* $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$. *Then the following conditions are equivalent if* U_1 *and* U_2 *are* $\mathcal{PFN}\delta\mathcal{S}U_{\frac{1}{2}}$ *(resp.* $\mathcal{PFN}\delta\mathcal{P}U_{\frac{1}{2}}$, $\mathcal{PFN}\delta\alpha U_{\frac{1}{2}}$ *and* $\mathcal{PFN}\delta\beta U_{\frac{1}{2}}$*)-spaces.*

(i) h_P is a $\mathcal{PFN}contra\delta\mathcal{S}Irr$ (resp. $\mathcal{PFN}contra\delta\mathcal{P}Irr$, $\mathcal{PFN}contra\delta\alpha Irr$ and $\mathcal{PFN}contra\delta\beta Irr$) map.

(ii) $h_P^{-1}(B)$ is a $\mathcal{PFN}\delta\mathcal{S}os$ (resp. $\mathcal{PFN}\delta\mathcal{P}os$, $\mathcal{PFN}\delta\alpha os$ and $\mathcal{PFN}\delta\beta os$) in U_1, for each $\mathcal{PFN}\delta\mathcal{S}cs$ (resp. $\mathcal{PFN}\delta\mathcal{P}cs$, $\mathcal{PFN}\delta\alpha cs$ and $\mathcal{PFN}\delta\beta cs$) B in U_2.

(iii) $\mathcal{PFN}contra\delta\mathcal{P}Irr, \mathcal{PFN}contra\delta\mathcal{S}Irr, \mathcal{PFN}contra\delta\alpha Irr$
$\mathcal{PFN}cl(h_P^{-1}(B)) \supseteq h_P^{-1}(\mathcal{PFN}int(B))$, for each pfs B of U_2.

Proof. (i) → (ii): Let B be any $\mathcal{PFN}\delta\beta cs$ in U_2. Then, B^c is a $\mathcal{PFN}\delta\beta os$ in U_2. Since h_P is $\mathcal{PFN}contra\delta\beta Irr$, $h_P^{-1}(B^c)$ is a $\mathcal{PFN}\delta\beta cs$ in U_1 . But $h_P^{-1}(B^c) = (h_P^{-1}(B))^c$. Therefore, $h_P^{-1}(B)$ is a $\mathcal{PFN}\delta\beta cs$ in U_1.

(ii) → (iii) : Let B be any pfs in U_2 and $\mathcal{PFN}int(B) \subseteq B$. Then, $h_P^{-1}(\mathcal{PFN}cl(B)) \subseteq h_P^{-1}(B)$. Since $\mathcal{PFN}int(B)$ is a $\mathcal{PFN}os$ in U_2 , $\mathcal{PFN}int(B)$ is a $\mathcal{PFN}\delta\beta os$ in U_2 . Therefore, $(\mathcal{PFN}int(B))^c$ is a $\mathcal{PFN}\delta\beta cs$ in U_2. By hypothesis, $h_P^{-1}((\mathcal{PFN}int(B))^c)$ is a $\mathcal{PFN}\delta\beta os$ in U_1. Since, $h_P^{-1}((\mathcal{PFN}int(B))^c) = (h_P^{-1}(\mathcal{PFN}int(B)))^c$, $h_P^{-1}(\mathcal{PFN}int(B))$ is a $\mathcal{PFN}\delta\beta os$ in U_1 . Since, U_1 is a $\mathcal{PFN}\delta\beta U_{1/2}$ -space, $h_P^{-1}(\mathcal{PFN}int(B))$ is a $\mathcal{PFN}os$ in U_1 . Hence, $\mathcal{PFN}cl(h_P^{-1}(B)) \supseteq \mathcal{PFN}cl(h_P^{-1}(\mathcal{PFN}int(B))) = h_P^{-1}(\mathcal{PFN}int(B))$. That is, $\mathcal{PFN}cl(h_P^{-1}(B)) \supseteq h_P^{-1}(\mathcal{PFN}int(B))$.

