Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

A New Stochastic Partial Differential Model for Image Restoration

[1]Radhia Halilou, [2]Fatma Zohra Nouri and [3]Mohamed Lakhdar Hadji,

[1,2,3]Mathematical Modeling and Numerical Simulation Laboratory, Department of Mathematics, Badji Mokhtar University, PO Box. 12, Annaba 23000-Algeria,

[1]e-mail: radhia.halilou@univ-annaba.dz, [2]email: tassili.nan09@gmail.com, [3]e-mail: ml_hadji@yahoo.fr,

Article History:

Received: *12-01-2025*

Revised: *15-02-2025*

Accepted: *01-03-2025*

Abstract:

In this paper, we propose a stochastic partial differential equation model for the restoration of noisy images. This proposed model is based on the well-known Perona-Malik one, to which we add a stochastic process. The mathematical analysis of the proposed model is carried out within a well-defined framework, considering the specific conditions associated with an anisotropic diffusion. For the numerical approximation, we employ a stable finite difference scheme, ensuring its stability through Fourier analysis. The obtained numerical results demonstrate that our approach enhances images while preserving their important structure details, proving efficiency and competiveness compare to other approaches for image restoration.

Keywords: Stochastic Partial Differential Equations, Image Denoising, Anisotropic Diffusion.

1. Introduction

Mathematics plays an important role in image processing, providing effective tools for various tasks such as noise removal, edge detection, and feature enhancement. Various techniques, including, variational methods, Fourier-based approaches, deterministic and stochastic differential equations have been utilized to tackle these problems effectively.

Recently, stochastic partial differential equations (SPDEs) have gained attention in image processing as a promising framework for removing noise from images and enhancing their structures. The SPDEs are build on the foundational work of stochastic differential equations (SDEs). Early research in this field focused on employing different SDEs using various diffusion processes for image noise reduction. The authors Descombes and Zhizhina in 2003 [10] explored the application of SDEs, demonstrating their potential in image denoising. Later, Barbu in 2016 [1] introduced a novel SDE-based image restoration technique, which significantly improved noise reduction while preserving important image structures. More recent advancements include the use of backward stochastic differential equations (BSDEs) for image denoising by Borkowski et al [5], [2] and [3] applied BSDEs to reconstruct RGB images affected by additive gaussian noise, achieving a great success in smoothing noisy pixels while enhancing edges.

The evolution of SPDEs in image denoising reflects a broader trend in applying stochastic models to image processing tasks. By integrating randomness into the model, the SPDE-based approaches offer enhanced flexibility and robustness, leading to more effective noise reduction and better preservation of critical image features.

In this work, we focus on the restoration of noisy images, and propose a novel SPDE model that extends Perona- Malik (PM) [17] model by including a stochastic process. The PM model [17], though effective, is known to be ill-posed, which limits its applicability. To address the well-known ill-posedness of the PM model, we adopt the regularization approach of Catté et al. introduced in 1992 [8] and [15], replacing the gradient $|\nabla u|$ by $|\nabla G_\sigma * u|$, to ensure the consistency and well-posedness of the model. The main idea here is that this proposed SPDE captures the stochastic nature of noise more effectively.

We build the mathematical study of this SPDE by combining the deterministic analysis of Catté et al. [8] and the weak solution theory developed by Benssoussen and Temam (1971) [4]. To approximate the solution, we employ an appropriate finite difference scheme and verify its stability via Fourier analysis. Numerical experiments demonstrate the efficiency of the proposed model in enhancing image quality, while preserving structure details, showing significant improvements in noise removal and feature refinement.

This paper is organized as follows: Section 2 introduces the proposed SPDE model. Section 3 provides the mathematical analysis of the model. In Section 4, the numerical discretisation is detailed with a stability analysis, and results. Finally, concluding remarks are presented in Section 5.

2. Proposed Model

In this section, we use a 2D standard Brownian motion to perturb a regularised PDE due to PM [17] proposed by F. Catté, P.L. Lions, J.M. Morel and T. Coll [8] and [15]. The problem can be written as follows

$$\begin{cases} \frac{\partial u}{\partial t} = div(g(|\nabla G_\sigma * u|)\nabla u) + dW_t & in\ ]0,T[\times D, \\ u(0,x) = u_0(x), \forall x \in D, \end{cases} \tag{1}$$

with

$$g(|\nabla G_\sigma * u|) = e^{-\frac{|\nabla G_\sigma * u|^2}{k^2}} \quad or\ g(|\nabla G_\sigma * u|) = \frac{1}{1+\frac{|\nabla G_\sigma * u|^2}{k^2}}\ , \quad k > 0 \tag{2}$$

In the problem *(1)* we have the following denotes

- $u_0 : D \subset \mathbb{R}^2 \to \mathbb{R}$ be the initial image.
- $D \subset \mathbb{R}^2$ is a bounded domain with a Lipschitz boundary and $0 < T < \infty$.
- G_σ is a Gaussian filter (GF), $G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{|x|^2}{4\sigma}\right), \quad \sigma > 0, \quad x = (x_1, x_2) \in R^2$

- $(D, \mathcal{F}, \mathcal{P})$ a completely regular topological space equipped with its Borel σ-algebra $\mathcal{F}$, and $\mathcal{P}$ is a Radon probability measure (i.e., an abstract measure on $\mathcal{F}$ that is inner regular) [19] and [13].

- W_t be a Wiener process defined on $(D, \mathcal{F}, \mathcal{P})$ and taking values in the separable Hilbert space H, with incremental covariance operator w. Let $(\mathcal{F}_t)_{t>0}$ be the σ-algebra generated by $W_s, 0 \leq s \leq t$ then W_t is a martingale relative to $((\mathcal{F}_t)_{t>0}$ and we have the following representation of W_t :

$$W_t = \sum_{i=1}^{\infty} \beta_t^i \, e_i$$

where e_i is an orthonormal set of eigenvectors of w, β_t^i are mutually independent real Wiener processes with incremental covariance $\lambda > 0, we_i \ and \ \ trw = \sum_{i=1}^{\infty} \lambda_i$ (tr denotes the trace of an operator, see [6] and [3] for more details on stochastic analysis).

In the next sections, we consider processes as: $W = W(t, x)$ of the form of the following proposition.

Proposition 1 *Under the previous assumptions,*

$$W_t \in C([0, T]; L^2(D, \mathcal{P}; H)) \tag{3}$$

Proof. For the details of the proof of this result, the reader is referred to [4] ■

3. Mathematical Study

In this section, we investigate the well posedness of *(1)*, according to C. Catté et al. [8] and [15], A. Bensoussan, R. Temam [4], E. Pardoux [16] and T.C. Garrido [11]. The variational method we will employ is defined within two fundamental spaces: a real, reflexive, and separable Banach space V, and a real Hilbert space H. We identify H with its dual and denote the dual space of $V \ by \ V\,'$. The embedding $V \hookrightarrow H$ is continuous, and V is dense in H. These relationships are summarised as:

$$V \subset H \subset V\,'$$

We will denote by $\|.\|, |\,.\,|$ and $\|.\|_*$ the norms in V , H and $V\,'$ respectively; by $\langle .,. \rangle$ the duality product between $V, V\,'$. We introduce H, V , $\mathcal{H}$ and $\mathcal{V}$ as follow

- $H = L^2(D)$, the space of square-integrable functions over D, with the inner product

$$(u, v) = \int_D u(x) v(x) dx$$

- $V = H_0^1(D) = \left\{ v \in H, \ \frac{\partial v}{\partial x} \in H, \ with \ v = 0 \ on \ the \ boundary \ \Gamma \right\}$
- $\mathcal{H} = L^2(D, \mathcal{P}; H)$, Hilbert space with scalar product

$$(u, v)_{\mathcal{H}} = E\big(u(.), v(.)\big) = \big(u(x), v(x)\big) d\mathcal{P}(x) \tag{4}$$

