Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

Generalization of Open Sets in Pythagorean Fuzzy Nano Topological Spaces and its Real Application

X. Arul Selvaraj[1] and N. Prabavathy[2]
[1]Department of Mathematics, DDE, Annamalai University, Annamalai Nagar - 608 002, India;
(Deputed to) Periyar Arts Collge, Cuddalore-607 001, Tamil Nadu, India.
xaselvarajmaths@gmail.com
[2] Department of Mathematics, Arignar Anna Gov. Arts College, Musiri-621 211, Trichy(D.t.), Tamil Nadu, India.
[1,2] Department of Mathematics, Annamalai University, Annamalai Nagar - 608 002, India.
online2020av@gmail.com , prabamarch23@gmail.com
(Corresponding Author: N. Prabavathy)

Article History:
Received: 12-01-2025
Revised: 15-02-2025
Accepted: 01-03-2025

Abstract: The purpose of this paper is to define and study a new class of sets called Pythagorean fuzzy nano Z (resp. δ, $\delta\mathcal{S}$ and pre)-open sets in Pythagorean fuzzy nano topological spaces. Basic properties of Pythagorean fuzzy nano Z (resp. δ, $\delta\mathcal{S}$ and pre) -open and their respective closed sets are analysed. We also used them to introduce the new notions like Pythagorean fuzzy nano Z (resp. δ, $\delta\mathcal{S}$ and pre)-closure (resp. interior) and their relations with already existing well known sets are also investigated.

Keywords: Pythagorean fuzzy nano open set, Pythagorean fuzzy nano pre open set, Pythagorean fuzzy nano δ semi open set, Pythagorean fuzzy nano Z open set.

AMS(2000) Subject classification: 03E72, 54A05, 54A40.

1 Introduction

Zadeh in [21] established the idea of fuzzy in 1965, which is a generalization of usual set using fuzzy where each element has a membership degree in $[0,1]$. The subsequent advanement of fuzzy subsets was the intuitionistic fuzzy set published by Atanassov, [4] in 1983, which has elements having membership and non-membership degree. In 1968, Chang in [5] defined fuzzy topological space and fundamental results such as continuity, open and closed set. Following this, Lowen in [7] defined fuzzy topological space in other form. Coker introduced the idea of intuitionistic fuzzy topological space with few properties, see [6]. The concept of Pythagorean fuzzy subset which is a typical fuzzy subset was presented by Yager, see [18, 19, 20]. Pythagorean fuzzy topological space was introduced by Olgun in [11] by taking the lead as from Chang.

In 2013, a new topology called Nano topology was introduced by Lellis Thivagar [9] which is an extension of rough set theory. He also introduced Nano topological spaces which were defined in terms of approximations and boundary region of a subset of a universe using an equivalence relation on it. The elements of a Nano topological space are called the Nano open sets and its complements are called the Nano closed sets. Nano means something very small. Nano topology thus literally means the study of very small surface. The fundamental ideas in Nano topology are those of approximations and indiscernibility relation. Furthermore nano δ open sets in nano topological space was studied in [13].

Recently, Lellis Thivagar et. al [10] explored a new concept of neutrosophic nano topology, and also Z-open sets in topological spaces by El-Magharabi and Mubarki [8], M-open sets in a nano topological spaces by Padma et. al [12], Z-open sets in nano topological spaces by Selvaraj and Balakrishna [3] and fuzzy Z-closed sets and generalized fuzzy Z-closed sets in double fuzzy topological spaces by Shiventhiradevi et. al in [14, 15]. Thangammal et. al [16] introduced fuzzy nano Z-open sets in fuzzy nano topological spaces. Vadivel et al. [17] discussed some open sets in fuzzy nano topological spaces.

Research Gap: No investigation on some stronger and weaker forms of Pythagorean fuzzy nano open sets such as Pythagorean fuzzy nano δ open set, Pythagorean fuzzy nano δ-semi open set, Pythagorean fuzzy nano pre open set and Pythagorean fuzzy nano Z open sets on Pythagorean fuzzy nano topological space has been reported in the Pythagorean fuzzy literature.