(iii) → (i) : Let B be any $\mathcal{PFN}\delta\beta cs$ in U_2. Since U_2 is a $\mathcal{PFN}\delta\beta U_{1/2}$-space, B is a $\mathcal{PFN}cs$ in U_2 and $\mathcal{PFN}cl(B) = B$. Hence, $h_P^{-1}(B) = h_P^{-1}(\mathcal{PFN}\delta\beta cl(B)) \supseteq \mathcal{PFN}\delta\beta int(h_P^{-1}(B))$. But clearly, $h_P^{-1}(B) \supseteq \mathcal{PFN}int(h_P^{-1}(B))$. Therefore, $\mathcal{PFN}int(h_P^{-1}(B)) = h_P^{-1}(B)$. This implies, $h_P^{-1}(B)$ is a $\mathcal{PFN}os$ and hence, it is a $\mathcal{PFN}\delta\beta os$ in X_1. Thus, h_P is a $\mathcal{PFN}contra\delta\beta Irr$ map. The proof of the others are similar. width 0.22 true cm height 0.22 true cm depth 0pt

5 Application

Measures of similarity is a real-valued function which quantifies the similarity between two pythagoren fuzzy sets and its value is always expressed as a number between 0 and 1. 0, a low level of similarity and 1, a high level of similarity.

Example 5.1

Gemstones within a species can exhibit distinct colors or optical characteristics, which are prized by gemologists and enthusiasts alike. These unique features play a crucial role in determining the gemstone's beauty, rarity, and ultimately, its value. A vast array of vibrant and colorful gemstones, representing various types, can be found in different regions of particular area. A panel of experts specializing in gemology aims to rank several of these gemstone varieties in order of preference based on a set of predetermined criteria. After thorough discussion and deliberation, the experts have identified five gemstone varieties S_1, S_2, S_3, S_4 and S_5 as alternatives for ranking. The ranking decision will be based on the following four key criteria's C_j, $j = 1, 2, 3, 4$: Color and clarity is C_1, Cut and polish is C_2, Carat and luster or originality is C_3 and hardness is C_4. Now, the experts evaluate the alternatives S_i, $i = 1,2,3,4$ under the criteriaC_j, $j = 1,2,3,4$ which can be represented by the following pfs's.

$$S_1 = \{< C_1; 0.8,0.4 >, < C_2; 0.3,0.8 >, < C_3; 0.6,0.3 >, < C_4; 0.6,0.4 >\}$$

$$S_2 = \{< C_1; 0.4,0.7 >, < C_2; 0.3,0.8 >, < C_3; 0.5,0.6 >, < C_4; 0.6,0.5 >\}$$

$$S_3 = \{< C_1; 0.6,0.5 >, < C_2; 0.7,0.4 >, < C_3; 0.3,0.8 >, < C_4; 0.3,0.5 >\}$$

$$S_4 = \{< C_1; 0.7,0.4 >, < C_2; 0.4,0.7 >, < C_3; 0.5,0.7 >, < C_4; 0.6,0.4 >\}$$

$$S_5 = \{< C_1; 0.9,0.2 >, < C_2; 0.4,0.8 >, < C_3; 0.6,0.3 >, < C_4; 0.7,0.2 >\}$$

These pfss are shown with the help of decision matrix in Table 1. The pfs of positive index set and pfs of negative index set are constructed respectively as follows:

$$S_p = \{< C_1; 1,0 >, < C_2; 1,0 >, < C_1; 1,0 >, < C_2; 1,0 >\}$$

$$S_n = \{< C_1; 0,1 >, < C_2; 0,1 >, < C_1; 0,1 >, < C_2; 0,1 >\}$$

Next, calculating the Zhang similarity between each alternatives S_i to pfs of positive index set and pfs of negative index set respectively. The the results are shown in the Table 2.The similarity between each alternative S_i to pfs of positive index set and pfs of negative index set respectively are utilized to calculate the degree of closeness [41] $\mathcal{DC}(S_i)$ is calculated as follows. $\mathcal{DC}(S_i) = \frac{\mathcal{S}_Z(S_n,S_i)}{\mathcal{S}_Z(S_p,S_i)+\mathcal{S}_Z(S_n,S_i)}$ and the results are display in the Table 3.