- $\mathcal{V} = L^2(D, \mathcal{P}; V)$, Banach space equipped with the norm

$$\|u\|_{\mathcal{V}} = \left\{ \int_D \|u\|_V^2 d\mathcal{P}(x) \right\}^{\frac{1}{2}} \tag{5}$$

- $\mathcal{V}'$ is the dual space of $\mathcal{V}$, by the Riesz representation theorem, the norm of $\mathcal{A}$(u) in $\mathcal{V}'$ is

$$\|\mathcal{A}(u)\|_{\mathcal{V}'} = \sup_{\|v\|_{\mathcal{V}} \leq 1} |\langle A(u), v\rangle| \tag{6}$$

We introduce the following notation

$$\mathcal{A}(u) = -div\,(g(|\nabla G_\sigma * u|)\nabla u) \tag{7}$$

The next lemma plays a crucial role in establishing the well-posedness of the proposed model *(1).*

Lemma 1 *Let $\mathcal{A}(.) : \mathcal{V} \to \mathcal{V}'$ be a nonlinear operator defined almost everywhere in t. We assume the following hypotheses:*

1. *Coercivity:* $\exists\, \rho > 0$ *such that*

$$\langle \mathcal{A}(u), v\rangle_{\mathcal{V},\mathcal{V}'} \geq \rho\|u\|_{\mathcal{V}}^2, \quad \forall u, v \in \mathcal{V},\ a.e.t. \tag{8}$$

2. *Monotonicity:*

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle_{\mathcal{V},\mathcal{V}'} \geq 0, \quad \forall u, v \in \mathcal{V},\quad a.e.t. \tag{9}$$

3. *Boundedness:* $\exists \beta > 0$:

$$\|\mathcal{A}(u)\|_{\mathcal{V}'} \leq \beta\|u\|_{\mathcal{V}}\ ,\ \forall u \in \mathcal{V}, a.e.t. \tag{10}$$

4. *Hemi-continuity:* $\forall u, v \in \mathcal{V}, and\ \psi \in \mathcal{V}'\ a.e.t \in [0, T]$

$$\Theta \in \mathbb{R} \longrightarrow \langle \mathcal{A}(u + \Theta v), \psi\rangle_{\mathcal{V},\mathcal{V}'}\ \ is\ continuous\ in\ \mathbb{R} \tag{11}$$

Proof.

According to [8], g, G are infinitely differentiable in D. So, g(.)∈ L^∞(0,T ;C^∞(D)). Thus, since g is decreasing, there exists a constant ρ such that

$$g(|\nabla G_\sigma * \varpi|) \geq \rho\ ,\ a.e. \in]0, T[\times D, \tag{12}$$

where $\|\varpi\|_{L^\infty(0,T:L^2(D))} \leq \|u_0\|_{L^2(D)}$.

1. Coercivity

$$\begin{aligned}\langle \mathcal{A}(u), u\rangle &= -[div(g(|\nabla G_\sigma * u|)\nabla u)u]d\mathcal{P}(x)\\ &= g(|\nabla G_\sigma * u|)(\nabla u)^2 d\mathcal{P}(x)\\ &\geq \rho\|u\|_{\mathcal{V}}^2\end{aligned} \tag{13}$$

Hence, *(8)* hold.

2. Monotonicity

$$\begin{aligned}&\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle\\ &= -\int_D [(div(g(|\nabla G_\sigma * u|)\nabla u) - div(g(|\nabla G_\sigma * v\,|)\nabla v))(u - v)]d\mathcal{P}(x)\\ &= -\int_D (div(g(|\nabla G_\sigma * u|)\nabla u)u)d\mathcal{P}(x) + \int_D (div(g(|\nabla G_\sigma v|)\nabla v)u)d\mathcal{P}(x)\\ &\quad + \int_D (div(g(|\nabla G_\sigma * u|)\nabla u)v)d\mathcal{P}(x) - \int_D (div(g(|\nabla G_\sigma * v|)\nabla v)v)d\mathcal{P}(x)\\ &= \int_D g(|\nabla G_\sigma * u|)(\nabla u)^2 d\mathcal{P}(x) - \int_D g(|\nabla G_\sigma * v|)\nabla u \nabla v\, d\mathcal{P}(x)\\ &\quad - \int_D g(|\nabla G_\sigma * u|)\nabla u \nabla v\, d\mathcal{P}(x) + \int_D g(|\nabla G_\sigma * v|)(\nabla v)^2 d\mathcal{P}(x)\end{aligned}$$

$$= \int_D g(|\nabla G_\sigma * u|)(\nabla u^2 - \nabla u \nabla v)d\mathcal{P} + \int_D g(|\nabla G_\sigma * v|)(\nabla v^2 - \nabla u \nabla v)d\mathcal{P}(x)$$
$$\geq \min_{\rho,\rho' \geq 0} (\rho, \rho') \left(\int_D (\nabla u - \nabla v)^2 d\mathcal{P}(x) \right)$$
$$= \min_{\rho,\rho' \geq 0} (\rho, \rho') \, \|u - v\|_{\mathcal{V}}^2 \geq 0 \,, \qquad \forall u, v \in \mathcal{V} \tag{14}$$

Then, the condition *(9)* hold.

3. Boundedness

Now, we verify the boundedness condition, we use the weak form of $\mathcal{A}(u)$ and the Cauchy-Schwartz
inequality, we obtain

$$\|\mathcal{A}(u)\|_{\mathcal{V}'} = \sup_{\|v\|_{\mathcal{V}'} \leq 1} |\langle \mathcal{A}(u), v \rangle|$$
$$= \sup_{\|v\|_{\mathcal{V}'} \leq 1} |\int_D (g(|\nabla G_\sigma * u|) \nabla u \nabla v) d\mathcal{P}|$$
$$\leq \sup_{\|v\|_{\mathcal{V}'} \leq 1} \| g(|\nabla G_\sigma * u|) \|_{L^2(D)} \| \nabla u \|_{L^2(D)} \| \nabla v \|_{L^2(D)} \tag{15}$$

By using the Poincaré inequality,
we obtain

$$\|\mathcal{A}(u)\|_{\mathcal{V}'} \leq \beta \|u\|_{\mathcal{V}} \quad , \beta > 0. \tag{16}$$

Hence, *(10)* holds. ■

4. Hemi-continuity: in [8], we find the weak continuity of the deterministic case of *(1)*, for $\Theta \to 0$

$$\langle \mathcal{A}(u(x) + \Theta v(x)), \psi(x) \rangle \longrightarrow \langle \mathcal{A}(u(x)), \psi(x) \rangle \quad a.s \tag{17}$$

by using the boundedness *(14)*, we obtain

$$|\langle \mathcal{A}(u(x) + \Theta v(x)), \psi(x) \rangle| < \|\mathcal{A}(u(x)) + \Theta v(x))\|_{\mathcal{V}} \|\psi(x)\|_{\mathcal{V}}$$
$$\leq \beta \|u(x) + \Theta v(x)\|_{\mathcal{V}'}^{p-1} \|\psi(x)\|_{\mathcal{V}}$$
$$\leq \beta \, (\|u(x)\|_{\mathcal{V}'} + \|v(x)\|_{\mathcal{V}'})^{p-1} \, \|\psi(x)\|_{\mathcal{V}} \tag{18}$$

(By limiting to $|\Theta| \leq 1$, it is sufficient).
According to Lebesgue's theorem, we obtain

$$\int_D \langle \mathcal{A}(u(x) + \Theta v(x)), \psi(x) \rangle d\mathcal{P}(x) \to \int_D \langle \mathcal{A}(u(x)), \psi(x) \rangle d\mathcal{P}(x) \tag{19}$$

Hence, *(10)* holds. ■

After proving lemma 1, we need to prove the following propositions 2 and 3 to verify existence and uniqueness in the weak sense.

Proposition 2 *if $t \to u(t)$ is a measurable mapping with values in $\mathcal{V}$, then $t \to \mathcal{A}(u(t))$ is measurable with values in $\mathcal{V}'$.*

Proof. We have proved above that the operator $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ is hemi-continuous, monotone, and coercive.