In this paper some preliminary concepts required in our work are briefly recalled in section 2. In section 3, we introduce the concept of $\mathcal{PFN}o$, $\mathcal{PFN}\delta o$, $\mathcal{PFN}\delta\mathcal{S}o$, $\mathcal{PFNP}o$ and $\mathcal{PFN}Zo$ sets and studied some of their properties. Also, we discuss on Pythagorean fuzzy nano Z-interior and Pythagorean fuzzy nano Z-closure operators in Pythagorean fuzzy nano topological spaces.

2 Preliminaries

Definition 2.1 *[21] A function λ from X into the unit interval I is called a fuzzy set in X. For every $x \in X, \lambda(x) \in I$ is called the grade of membership of x in λ. Some authors say that λ is a fuzzy subset of X instead of saying that λ is a fuzzy set in X. The class of all fuzzy sets from X into the closed unit interval I will be denoted by I^X.*

Definition 2.2 *[21] If λ and ξ are any two fuzzy subsets of a set X, then λ is said to be included in ξ or λ is contained in ξ or λ is less than or equal to ξ iff $\lambda(x) \leq \xi(x)$ for all x in X and is denoted by $\lambda \leq \xi$. Equivalently, $\lambda \leq \xi$ iff $\mu_\lambda(x) \leq \mu_\xi(x)$ for all x in X.*

Note that every fuzzy subset is included in itself and empty fuzzy subset is included in every fuzzy subset.

Definition 2.3 *[21] Two fuzzy subsets λ and μ of a set X are said to be equal, written $\lambda = \mu$, if $\lambda(x) = \mu(x)$ for every x in X.*

Definition 2.4 *[21] The complement of a fuzzy subset λ in a set X, denoted by $1 - \lambda$, is the fuzzy subset of X defined by $1 - \lambda(x)$ for all x in X. Note that $1 - (1 - \lambda) = \lambda$.*

Definition 2.5 *[21] The union of two fuzzy subsets λ and μ in a set X, denoted by $\lambda \vee \mu$, is fuzzy subset in X defined by $(\lambda \vee \mu)(x) = max\{\lambda(x), \mu(x)\}$, for all x in X.*

In general, the union of a family of fuzzy subsets $\{\xi_i : i \in I\}$ is a fuzzy subset denoted by $\vee_{i\in I} \xi_i$ and defined by $(\vee_{i\in I} \xi_i)(x) = \sup\{\xi_i(x) : i \in I\}$, for all x in X.

Definition 2.6 *[21] The intersection of two fuzzy subsets λ and μ in a set X, denoted by $\lambda \wedge \mu$, is fuzzy subset in X defined by $(\lambda \wedge \mu)(x) = min\{\lambda(x), \mu(x)\}$, for all x in X.*

In general, the intersection of a family of fuzzy subsets $\{\xi_i : i \in I\}$ is a fuzzy subset denoted by $\wedge_{i\in I} \xi_i$ and defined by $(\wedge_{i\in I} \xi_i)(x) = \inf\{\xi_i(x) : i \in I\}$, for all x in X.

Definition 2.7 *[18, 19, 20] Let X be a universal set. Then, a Pythagorean fuzzy set A, which is a set of ordered pairs over X, is defined by the following: $A = \{< x, \mu_A(x), \lambda_A(x) | x \in X\}$ or $A = \left\{\left\langle \frac{\mu_A(x), \lambda_A(x)}{x} \right\rangle | x \in X\right\}$, where the functions $\mu_A(x): X \to [0,1]$ and $\lambda_A(x): X \to [0,1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$, $0 \leq (\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$. Supposing $(\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$, then there is a degree of indeterminacy of $x \in X$ to A defined by $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\lambda_A(x))^2]}$ and $\pi_A(x) \in [0,1]$. In what follows, $(\mu_A(x))^2 + (\lambda_A(x))^2 + (\pi_A(x))^2 = 1$. Otherwise, $\pi_A(x) = 0$ whenever $(\mu_A(x))^2 + (\lambda_A(x))^2 = 1$. We denote the set of all PFS's over X by $pfs(X)$.*

Definition 2.8 *[20] Let A and B be pfs's of the forms $A = \{< a, \mu_A(a), \lambda_A(a) > | a \in X\}$ and $B = \{< a, \mu_B(a), \lambda_B(a) > | a \in X\}$. Then*