Table 3, shows the degree of closeness of each alternative S_i over the criteria C_j:The degree of closeness is used to rank the alternative S_i in preference order. The results are exhibited in Table 4. Table 4, shows the rank of alternative S_i according to the degree of closeness in preferred order.The alternative S_i with the largest degree of closeness is considered as the best alternative.

Table 1:pfs's of alternatives

Alternative	Criteria 1 (C_1)	Criteria 2 (C_2)	Criteria 3 (C_3)	Criteria 4 (C_4)
S_1	$< S_1, C_1; 0.8,0.4 >$	$< S_1, C_2; 0.3,0.8 >$	$< S_1, C_3; 0.6,0.3 >$	$< S_1, C_4; 0.6,0.4 >$
S_2	$< S_2, C_1; 0.4,0.7 >$	$< S_2, C_2; 0.3,0.8 >$	$< S_2, C_3; 0.5,0.6 >$	$< S_2, C_4; 0.6,0.5 >$

S_3	$< S_3, C_1; 0.6,0.5 >$	$< S_3, C_2; 0.7,0.4 >$	$< S_3, C_3; 0.3,0.8 >$	$< S_3, C_4; 0.3,0.5 >$
S_4	$< S_4, C_1; 0.7,0.4 >$	$< S_4, C_2; 0.4,0.7 >$	$< S_4, C_3; 0.5,0.7 >$	$< S_4, C_4; 0.6,0.4 >$
S_5	$< S_5, C_1; 0.9,0.2 >$	$< S_5, C_2; 0.4,0.8 >$	$< S_5, C_3; 0.6,0.3 >$	$< S_5, C_4; 0.7,0.2 >$

Table 2: Zhang similarity measure of each alternative with S_p and S_n

Alternative	$\mathcal{S}_Z(S_p, S_i)$	$\mathcal{S}_Z(S_n, S_i)$
S_1	0.476	1.000
S_2	0.618	1.000
S_3	0.536	1.000
S_4	0.495	0.181
S_5	0.588	0.090

Table 3: Degree of closeness of each alternative

Alternative	$\mathcal{DC}(S_i)$
S_1	0.68
S_2	0.62
S_3	0.65
S_4	0.27
S_5	0.13

Table 4:Ranking of each alternative

Similarity	Ranking
$\mathcal{S}_Z(S_i)$	$\mathcal{DC}(S_5) < \mathcal{DC}(S_4) < \mathcal{DC}(S_2) < \mathcal{DC}(S_3) < \mathcal{DC}(S_1)$

Hence, the alternative S_1 is selected as the best alternative.

6 Conclusion

In this paper, $\mathcal{PFN}contraCts$, $\mathcal{PFN}contra\delta Cts$, $\mathcal{PFN}contra\delta \mathcal{S}Cts$, $\mathcal{PFN}contra\delta \mathcal{P}Cts$, $\mathcal{PFN}contra\delta\alpha Cts$, and $\mathcal{PFN}\ contra\delta\beta Cts$ respective irresolute map is defined using $\mathcal{PFN}\delta o$, $\mathcal{PFN}\delta \mathcal{S}o$, $\mathcal{PFN}\delta \mathcal{P}o$, $\mathcal{PFN}\delta\alpha o$ and $\mathcal{PFN}\delta\beta o$ set and its properties are analyzed with the examples. Also we extended the concept of Pythagorean fuzzy contra irresolute maps in Pythagorean fuzzy topological spaces using above mentioned open sets. Some examples and basic relationships between the contra irresolute mappings were also discussed. In future, these can be extended to Pythagorean fuzzy open, closed, homeomorphism and contra maps. Application for Economical decision making problem was solved with the proposed similarity measure. In future, MCDM to the field of medical diagnostic can be develope to the $\mathcal{PFN}ts$.

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