Consequently, $\mathcal{A}$ is continuous from strong $\mathcal{V}$ to weak $\mathcal{V}'$. Hence, the proposition 2 holds. (see [4], p.105) ■

Proposition 3 *If* $u \in L^2(\mathcal{V})$, *then*

$$\{t \to \mathcal{A}(u(t))\} \in L^2(\mathcal{V}') \quad a.e.t. \tag{20}$$

Proof. If $u \in L^2(\mathcal{V})$, then by the proposition 2, $\mathcal{A}(u(t))$ is measurable with values in $\mathcal{V}'$, and by *(14)*,
$\|\mathcal{A}(u)\|_{\mathcal{V}'} \leq \beta\|u\|_{\mathcal{V}}$. Raising this inequality to the power 2 and integrating, it follows that $\boldsymbol{\mathcal{A}}(u(t)) \in L^2\left(\mathcal{V}'\right)$. The other properties stated above, such as monotonicity, hemi-continuity, and coercivity, are similarly verified using the assumptions and Lebesgue's theorem. Therefore, all hypotheses are satisfied (see [4]). ■

Theorem 1 *Under the hypotheses of lemma 1 of* $\mathcal{A}(.)$, *and propositions 1, 2 and 3, then there exists a unique solution* $u \in L^2(\mathcal{V}) \cap C(H)$, for $t \in]0,T[$, *verifying*

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}(u) = \frac{\partial W}{\partial t} & in\ ]0,T[\times D \\ u(0,x) = u_0(x), & \forall x \in D \end{cases} \tag{21}$$

Proof. Let us consider the operator $\mathcal{A}_W : \mathcal{V} \to \mathcal{V}'$ defined by

$$\mathcal{A}_W(v) = \mathcal{A}(v + W_t),\ for\ v \in \mathcal{V} \tag{22}$$

So, *(21)* became as follow

$$\begin{cases} \frac{\partial v}{\partial t} + \mathcal{A}_W(v(x)) = 0 \\ u(0,x) = u_0(x), \ \forall x \in D \end{cases} \tag{23}$$

The necessary and sufficient condition for *(21)* to have a unique solution is that *(23)* has a unique solution in $L^2(\mathcal{V}) \cap L^\infty(\mathcal{H})$, such that

$$\frac{\partial v}{\partial t} \in L^2(\mathcal{V}') + L^1(\mathcal{H})$$

Indeed, if $v \in L^2(\mathcal{V}) \cap L^\infty(\mathcal{H}), \frac{\partial v}{\partial t} \in L^2(\mathcal{V}') \cap L^1(\mathcal{H})$, then $v \in C(\mathcal{H})$ If we set $u = v + W$, then $u \in L^2(\mathcal{V}) \cap C(\mathcal{H})$ (thanks to *(3)*), and it is clear that *u* is a solution of *(1)*.

The reverse implication shows that if *u* is a solution of *(21)*, then $v = u - W$ is a solution of *(23)*, and

$$\frac{\partial v}{\partial t} = -\mathcal{A}(u) \in L^2(\mathcal{V}') \tag{24}$$

Lemma 2 *Let* $\mathcal{A}_W : \mathcal{V} \to \mathcal{V}'$ *satisfied for a.e.* $t \in [0,T]$

1. $\mathcal{A}_W(.)$ *is hemi-continuous;*

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

2. $\mathcal{A}_W(.)$ *is monotone;*

3. *If* $u \in L^2(\mathcal{V})$, *then* $\{t \rightarrow \mathcal{A}_W(u(x)) \in L^2(\mathcal{V}')\}$

Proof. Let $u, v \in \mathcal{V}$ and $\psi \in \mathcal{V}'$; we have $\Theta \in R$

$$\langle \mathcal{A}_W(u + \Theta v), \psi \rangle_{\mathcal{V},\mathcal{V}'} = \langle \mathcal{A}(u + W_t + \Theta v), \psi \rangle_{\mathcal{V},\mathcal{V}'} \tag{24}$$

and according to *(19)*, when $\Theta \longrightarrow 0$, the second member of *(25)* converges to

$$\langle \mathcal{A}(u + W_t), \psi \rangle_{\mathcal{V},\mathcal{V}'} = \langle \mathcal{A}_W(u), \psi \rangle_{\mathcal{V},\mathcal{V}'}. \tag{25}$$

Hence, $\mathcal{A}_W(.)$ is hemi-continuous.

To prove the monotonicity of $\mathcal{A}_W(u)$, we write

$\langle \mathcal{A}_W(u) - \mathcal{A}_W(v), u - v \rangle = \langle \mathcal{A}(u + W_t) - \mathcal{A}(v + W_t), (u + W_t) - (v + W_t) \rangle \geq 0$, based on *(9)*.

Finally, if $u \in L^2(\mathcal{V})$ and according to *(3)*, then $u(.) + W(.) \in L^2(\mathcal{V})$, and therefore

$$\mathcal{A}_W u(.) = \mathcal{A}(u(.) + W(.)) \in L^2(\mathcal{V}') \tag{26}$$

■

3.1 Approximation

Let N an integer intended to approach infinity and $k = \frac{T}{N}$. We consider a partition of the interval $[0, T]$, $0, k, 2k, \ldots, Nk$. We propose

$$W_n = W(nk) \in \mathcal{V} \tag{27}$$

and introduce the family of operators $\mathcal{A}_W^n : \mathcal{V} \longrightarrow \mathcal{V}'$ defined by

$$\mathcal{A}_W^n \psi = \frac{1}{k} \int_{(n-1)k}^{nk} \mathcal{A}(\psi + W^n) dt \tag{28}$$

Consider the recurrence relations

$$\begin{cases} \frac{v^n - v^{n-1}}{k} + \mathcal{A}_W v^n = 0 \\ v(0, x) = u_0 - W_0 \end{cases} \tag{29}$$

First, note that *(30)* uniquely defines a sequence v_n of elements in $\mathcal{V}$ (except for $n = 0$, where $v_n \in \mathcal{H}$). Indeed, introduce $\mathcal{A}_n \mathcal{V} \longrightarrow \mathcal{V}'$, defined as

$$\mathcal{A}^n \psi = \frac{1}{k} \int_{(n-1)k}^{nk} \mathcal{A} \psi dt, \quad \forall \psi \in \mathcal{V} \tag{30}$$

Then *(30)* can be written as

$$\frac{v^n - v^{n-1}}{k} + \mathcal{A}^n(v^n + W^n) = 0 \tag{32}$$

But then, by setting $u^n = v^n + W^n$, we see that u^n must satisfy the recurrence relations

$$\begin{cases} \frac{u^n - u^{n-1}}{k} + \mathcal{A}^n u^n = \frac{W^n - W^{n-1}}{k} \\ u^0 = u_0 \end{cases} \tag{33}$$

It follows from the properties of $\mathcal{A}(.)$ (lemma 1) that $\mathcal{A}^n$ is monotone, hemi-continuous, and coercive from $\mathcal{V}$ to $\mathcal{V}'$, and consequently (cf. Lions [11]) $(I + k^n)$ is invertible. Thus, in *(33)*, when u^{n-1} is known, u^n is uniquely defined as an element of $\mathcal{V}$. We now introduce the step functions.

$$\begin{cases} W_k(t) = W^n \ \ dans\ [nk, (n+1)k[\\ u_k(t) = u^n \ \ dans\ [nk, (n+1)k[\\ v_k(t) = v^n \ \ dans\ [nk, (n+1)k[\end{cases} \tag{34}$$

Lemma 3 $u_k(.)$ *and* $\mathrm{v}_k(.)$ *remain, as* $k \longrightarrow 0$, *within bounded subsets of* $L^\infty(\mathcal{H})$ *and* $L^2(\mathcal{V})$.