1. $A \subseteq B$ if and only if $\mu_A(a) \leq \mu_B(a)$ and $\lambda_A(a) \geq \lambda_B(a)$ for all $a \in X$.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $\bar{A} = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$.
4. $A \cap B = \{< a, \mu_A(a) \wedge \mu_B(a), \lambda_A(a) \vee \lambda_B(a) > | a \in X\}$.
5. $A \cup B = \{< a, \mu_A(a) \vee \mu_B(a), \lambda_A(a) \wedge \lambda_B(a) > | a \in X\}$.
6. $0_P = \{< a, 0,1 > | a \in X\}$ and $1_P = \{< a, 1,0 > | a \in X\}$.
7. $\overline{1}_P = 0_P$ and $\overline{0}_P = 1_P$.

Definition 2.9 *[1] Let U be a non-empty set and R be an equivalence relation on U. Let A be a Pythagorean fuzzy set in U with the membership function $\mu_A(x)$ and non membership function $\lambda_A(x)$, $\forall\ x \in U$. The Pythagorean fuzzy nano lower, Pythagorean fuzzy nano upper approximation and Pythagorean fuzzy nano boundary of A in the approximation (U, R) denoted by $\underline{\mathcal{PFN}}(A), \overline{\mathcal{PFN}}(A)$ and $B_{\mathcal{PFN}}(A)$ are respectively defined as follows:*

1. $\underline{\mathcal{PFN}}(A) = \left\{\langle x, \mu_{\underline{R}(A)}(x), \lambda_{\overline{R}(A)}(x)\rangle / y \in [x]_R, x \in U\right\}$
2. $\overline{\mathcal{PFN}}(F) = \left\{\langle x, \mu_{\overline{R}(A)}(x), \lambda_{\underline{R}(A)}(x)\rangle / y \in [x]_R, x \in U\right\}$
3. $B_{\mathcal{PFN}}(F) = \overline{\mathcal{PFN}}(F) - \underline{\mathcal{PFN}}(F)$

where $\mu_{\underline{R}(A)}(x) = \wedge_{y \in [x]_R} \mu_A(y)$

$$\lambda_{\underline{R}(A)}(x) = \wedge_{y \in [x]_R} \lambda_A(y),$$

$$\mu_{\overline{R}(A)}(x) = \vee_{y \in [x]_R} \mu_A(y),$$

$$\lambda_{\overline{R}(A)}(x) = \vee_{y \in [x]_R} \lambda_A(y).$$

Definition 2.10 *[1] Let U be an universe of discourse, R be an equivalence relation on U and A be a Pythagorean fuzzy set in U and if the collection $\tau_{\mathcal{R}}(A) = \{0_{\mathcal{P}},\ 1_{\mathcal{P}},\ \underline{\mathcal{PFN}}(A),$*

$\overline{\mathcal{PFN}}(A)$, $B_{\mathcal{PFN}}(A)\}$ *forms a topology then it is said to be a Pythagorean fuzzy nano topology. We call* $(U,\tau_{\mathcal{R}}(A))$ *(or simply* U*) as the Pythagorean fuzzy nano topological space. The elements of* $\tau_{\mathcal{R}}(A)$ *are called Pythagorean fuzzy nano open (briefly,* $\mathcal{PFN}o$*) sets.*

Remark 2.1 *[1]* $[\tau_{\mathcal{R}}(A)]^c$ *is called the dual fuzzy nano topology of* $\tau_{\mathcal{R}}(A)$. *Elements of* $[\tau_{\mathcal{R}}(A)]^c$ *are called Pythagorean fuzzy nano closed (briefly,* $\mathcal{PFN}c$*) sets. Thus, we note that a Pythagorean fuzzy set* G *of* U *is Pythagorean fuzzy nano closed in* $\tau_{\mathcal{R}}(A)$ *if and only if* $1_P - G$ *is Pythagorean fuzzy nano open in* $\tau_{\mathcal{R}}(A)$.

Definition 2.11 *[1, 2] Let* $(U,\tau_{\mathcal{R}}(A))$ *be a* $\mathcal{PFN}ts$ *with respect to* A *where* A *is a Pythagorean fuzzy subset of* U. *Let* S *be a Pythagorean fuzzy subset of* U. *Then Pythagorean fuzzy nano*

1. interior of S (briefly, $\mathcal{PFN}int(S)$) is defined by $\mathcal{PFN}int(S) = \cup\{I: I \subseteq S \,\&\, I \text{ is a } \mathcal{PFN}o \text{ set in } U\}$.
2. closure of S (briefly, $\mathcal{PFN}cl(S)$) is defined by $\mathcal{PFN}cl(S) = \cap\{A: S \subseteq A \,\&\, A \text{ is a } \mathcal{PFN}c \text{ set in } U\}$.
3. regular open (briefly, $\mathcal{PFN}ro$) set if $S = \mathcal{PFN}int(\mathcal{PFN}cl(S))$.
4. regular closed (briefly, $\mathcal{PFN}rc$) set if $S = \mathcal{PFN}cl(\mathcal{PFN}int(S))$.

3 Pythagorean fuzzy nano Z (resp. δ, $\delta\mathcal{S}$ and pre)-open sets

Definition 3.1 *Let* $(U,\tau_{\mathcal{R}}(A))$ *be a* $\mathcal{PFN}ts$ *with respect to* A *where* A *is a* pfs *of* U. *Let* S *be a* pfs *of* U. *Then Pythagorean*

1. fuzzy nano δ interior of S (briefly, $\mathcal{PFN}\delta int(S)$) is defined by $\mathcal{PFN}\delta int(S) = \cup\{I: I \subseteq S \,\&\, I \text{ is a } \mathcal{PFN}ro \text{ set in } U\}$.
2. fuzzy nano δ closure of S (briefly, $\mathcal{PFN}\delta cl(S)$) is defined by $\mathcal{PFN}\delta cl(S) = \cap\{A: S \subseteq A \,\&\, A \text{ is a } \mathcal{PFN}rc \text{ set in } U\}$.

Definition 3.2 *Let* $(U,\tau_{\mathcal{R}}(A))$ *be a* $\mathcal{PFN}ts$ *with respect to* A *where* A *is a* pfs *of* U. *Then a* pfs S *in* U *is said to be Pythagorean :*

1. fuzzy nano δ-open (briefly, $\mathcal{PFN}\delta o$) set if $S = \mathcal{PFN}\delta int(S)$.
2. fuzzy nano δ-α-open (or) fuzzy nano α-open (briefly, $\mathcal{PFN}\delta\alpha o$ (or) $\mathcal{PFN}\alpha o$) set if $S \subseteq \mathcal{PFN}int(\mathcal{PFN}cl\ (\mathcal{PFN}\delta int\ (S)))$.
3. fuzzy nano δ-semi open (briefly, $\mathcal{PFN}\delta\mathcal{S}o$) set if $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(S))$.
4. fuzzy nano pre open (briefly, $\mathcal{PFN}\mathcal{P}o$) set if $S \subseteq \mathcal{PFN}int(\mathcal{PFN}cl(S))$.

The complement of an $\mathcal{PFN}\delta o$ (resp. $\mathcal{PFN}\delta\alpha o$, $\mathcal{PFN}\delta\mathcal{S}o$ & $\mathcal{PFN}\mathcal{P}o$) set is called a Pythagorean fuzzy nano δ (resp. Pythagorean fuzzy nano δ-α, Pythagorean fuzzy nano δ-semi & Pythagorean fuzzy nano pre) closed (briefly, $\mathcal{PFN}\delta c$ (resp. $\mathcal{PFN}\delta\alpha c$, $\mathcal{PFN}\delta\mathcal{S}c$ & $\mathcal{PFN}\mathcal{P}c$)) in U.

Definition 3.3 *Let* $(U,\tau_{\mathcal{R}}(A))$ *be a* $\mathcal{PFN}ts$ *with respect to* A *where* A *is a* pfs *of* U. *Let* S *be a* pfs *of* U. *Then Pythagorean fuzzy nano*

1. δ semi interior of S (briefly, $\mathcal{PFN}\delta\mathcal{S}int(S)$) is defined by $\mathcal{PFN}\delta\mathcal{S}int(S) = \cup\{I: I \subseteq S \;\&\; I \text{ is a } \mathcal{PFN}\delta\mathcal{S}o \text{ set in } U\}$.