Proof. Let us consider relation *(32)*, which is written as

$$v^n - v^{n-1} + k\mathcal{A}^n u^n = 0 \tag{35}$$

So,

$$(v^n - v^{n-1}, v^n) + k\langle \mathcal{A}^n u^n, u^n - W^n \rangle = 0 \tag{36}$$

Let

$$(v^n - v^{n-1}, v^n) + k\langle \mathcal{A}^n u^n, u^n \rangle = k\langle W^n, \mathcal{A}^n u^n \rangle \tag{37}$$

But

$$(v^n - v^{n-1}, v^n) = \frac{1}{2}(|v^n|^2 - |v^{n-1}|^2) + \frac{1}{2}|v^n - v^{n-1}|^2 \tag{38}$$

Thus *(37)* implies

$$|v^n|^2 - |v^{n-1}|^2 + |v^n - v^{n-1}|^2 + 2k\,\langle \mathcal{A}^n u^n, u^n \rangle = 2k\langle W^n, \mathcal{A}^n u^n \rangle \tag{39}$$

According to properties *(8)* and *(10)* of $\mathcal{A}$, which lead to the same for $\mathcal{A}^n$, we deduce from *(39)* the following estimate

$$|v^n|^2 - |v^{n-1}|^2 + 2k\rho\|u^n\|^2 \leq 2k\beta\|W^n\|_{\mathcal{V}}\|u^n\|_{\mathcal{V}} \tag{40}$$

We then use the following classic inequality: if $i, j > 0$ satisfy $\frac{1}{i} + \frac{1}{j} = 1$, then

$$ab \leq \frac{a^i c^i}{i} + \frac{b^j}{jc^j}, \forall a, b, c > 0.$$

Therefore, we have

$$\|W^n\|_{\mathcal{V}}\|u^n\|_{\mathcal{V}}^{p-1} \leq \|u^n\|_{\mathcal{V}}^p \frac{l^{p'}}{p'} + \|W^n\|_{\mathcal{V}}^p \frac{1}{pl^p}, \quad \forall l > 0. \tag{41}$$

Let us choose l such that $C = 2\rho + 2\beta\frac{l^2}{2}$. Then, taking into account *(41)*, we deduce from *(36)* the following inequality

$$|v^n|^2 - |v^{n-1}|^2 + kC\|u^n\|^2 \leq \frac{2k\beta}{2l^2}\|W^n\|_{\mathcal{V}} \tag{42}$$

Since $W_t \in C(\mathcal{V})$

$$\|W^n\|_{\mathcal{V}} \leq C_1, \qquad \forall n \tag{43}$$

From *(43)*, we finally obtain the upper bound

$$|\mathcal{v}^n|^2_{\mathcal{H}} - |\mathcal{v}^{n-1}|^2_{\mathcal{H}} + kC\|u^n\|^2_{\mathcal{V}} \leq kC_2, \tag{44}$$

where $C_2 = 2C_1\frac{\beta}{2l^2}$.

By summing, we easily deduce the following bounds from *(44)*.

$$\begin{cases} \forall n, |\mathcal{v}^n|^2_{\mathcal{H}} \leq |u_0 - W_0|^2_{\mathcal{H}} + TC_2 \\ C\sum_{n=1}^{N} k\,\|u^n\|^2_{\mathcal{V}} \leq |u_0 - W_0|^2_{\mathcal{H}} + TC_2 \end{cases} \tag{45}$$

But *(45)* means that u_k remains bounded in $L^2(\mathcal{V})$ and $\mathcal{v}_k$ remains bounded in $L^\infty(\mathcal{H})$. Since W_k remains bounded in $L(\mathcal{V})$, the lemma follows.

Let t be a fixed value in $[0, T]$, and define $n_t = \frac{t}{k}$. By summing the discret relations for n from 1 to n_t, we obtain

$$\mathcal{v}^{n_t} + k\sum_{n=1}^{n_t}\mathcal{A}^n u^n = u_0 - W_0, \tag{46}$$

Which is also written as

$$\mathcal{v}_k(t) + \int_0^{n_t k}\mathcal{A}_k u_k\,(s)ds = u_0 - W_0 \tag{47}$$

Also, $\mathcal{A}_k$ can be replaced by $\mathcal{A}$ in this equality (see [9]). Now, let $\chi \in \mathcal{V}$, *(47)* gives the following

$$(\mathcal{v}_k(t), \chi) + \int_0^T\langle\mathcal{A}u_k(s), \zeta_{n_t k}(s)\chi\rangle ds = u_0, \tag{48}$$

where $\zeta_{n_t k}(s)$ denotes the characteristic function of $]0, n_t k[$.

Lemma *3* allows us to extract sequences u_k and $\mathcal{v}_k$ from subsequences denoted in the same way, which converge to elements u and $\mathcal{v}$ in weak $L^2(\mathcal{V})$ and weak-star $L^\infty(\mathcal{H})$. It is evident that $u = \mathcal{v} + W$.

Furthermore, according to property *(10)* of $\mathcal{A}$, we deduce from lemma *3* that $\mathcal{A}u_k(s)$ remains bounded in $L^2(\mathcal{V}')$ and by extracting a new subsequence if necessary, we can always assume that:

$$\mathcal{A}u_k(.) \to \Psi(s) \text{ weakly in } L^2(\mathcal{V}').$$

We can then take the limit in *(48)* and obtain

$$(\mathcal{v}_k(t), \chi) + \int_0^t\langle\Psi(s), \chi\rangle ds = u_0\ . \tag{49}$$

Then

$$\frac{d\mathcal{v}}{dt} + \Psi(s) = 0, \;\; a.e.\, t \in [0,T] \;\; (equality\ in\ \mathcal{V}') . \tag{50}$$

Then, we have

Lemma 4

$$\Psi(\mathrm{t}) = \mathcal{A}\mathrm{u}(\mathrm{t}), \;\; \mathrm{a.e.\,t} \in [0,\mathrm{T}] \tag{51}$$

Proof. First, let's prove that

$$\int_0^T \langle \mathcal{A}u_k(s), u_k(s)\rangle ds \xrightarrow{k\to 0} \int_0^T \langle \Psi(s), u(s)\rangle ds. \tag{52}$$

Indeed, *(50)* shows that $\frac{\mathrm{d}\mathcal{v}}{\mathrm{dt}} \in L^2\,(\mathcal{V}')$. Since $\mathcal{v} \in L^2\,(\mathcal{V})$, it follows (cf. [16]) that we have

$$\frac{d}{dt}|\mathcal{v}(t)|_{\mathcal{H}}^2 = 2\,\langle \mathcal{v}(t), \frac{d\mathcal{v}}{dt}(t)\rangle_{\mathcal{V},\mathcal{V}'} \quad a.e.\,t. \tag{53}$$

Let

$$|\mathcal{v}(T)|^2 = |\mathcal{v}(0)|^2 + 2\int_0^T \langle \mathcal{v}(s) - \Psi(s)\rangle_{\mathcal{V},\mathcal{V}'} ds. \tag{54}$$

Next, consider the equalities *(39)*, which we sum over n from 1 to N. From this, we easily deduce the inequality

$$|\mathcal{v}_k(T)|^2 + 2\int_0^T \langle \mathcal{A}u_k, u_k(s)\rangle ds \;\le |\mathcal{v}(0)|^2 + 2\int_0^T \langle W_k(s), \mathcal{A}u_k\rangle ds. \tag{55}$$

By taking the upper limit, we obtain

$$\limsup_{k\to 0} \int_0^T \langle \mathcal{A}u_k, u_k(s)\rangle ds \le \frac{1}{2}|\mathcal{v}(0)|^2 + \int_0^T \langle W_k(s), \Psi(s)\rangle ds - \liminf_{k\to 0} \frac{1}{2}|\mathcal{v}_k(T)|^2, \tag{56}$$

but,

$$\frac{1}{2}|\mathcal{v}(T)|^2 \le \liminf_{k\to 0} \frac{1}{2}|\mathcal{v}_k(T)|^2. \tag{57}$$