2. δ semi closure of S (briefly, $\mathcal{PFN}\delta\mathcal{S}cl(S)$) is defined by $\mathcal{PFN}\delta\mathcal{S}cl(S) = \cap\{A: S \subseteq A \;\&\; A \text{ is a } \mathcal{PFN}\delta\mathcal{S}c \text{ set in } U\}$.

3. pre interior of S (briefly, $\mathcal{PFNP}int(S)$) is defined by $\mathcal{PFNP}int(S) = \cup\{I: I \subseteq S \;\&\; I \text{ is a } \mathcal{PFNP}o \text{ set in } U\}$.

4. pre closure of S (briefly, $\mathcal{PFNP}cl(S)$) is defined by $\mathcal{PFNP}cl(S) = \cap\{A: S \subseteq A \;\&\; A \text{ is a } \mathcal{PFNP}c \text{ set in } U\}$.

Definition 3.4 *Let $(U, \tau_{\mathfrak{R}}(A))$ be a $\mathcal{PFN}ts$ with respect to A where A is a pfs of U. Then a pfs S in U is said to be a Pythagorean fuzzy nano*

1. Z-open (briefly, $\mathcal{PF}\mathfrak{N}Zo$) set if $S \subseteq \mathcal{PF}\mathfrak{N}cl\big(\mathcal{PF}\mathfrak{N}\delta int(S)\big) \cap \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(S))$,

2. Z-closed (briefly, $\mathcal{PF}\mathfrak{N}Zc$) set if $\mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\delta cl(S)) \cap \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}int(S)) \subseteq S$.

The family of all $\mathcal{PF}\mathfrak{N}Zo$ (resp. $\mathcal{PF}\mathfrak{N}Zc$) sets of a space $(U, \tau_{\mathfrak{R}}(A))$ will be as always denoted by $\mathcal{PF}\mathfrak{N}ZO(U, A)$ (resp. $\mathcal{PF}\mathfrak{N}ZC(U, A)$).

Definition 3.5 *Let $(U, \tau_{\mathfrak{R}}(A))$ be a $\mathcal{PFN}ts$ with respect to A where A is a pfs of U. Then a pfs K in U, then the Pythagorean*

1. fuzzy nano Z-interior of K is the union of all $\mathcal{PF}\mathfrak{N}Zo$ sets contained in K and denoted by $\mathcal{PF}\mathfrak{N}Zint(K)$.

2. fuzzy nano Z-closure of K is the intersection of all $\mathcal{PF}\mathfrak{N}Zc$ sets containing K and denoted by $\mathcal{PF}\mathfrak{N}Zcl(K)$.

Remark 3.1 *Let K be a subset of a $\mathcal{PFN}ts$ $(U, \tau_{\mathfrak{R}}(A))$. Then $(\mathcal{PF}\mathfrak{N}Zcl(K))^c = \mathcal{PF}\mathfrak{N}Zint(K^c), (\mathcal{PF}\mathfrak{N}Zint(K))^c = \mathcal{PF}\mathfrak{N}Zcl(K^c)$.*

Theorem 3.1 *Let $(U, \tau_{\mathfrak{R}}(A))$ be a $\mathcal{PFN}ts$. Then,*

(i) Every $\mathcal{PF}\mathfrak{N}\delta o$ set is $\mathcal{PF}\mathfrak{N}o$ set.

(ii) Every $\mathcal{PF}\mathfrak{N}o$ set is $\mathcal{PF}\mathfrak{N}\mathcal{P}o$ set.

(iii) Every $\mathcal{PF}\mathfrak{N}\delta o$ set is $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}o$ set.

(iV) Every $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}o$ set is $\mathcal{PF}\mathfrak{N}Zo$ set.

(V) Every $\mathcal{PF}\mathfrak{N}\mathcal{P}o$ set is $\mathcal{PF}\mathfrak{N}Zo$ set.