And considering *(54)*, we then deduce from *(56)*

$$\begin{aligned}\limsup_{\mathrm{k}\to 0} \int_0^{\mathrm{T}} \langle \mathcal{A}\mathrm{u_k}, \mathrm{u_k(s)}\rangle \mathrm{ds} &\le \frac{1}{2}|\mathcal{v}(0)|^2 + \int_0^{\mathrm{T}} \langle \mathrm{W_k(s)}, \Psi(s)\rangle \mathrm{ds} - \frac{1}{2}|\mathcal{v}(0)|^2 \\ &\quad - \int_0^T \langle u(s) - W(s), \Psi(s)\rangle ds \\ &= \int_0^T \langle u(s), \Psi(s)\rangle ds\end{aligned} \tag{58}$$

And as $\mathcal{A}$ is monotone, we deduce that

$$\liminf_{k\to 0} \int_0^T \langle \mathcal{A}u_k(s), u_k(s)\rangle ds \ge \int_0^T \langle \Psi(s), u(s)\rangle ds, \tag{59}$$

which, compared with *(58)*, clearly shows *(52).*

To prove lemma 4. from *(52)*, we use a classical technique based on the monotonicity and hemi-continuity of $\mathcal{A}$ (cf. Lions [12], Brézis [7], Da. Prato[9] and Minty [14]).

According to the monotonicity of $\mathcal{A}$, we have, for all $\emptyset \in L^2(\mathcal{V})$

$$\int_0^T \langle \mathcal{A}u_k(s) - \mathcal{A}\emptyset(s), u_k(s) - \emptyset(s)\rangle ds \geq 0 \tag{60}$$

Thus, by passing to the limit, from *(52)*, it follows that

$$\int_0^T \langle \Psi(s) - \mathcal{A}\emptyset(s), u(s) - \emptyset(s)\rangle ds \geq 0 \tag{61}$$

We then choose, $\emptyset(\mathrm{t}) = \mathrm{u}(\mathrm{t}) - \gamma\varphi(\mathrm{t})$, where $\varphi \in L^2(\mathcal{V})$ and $\lambda > 0$ are arbitrary; *(61)* became as follow

$$\int_0^T \langle \Psi(s) - \mathcal{A}\big(u(s) - \gamma\varphi(s)\big), \gamma\varphi(s)\rangle ds \geq 0 \tag{62}$$

we devise then by γ, we obtain

$$\int_0^T \langle \Psi(s) - \mathcal{A}\big(u(s) - \gamma\varphi(s)\big), \varphi(s)\rangle ds \geq 0 \tag{63}$$

We then let γ tend to 0 in *(63)*. The property of hemi-continuity leads to

$$\int_0^T \langle \Psi(s) - \mathcal{A}u(s), \varphi(s)\rangle ds \geq 0\,. \tag{64}$$

So, the result holds.

3.2 Prove of the Theorem 1

We summarize the results obtained. There exists $\mathcal{v} \in L^2(\mathcal{V})$, $\frac{\mathrm{d}\mathcal{v}}{\mathrm{dt}} \in L^2(\mathcal{V}')$ and $\mathrm{u} \in L^2(\mathcal{V})$ such that

$\mathrm{u} = \mathcal{v} + \mathrm{W}$. Moreover *(50)* and the lemma 4 give

$$\frac{d\mathcal{v}}{dt} + \mathcal{A}u(t) = 0 \tag{65}$$

Finally, $\mathcal{v}(0) = \mathrm{u}(0) - \mathrm{W}_0$ (according to *(49)*). Then $\mathcal{v}$ is a solution of *(23)*, and consequently u of *(21)*.

Uniqueness follows from the monotonicity property of the operator $\mathcal{A}_{\mathrm{W_t}}$. Indeed, suppose that *(23)* has two solutions $\mathcal{v}_1, \mathcal{v}_2$, and let $\chi = \mathcal{v}_1 - \mathcal{v}_2$. We then have by subtraction,

$$\begin{cases} \frac{d\chi}{dt} + \mathcal{A}_W\mathcal{v}_1 - \mathcal{A}_W\mathcal{v} = 0 \quad a.e. \quad t \\ \chi\,(0) = 0 \end{cases} \tag{66}$$

And thus, by multiplying by χ and integrating, we get

$\forall \mathrm{t} \in [0, \mathrm{T}]$

$$\frac{1}{2}\,|\chi\,(t)|^2 + \int_0^t \langle\, \mathcal{A}_{W_s}\mathcal{v}_1(s) - \mathcal{v}(s),\, \mathcal{v}_1(s) - \mathcal{v}_2(s)\rangle\, ds = 0, \tag{67}$$

And since $\mathcal{A}_{W_t}$ is monotone, we see that $\chi(\mathrm{t}) = 0, \forall \mathrm{t}$. ■

3.3 Energy Estimates and Stability Analysis

This section establishes energy estimates to verify the stability of solutions. We will need to consider process of the form W, which satisfy condition *(3)*. As for the correlations between W, u, we will assume the following hypothesis.

Hypothesis:

$\forall t_1, t_2\; with\; 0 \le t_1 \le t_2 \le T, W_{t_1} - W_{t_2}$ is a random variable taking values in H independent of the random variable $\left\{u_0, W\left(t_{j_1}\right), \dots, W\left(t_{j_q}\right)\right\}$ taking values in $H \times H^q\; \forall q\; and\; t_{j_1}, \dots, t_{j_q} \le t_1$.

Based on the previously stated hypotheses, we derive the following theorem.

Theorem 2 *Under the assumptions stated in lemma 2 and hypothesis above, the following energy equality hold*

$$E\|u(t)\|_H^2 + 2E\int_0^t \langle \mathcal{A}(s)u(s), u(s)\rangle\, ds = E\|u(0)\|_H^2 + E\|W(t)\|_H^2 \quad \forall t \in [0,T]. \tag{68}$$

Proof. See [4] for details. ∎

Now, we prove the following stability result.

Proposition 4 *Consider $W \in C\big([0,T];\, L^2(D,\mathcal{P},H)\big), u_0 \in H_0^1(D)$ and u associated solution, then for any t,*

$$E\|u(t)\|_H^2 \le E\|u(0)\|_H^2 + E\|W(t)\|_H^2 \quad \forall t \in [0,T]. \tag{69}$$

Proof.

Since the operator $\mathcal{A}(t)$ verify the coersivity condition, i.e. $\langle \mathcal{A}(t)u(t), u(t)\rangle \ge \rho\|u(t)\|^2$, the energy equality *(68)* became as follow

$$E\|u(t)\|_H^2 + 2\rho E\int_0^t \|u(s)\|^2\, ds \le E\|u(0)\|_H^2 + E\|W(t)\|_H^2 \tag{70}$$

we know $2\rho E\int_0^t \|u(s)\|^2\, ds \ge 0$, we substitute that in *(65)*, we obtain *(69)*. Hence, the proposition 4 hold.

4. Numerical Scheme

The SPDE-based denoising scheme is approximated by applying a finite-difference based method. Thus, we put a space grid size of $\Delta x = \Delta y = 1$ and a time step $\Delta t = \frac{T}{N}$, where T and N are the final time and the number of iterations respectively.

$$\begin{aligned} u_{i,j}^{n+1} = u_{i,j}^n + \Delta t + \frac{\Delta t}{4}\left[(g_{i+1,j}^n - g_{i-1,j}^n)(u_{i+1,j}^n - u_{i-1,j}^n)\right] \\ + \tfrac{\Delta t}{4}\left[\left(g_{i,j+1}^n - g_{i,j-1}^n\right)\left(u_{i,j+1}^n - u_{i,j-1}^n\right) + (W_{i,j}^{n+1} - W_{i,j}^n)\right] \end{aligned} \tag{71}$$

Iterative algorithm given by *(71)*, it begins by inputting the initial conditions which is the noisy image u^0. Then, we define the continuous Wiener process $W_t, a.e.\,t$ and the function g. Next, we repeat $N+1$ times by using the numerical scheme *(71)*. Finally, we get the restored image u^{N+1} without noise.