Proof. (i) If K is a $\mathcal{PF}\mathfrak{N}\delta os$ in U, then $K = \mathcal{PF}\mathfrak{N}\delta int(K) \subseteq \mathcal{PF}\mathfrak{N}int(K)$. Therefore, K is a $\mathcal{PF}\mathfrak{N}os$.

(ii) If K is a $\mathcal{PF}\mathfrak{N}os$ in U, then $K = \mathcal{PF}\mathfrak{N}int(K)$. So, $K = \mathcal{PF}\mathfrak{N}int(K) \subseteq \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(K))$. Therefore, K is a $\mathcal{PF}\mathfrak{N}\mathcal{P}os$.

(iii) If K is a $\mathcal{PF}\mathfrak{N}\delta os$ in U, then K is $\mathcal{PF}\mathfrak{N}os$ by (i) . So, $K \subseteq \mathcal{PF}\mathfrak{N}int(K) \subseteq \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\delta cl(K))$. Therefore, K is a $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}os$.

(iv) K is a $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}os$, then $K \subseteq \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(K))$ and so $K \subseteq \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(K)) \subseteq \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(K)) \cup \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(K))$. $\therefore$ K is a $\mathcal{PF}\mathfrak{N}Zos$.

(v) K is a $\mathcal{PF}\mathfrak{N}\mathcal{P}os$, then $K \subseteq \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(K))$ and so $K \subseteq \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(K)) \subseteq \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(K)) \cup \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(K))$. $\therefore$ K is a $\mathcal{PF}\mathfrak{N}Zos$.

The converse of the above propositions need not to be true. The following examples show it.

Example 3.1 *Assume* $U = \{s_1, s_2, s_3, s_4\}$ *be the universe set and the equivalence relation is* $U/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$. *Let* $A = \left\{\left\langle\frac{s_1}{0.3,0.1}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle, \left\langle\frac{s_4}{0.4,0.25}\right\rangle\right\}$ *be a* pfs *of* U.

$$\underline{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1,s_4}{0.3,0.25}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\},$$

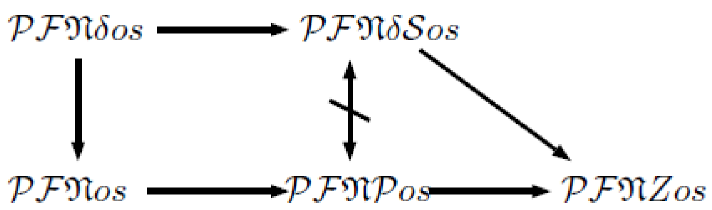
$$\overline{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1,s_4}{0.4,0.1}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1,s_4}{0.25,0.3}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\}.$$

Thus $\tau_{\mathcal{R}}(A) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PF}\mathfrak{N}}(A), \overline{\mathcal{PF}\mathfrak{N}}(A), B_{\mathcal{PF}\mathfrak{N}}(A)\}$. Then

1. $\left\{\left\langle\frac{s_1,s_4}{0.4,0.1}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\}$ is a $\mathcal{PF}\mathfrak{N}o$ (resp. $\mathcal{PF}\mathfrak{N}Zo$) set but not $\mathcal{PF}\mathfrak{N}\delta o$ (resp. $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}o$) set.
2. $\left\{\left\langle\frac{s_1,s_4}{0.25,0.3}\right\rangle, \left\langle\frac{s_2}{0.5,0.1}\right\rangle, \left\langle\frac{s_3}{0.45,0.2}\right\rangle\right\}$ is a $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}o$ (resp. $\mathcal{PF}\mathfrak{N}Zo$) set but not $\mathcal{PF}\mathfrak{N}\delta o$ (resp. $\mathcal{PF}\mathfrak{N}\mathcal{P}o$) set.
3. $\left\{\left\langle\frac{s_1,s_4}{0.45,0.35}\right\rangle, \left\langle\frac{s_2}{0.25,0.45}\right\rangle, \left\langle\frac{s_3}{0.3,0.2}\right\rangle\right\}$ is a $\mathcal{PF}\mathfrak{N}\mathcal{P}o$ set but not $\mathcal{PF}\mathfrak{N}o$ set.