4.1 Stability analysis

In this section, we study the stability of the numerical scheme using the Fourier transform method. Specifically, we implement this change in *(71)*, given by:

$$u_{i,j}^{n} = \hat{u}^{n} e^{I\pi(ki+mj)} \tag{72}$$

Definition 1 *Scheme (67) said to be stable, if there exists a constant C such that*

$$\left|\frac{\hat{u}^{n+1}}{\hat{u}^{n}}\right| \leq C\Delta t \tag{73}$$

let us find the stability condition for *(71)*, i.e. find the constant *C* in *(73)*.

Proposition 5 *If (2) satisfied g, then*

$$\Delta t \leq \frac{2}{8-\xi_n}, \quad \xi_n = W_{i,j}^{n+1} - W_{i,j}^{n} \tag{74}$$

Proof. We substitute *(72)* in *(71)* as follow

$$\begin{aligned}
\hat{u}^{n+1} e^{I\pi(ki+mj)} &= \hat{u}^{n} e^{I\pi(ki+mj)} + \Delta t g_{i,j}^{n}(\hat{u}^{n} e^{I\pi(k(i+1)+mj)} + \hat{u}^{n} e^{I\pi(k(i-1)+mj)}) \\
&\quad + \Delta t g_{i,j}^{n}\left(\hat{u}^{n} e^{I\pi(ki+m(j+1))} + \hat{u}^{n} e^{I\pi(ki+m(j-1))} - 4\hat{u}^{n} e^{I\pi(ki+mj)}\right) \\
&\quad + \frac{\Delta t}{4}\left[\left(g_{i+1,j}^{n} - g_{i-1,j}^{n}\right)\left(\hat{u}^{n} e^{I\pi(k(i+1)+mj)} + \hat{u}^{n} e^{I\pi(k(i-1)+mj)}\right)\right] \\
&\quad + \frac{\Delta t}{4}\left(g_{i,j+1}^{n} - g_{i,j-1}^{n}\right)\left(\hat{u}^{n} e^{I\pi\left(ki+m(j+1)\right)} + \hat{u}^{n} e^{I\pi\left(ki+m(j-1)\right)}\right) \\
&\quad + \Delta t(W_{i,j}^{n+1} - W_{i,j}^{n})
\end{aligned}$$

$$\begin{aligned}
\hat{u}^{n+1} e^{I\pi(ki+mj)} &= \hat{u}^{n} e^{I\pi(ki+mj)} \left(1 + \Delta t\left(g_{i,j}^{n}(e^{I\pi k} + e^{-I\pi k} + e^{I\pi m} + e^{-I\pi m} - 4)\right)\right) \\
&\quad + \hat{u}^{n} e^{I\pi(ki+mj)}\left(\frac{\Delta t}{4}\left(g_{i+1,j}^{n} - g_{i-1,j}^{n}\right)(e^{I\pi k} + e^{-I\pi k})\right) \\
&\quad + \hat{u}^{n} e^{I\pi(ki+mj)}\left(\frac{\Delta t}{4}\left(g_{i,j+1}^{n} - g_{i,j-1}^{n}\right)(e^{I\pi m} + e^{-I\pi m})\right) \\
&\quad + \Delta t(W_{i,j}^{n+1} - W_{i,j}^{n})
\end{aligned}$$

$$\begin{aligned}
\frac{\hat{u}^{n+1} e^{I\pi(ki+mj)}}{\hat{u}^{n} e^{I\pi(ki+mj)}} &= 1 + \Delta t(g_{i,j}^{n}(e^{I\pi k} + e^{-I\pi k} + e^{I\pi m} + e^{-I\pi m} - 4)) \\
&\quad + \frac{\Delta t}{4}\left(g_{i+1,j}^{n} - g_{i-1,j}^{n}\right)(e^{I\pi k} + e^{-I\pi k}) \\
&\quad + \frac{\Delta t}{4}\left(g_{i,j+1}^{n} - g_{i,j-1}^{n}\right)(e^{I\pi m} + e^{-I\pi m}) \\
&\quad + \Delta t\left(\frac{W_{i,j}^{n+1} - W_{i,j}^{n}}{\hat{u}^{n} e^{I\pi(ki+mj)}}\right),
\end{aligned}$$

such as $g_{i,j}^{n} \leq 1, g_{i+1,j}^{n} \leq 1, g_{i-1,j}^{n} \leq 1, g_{i,j+1}^{n} \leq 1$ and $g_{i,j-1}^{n}$, and

$$\begin{cases} e^{I\pi k} + e^{-I\pi k} = 2\cos(\pi k) \\ e^{I\pi m} + e^{-I\pi m} = 2\cos(\pi m) \end{cases} \tag{75}$$

$$\Rightarrow \begin{cases} 2\cos(\pi k) - 2 = -4\sin^2(\frac{\pi k}{2}) \\ 2\cos(\pi m) - 2 = -4\sin^2(\frac{\pi m}{2}) \end{cases}, \tag{76}$$

with $\sin^2\left(\frac{\pi k}{2}\right) \leq 1, \sin^2(\frac{\pi m}{2}) \leq 1$

$$\begin{aligned} \left|\frac{\hat{u}^{n+1}}{\hat{u}^n}\right| &\leq \left|1 - 8\Delta t + \Delta t\left(\frac{W_{i,j}^{n+1} - W_{i,j}^n}{\hat{u}^n e^{I\pi(ki+mj)}}\right)\right| \\ &\leq \left|1 - 8\Delta t + \Delta t\left(W_{i,j}^{n+1} - W_{i,j}^n\right)\right| \\ &\leq |1 - \Delta t(8 - \quad \xi_n)| \end{aligned}$$

$$|1 - \Delta t(8 - \quad \xi_n)| \leq 1 \Rightarrow -1 \leq 1 - \Delta t(8 - \quad \xi_n) \leq 1$$

$$\Rightarrow \Delta t \leq \frac{2}{8-\xi_n} \tag{77}$$

Then proposition 5 hold.

If we consider

$$\Delta t \leq \frac{2}{8-\xi_n} = \frac{1}{4-\frac{1}{2}\xi_n} = \frac{1}{C}. \quad C = 4 - \frac{1}{2}\xi_n \tag{78}$$

we obtain stability conditions *(78)* for a good choice of the time discretisation parameter to solve *(71)*.

4.2 Numerical results and Comments

In this section, we present the obtained results from our numerical experimentations, using MATLAB R2022b. We tested different approaches to evaluate the performance of our proposed SPDE model for image restoration. To measure the quality of the restored images, we calculated the Peak Signal-to-Noise Ratio (PSNR) and the Structural Similarity Index (SSIM). Both Gaussian and salt & pepper noises were considered, where the tests were carried out by varying the standard deviation (σ) of the Gaussian filter, while keeping the noise variance fixed at γ = 0.1 & 0.01. Note that we can evaluate how well the model is adapted to different smoothing conditions in image denoising, as shown in the results in table 3.

Table 1. PSNR values for denoised images with Gaussian noise

Model	PDE Kolmogorov	SDE Barbu	SDE Borkowski	PM1	PM2	SPDE 1	SPDE 2
γ = 0.1	21.3094	20.1880	24.6288	24.1495	24.1601	27.6852	27.7182
γ = 0.01	24.3094	30.1259	30.0948	30.2702	30.1051	33.3760	33.4251

Table 2. SSIM values for denoised Images with Gaussian noise

Models	PDE Kolmogorov	SDE Barbu	SDE Borkowski	PM1	PM2	SPDE 1	SPDE 2
$\gamma = 0.1$	0.6008	0.5391	0.6104	0.6769	0.6766	0.7452	0.7559
$\gamma = 0.01$	0.6017	0.7111	0.8160	0.8035	0.8040	0.8913	0.8926

Table 3. Impact of Gaussian filter variance on image quality metrics under fixed noise Level $\gamma = 0.1$ & 0.01

σ		0.45	0.9	1	1.6
PSNR	0.1	22.7798	**25.2683**	25.1102	24.6648
	0.01	**33.5837**	26.3696	27.9571	26.2707
SSIM	0.1	0.5513	**0.7466**	0.7153	0.7104
	0.01	**0.8915**	0.8404	0.8381	0.8188

Table 4. Performance of our SPDE model with salt & pepper noise compared to other approaches under a fixed noise level $\gamma = 0.1$ and $\sigma = 0.45$.