Remark 3.2 *According to Definition 3.4 and Theorem 3.1, the following diagram holds for any set in* $\mathcal{PFN}ts$.



Lemma 3.1 *Let* $(U, \tau_{\mathcal{R}}(A))$ *be a* $\mathcal{PF}\mathfrak{N}ts$. *Then the following statements are hold.*

(i) The union of arbitrary $\mathcal{PF}\mathfrak{N}Zo$ sets is $\mathcal{PF}\mathfrak{N}Zo$,

(ii) The intersection of arbitrary $\mathcal{PF}\mathfrak{N}Zc$ sets is $\mathcal{PF}\mathfrak{N}Zc$.

Proof. (i) Let $\{K_i, i \in I\}$ be a family of $\mathcal{PF}\mathfrak{N}Zo$ sets.

Then $K_i \subseteq \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(K_i)) \cup \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(K_i))$ and hence $\cup_i K_i \subseteq \cup_i \left(\mathcal{PF}\mathfrak{N}cl\left(\mathcal{PF}\mathfrak{N}\delta int(K_i)\right) \cap \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(K_i))\right) \subseteq \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(\cup_i K_i)) \cap$

$\mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(\cup_i K_i))$, for all $i \in I$. Thus $\cup_i K_i$ is $\mathcal{PF}\mathfrak{N}Zo$.

(ii) It follows from (i).

Remark 3.3 *By the following we show that the intersection of any two $\mathcal{PF}\mathfrak{N}Zo$ sets is not $\mathcal{PF}\mathfrak{N}Zo$.*

Example 3.2 *In Example 3.1, let $A = \left\{\left\langle\frac{s_1,s_4}{0.3}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.5}\right\rangle\right\}$ and $B = \left\{\left\langle\frac{s_1,s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.2}\right\rangle, \left\langle\frac{s_3}{0.7}\right\rangle\right\}$ are $\mathcal{PFN}Zo$ sets but $A \cap B = \left\{\left\langle\frac{s_1,s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.2}\right\rangle, \left\langle\frac{s_3}{0.5}\right\rangle\right\}$ is not $\mathcal{PFN}Zo$ set.*

Theorem 3.2 *Let K be a $\mathcal{PF}\mathfrak{N}Zc$ set in $(U, \tau_{\mathcal{R}}(A))$ then $\mathcal{PF}\mathfrak{N}Zcl(K) - K$ does not contain any non-empty $\mathcal{PF}\mathfrak{N}c$ set in $(U, \tau_{\mathcal{R}}(A))$.*

Proof. Let K be a $\mathcal{PF}\mathfrak{N}Zc$ set in $(U, \tau_{\mathcal{R}}(A))$ and S be a $\mathcal{PF}\mathfrak{N}c$ subset of $\mathcal{PF}\mathfrak{N}Zcl(K) - K$. That is, $S \subseteq \mathcal{PF}\mathfrak{N}Zcl(K) - K$ implies $S \subseteq \mathcal{PF}\mathfrak{N}Zcl(K) \cap (1_{\mathcal{P}} - K)$. That is $S \subseteq \mathcal{PF}\mathfrak{N}Zcl(K)$ and $S \subseteq (1_{\mathcal{P}} - K)$ which implies $K \subseteq (1_{\mathcal{P}} - S)$ where $1_{\mathcal{P}} - S$ is a $\mathcal{PF}\mathfrak{N}o$ set. Since K is $\mathcal{PF}\mathfrak{N}Zc$, $\mathcal{PF}\mathfrak{N}Zcl(K) \subseteq (1_{\mathcal{P}} - S)$. That is $S \subseteq (1_{\mathcal{P}} - \mathcal{PF}\mathfrak{N}Zcl(K))$. Thus $S \subseteq \mathcal{PF}\mathfrak{N}Zcl(K) \cap (1_{\mathcal{P}} - \mathcal{PF}\mathfrak{N}Zcl(K)) = 0_P$. Hence $S = 0_P$. Therefore $\mathcal{PF}\mathfrak{N}Zcl(K) - K$ does not contain any non-empty $\mathcal{PF}\mathfrak{N}c$ set in $(U, \tau_{\mathcal{R}}(A))$.