Model	PDE Kolmogorov	SDE Barbu	SDE Borkowski	PM 1	PM 2	SPDE 1	SPDE 2
PSNR	22.9429	34.3444	30.9117	31.6016	32.2173	35.4783	35.9521
SSIM	0.7748	0.9739	0.8999	0.9613	0.9642	0.9861	0.9881

We denote:

- PDE Kolmogorov: the partial differential equation (PDE) related to SDE of Barbu [1].
- SDE Barbu: the model introduced by Barbu in 2016 [17].
- PM 1 and PM 2: the model of PM [17] with their decreasing functions *(2)* (fractional and exponential respectively).
- SDE Borkowski: Borkowski's model, introduced in 2013 [6].
- SPDE 1 and SPDE 2: the used of exponential and fractional functions *(2)* respectively.

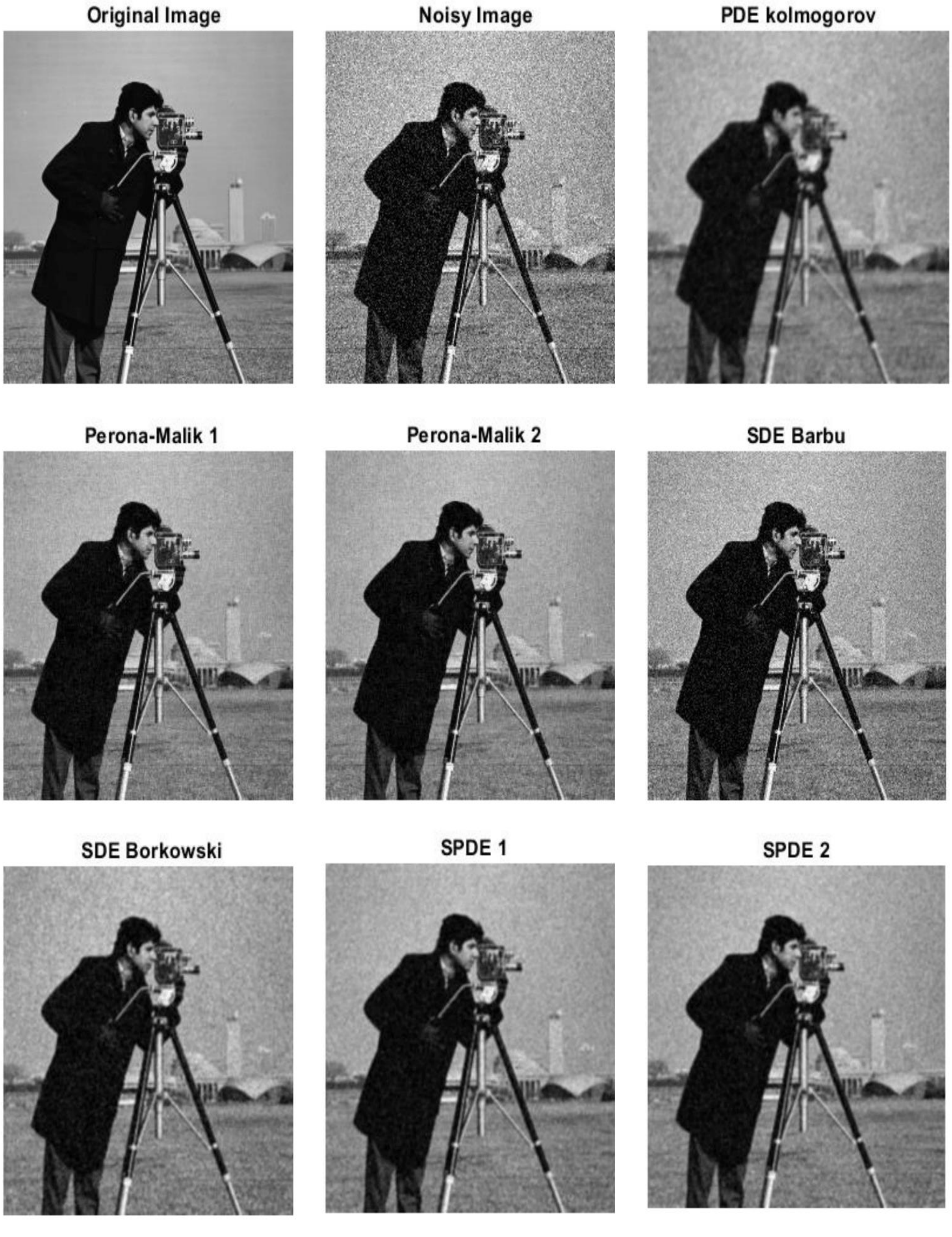

Figure 1. Restored cameraman image by using different approaches after additive Gaussian noise with $\Delta \boldsymbol{t} = \frac{\boldsymbol{T}}{\boldsymbol{N}}$, N=100, T=1, $\boldsymbol{\gamma} = \mathbf{0.1}$, $\boldsymbol{\sigma} = \mathbf{0.9}$.

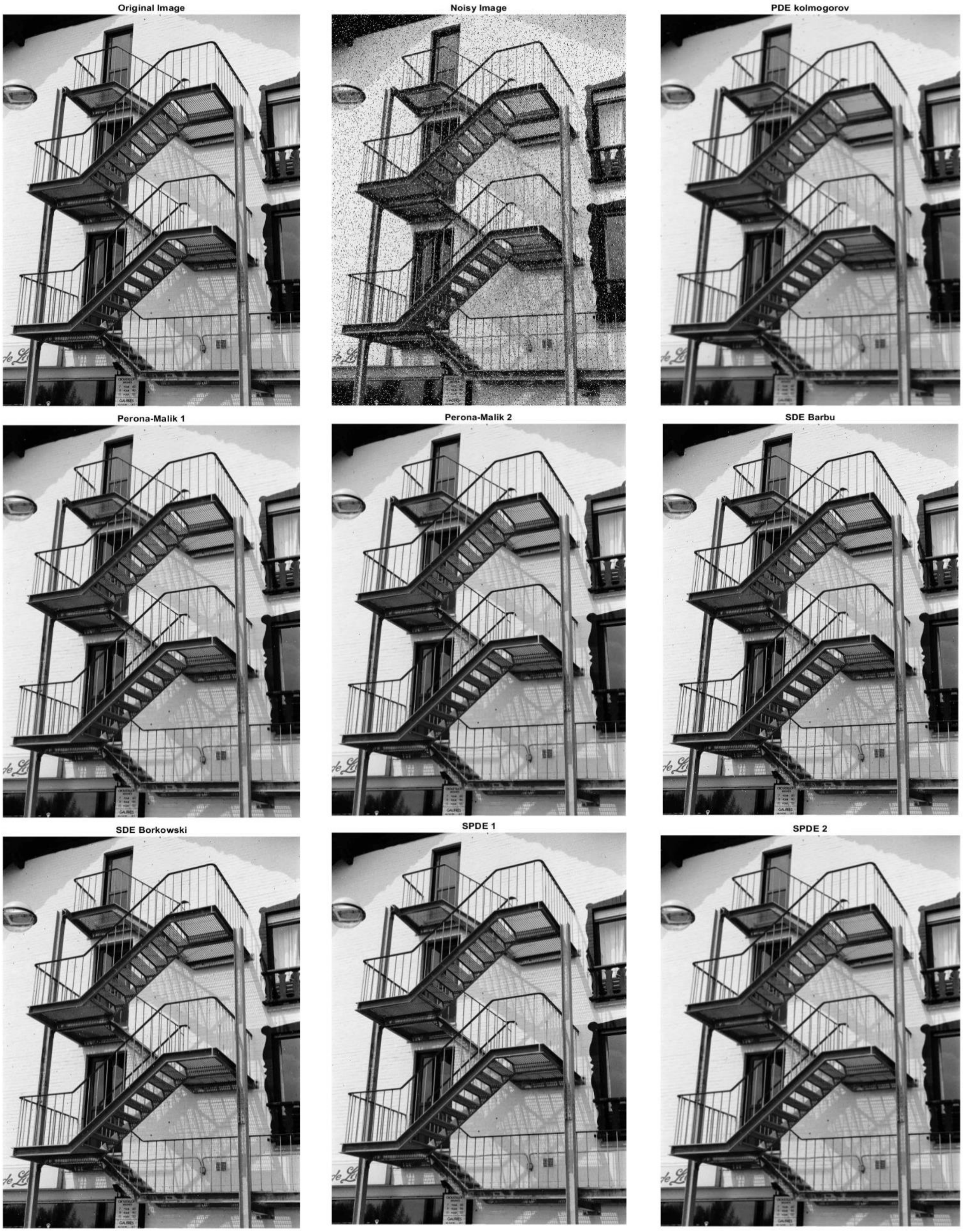

Figure 2. Restored image results after 'salt & Pepper' noise application with N = 5, T = 1, **γ = 0.1** and **σ = 0.45**

Comments on Numerical Results

The numerical results are resumed in the Tables 1,2,3 and 4 as well as the Figures 1 and 2.

By closely observing Tables 1,2, 3, and 4 and Figures 1 & 2, we notice that

- The restoration performance under Gaussian noise varies according the parameters choices. As shown in Tables 1 & 2, Barbu and its related PDE models (Kolmogorov's PDE) [1] exhibit less performant compare to PM1, PM2 [17] and SDE Borkowski models [5], as reflected in their PSNR and SSIM values. Specifically, when $\gamma = 0.1$, Barbu's SDE model [1] achieved a PSNR of 20.1880 dB and an SSIM of 0.5391, which is lower than PM1 (where PSNR=24.1495 dB and SSIM=0.6769) and PM2 (with PSNR=24.1601 dB and SSIM=0.6766). This discrepancy underscores the important role of the diffusion in image restoration, that has been neglected by Barbu et Al who relied solely on the drift term in their model. In contrast, our proposed model, which integrates (combines) both SDEs and PDEs, achieves better qualitative image restoration results, surpassing models that employ either SDEs or PDEs. As indicated in Tables 1 and 2 for our approach, we can confirm its effectiveness and further highlight the advantage of SPDEs in enhancing image restoration.

- Our proposed model incorporates regularisation through the convolution of the functions g and G_σ, which plays a crucial role in enhancing image quality while preserving image details. The precise selection of the parameter σ is essential for obtaining optimal restoration results, as demonstrated by the qualitative evaluations in Table 3. Specifically, selecting $\sigma = 0.9$ with $\gamma = 0.1$ leads to an increase in PSNR to 25.2683 dB and SSIM to 0.7466, and $\sigma = 0.45$ with $\gamma = 0.01$ to higher PSNR=33.5837 dB and SSIM=0.8915, reinforcing the importance of careful parameter choices in order to achieve a better denoising performance.

- For salt-and-pepper noise, all models give good performance improvements, achieving higher PSNR and SSIM values. However, while Barbu SDE model perform significantly better with salt & pepper noise than the white Gaussian noise, more competitive results are obtained by our proposed approach, as shown in Table 4 for the PSNR and SSIM values, and visually in Figure 2.

5. Conclusion

In this work, we introduced an image restoration technique based on SPDEs, integrating the PM equation with stochastic perturbations to enhance noise removal while preserving image details. The stochastic component introduces controlled randomness, preventing excessive smoothing while maintaining important image structures. This approach allows adaptive noise reduction, making the restoration process more resilient to varying noise levels. Through a detailed mathematical analysis, we established the theoretical and numerical stability of the proposed model, ensuring its robustness under different conditions. Numerical experimentations confirm its effectiveness, according to the obtained results for the PSNR and SSIM values, highlighting the potential of SPDE-based models for image restoration.

References

[1] T. Barbu, et A. Favini, "Novel stochastic differential model for image restoration", Processings of the Romanian Academy-Series A: Mathematics, Physics, Technical Sciences, Information Science. vol. 17, no. 2, pp. 109-116, 2016.

[2] M. Benseghir and F.Z. Nouri, "A Study of a Stochastic Differential Equation with Reflection for Image Processing", Journal of Applied Probability and Statistics 2022, Vol. 17, No. 2, pp. 035-046.

[3] M. Benseghir, F.-Z. Nouri, and P.-C. Tauber, "A new Partial Differential Equation for Image Inpainting", Bol. Soc. Paran. Mat. (3s.) v. 39 3, 137–155, 2021.

[4] A. Bensoussan and R. Temam, "Equations aux dérivées partielles stochastiques non lineaires (1)", 1971.

[5] D. Borkowski, and K. Jańczak-Borkowska, "Image Denoising Using Backward Stochastic Differential Equations", Advances in Intelligent Systems and Computing, pp. 185-194, 2017.

[6] D. Borkowski and K. Jańczak-Borkowska, "Image Restoration Using Anisotopic Stochastic Diffusion Collaborated with Non Local Means". In IFIP Internatinal Conference on Computer Information Systems and Industial Management. Springer, Berlin, Heidelberg, 2013.

[7] H. Brézis and M. Sibony, "Méthodes d'Approximation et d'Itération pour les Opérateurs Monotones",
Mémoire présenté par J.L. Lions, Pour les Opérateurs Monotones.

[8] C. Catté, P.L. Lions, J.M. Morel and T. Coll, "Image Selective Smoothing and Edge Detection by Nonlinear Diffusion", SIAM J. NUMER. ANAL, vol.29, No. 1, pp.182-193, 1992.

[9] G. Da Prato and J. Zabczyk, "Stochastic equations in infinite dimensions", Cambridge University Press, Series Encyclopedia of Mathematics and its Applications, 44, 1992.

[10] X. Descombes and E. Zhizhina, "Image Denoising using Stochastic Differential Equations", INRIA.

[11] T.C. Garrido, "Existence and Uniqueness of Solutions for Non-Linear Stochastic Partial Differential Equations", Departamento de Anàlisis Matemàtico, Universidad de Sevilla, Apartado de Correos 1.160. 41080-SEVILLA, Spain.

[12] J.L. Lions, "Quelques méthodes de résolutions des problèmes aux limites non linéaires", Dunod Gauthier Villars, Paris, 1969.

[13] P.A. Mayer, "Probabilités et Potentiel, Herman", Paris, (1966).

[14] G.J. Minty, "Monotone nonlinear operators in Hilbert spaces", Duke Math. J. 29,341-346, 1962.

[15] F.Z. Nouri, "Uniqueness and Existence Results for a Partial Differential Equation in Image Inpainting", Commun. Optim. Theory 2020 (2020), Article ID 7, pp. 1-17.

[16] E. Pardoux, "Stochastic Partial Differential Equations and Filtering of Diffusion Processes", Stochastic 3, 127-167, (1979).

[17] P. Perona and J. Malik, "Scale-space and edge detection using anisotropic diffusion", IEEE Trans. Pattern Anal. Machine Intell., vol. 12, pp. 629-639, 1990.

[18] C. Prévot and M. RÖCKNER, "A concise cours on stochastic partial differential equations", Springer, Lecture Notes in Mathematics, vol. 1905, 2007.

[19] L. Schwartz, "Radon measures on arbitrary topological spaces", à paraître, Tata institute of Fundamental Research Bombay.