Remark 3.4 *The converse of the theorem 3.2 need not be true as seen from the following example.*

Example 3.3 *In Example 3.1, let $K = \{l_2, l_3, l_4\}$ be any subset of U then $\mathcal{PF}\mathfrak{N}Zcl(K) - K = U - \{l_2, l_3, l_4\} = \{l_1\}$ which contain any non-empty $\mathcal{PF}\mathfrak{N}c$ in U, but K is not a $\mathcal{PF}\mathfrak{N}Zc$ in U.*

Theorem 3.3 *Let K be a $\mathcal{PF}\mathfrak{N}$ set in $(U, \tau_{\mathcal{R}}(A))$ then K is $\mathcal{PF}\mathfrak{N}Zcs$ if and only if $\mathcal{PF}\mathfrak{N}Zcl(K) - K$ is a $\mathcal{PF}\mathfrak{N}c$ set in $(U, \tau_{\mathcal{R}}(A))$.*

Proof. Let K be a $\mathcal{PF}\mathfrak{N}Zc$ set. Assume that K is $\mathcal{PF}\mathfrak{N}Zc$ then we have $\mathcal{PF}\mathfrak{N}Zcl(K) = K$, $\mathcal{PF}\mathfrak{N}Zcl(K) - K = 0_P$ which is $\mathcal{PF}\mathfrak{N}cs$.

Conversely, assume that $\mathcal{PF}\mathfrak{N}Zcl(K) - K$ be $\mathcal{PF}\mathfrak{N}cs$ and K is a $\mathcal{PF}\mathfrak{N}Zc$ set in U. Now $\mathcal{PF}\mathfrak{N}Zcl(K) - K$ is a $\mathcal{PF}\mathfrak{N}c$ subset of itself. Therefore by theorem 3.2, $\mathcal{PF}\mathfrak{N}Zcl(K) - K = 0_P$. That is $\mathcal{PF}\mathfrak{N}Zcl(K) = K$, implies K is $\mathcal{PF}\mathfrak{N}Zcs$.

Theorem 3.4 *If K is $\mathcal{PF}\mathfrak{N}Zc$ set in $(U, \tau_{\mathcal{R}}(A))$ and $K \subseteq L \subseteq \mathcal{PF}\mathfrak{N}Zcl(K)$, then L is also $\mathcal{PF}\mathfrak{N}Zc$ in $(U, \tau_{\mathcal{R}}(A))$.*

Proof. Let K be a $\mathcal{PF}\mathfrak{N}Zc$ set in U and $K \subseteq L \subseteq \mathcal{PF}\mathfrak{N}Zcl(K)$. Let $L \subseteq O$ where O be $\mathcal{PF}\mathfrak{N}c$ set in U. Since $K \subseteq L$, implies $K \subset O$ and K is $\mathcal{PF}\mathfrak{N}Zcs$,

implies $\mathcal{PF}\mathfrak{N}Zcl(K) \subseteq O$. By hypothesis $L \subseteq \mathcal{PF}\mathfrak{N}Zcl(K)$, implies $\mathcal{PF}\mathfrak{N}Zcl(L) \subseteq \mathcal{PF}\mathfrak{N}Zcl(\mathcal{PF}\mathfrak{N}Zcl(K)) = \mathcal{PF}\mathfrak{N}Zcl(K) \subseteq O$, which implies $\mathcal{PF}\mathfrak{N}Zcl(L) \subseteq O$. Therefore L is $\mathcal{PF}\mathfrak{N}Zcs$ in U.

Theorem 3.5 *If a subset K of U is $\mathcal{PF}\mathfrak{N}Zc$ set, then $\mathcal{PF}\mathfrak{N}cl(\{x_r\}) \cap K \neq 0_P$ for each $x_r \in \mathcal{PF}\mathfrak{N}Zcl(K)$.*

Proof. Suppose K is a $\mathcal{PF}\mathfrak{N}Zc$ set and $x_r \in \mathcal{PF}\mathfrak{N}Zcl(K)$. If possible $\mathcal{PF}\mathfrak{N}cl(\{x_r\}) \cap K = 0_P$. Then $K \subseteq 1_{\mathcal{P}} - \mathcal{PF}\mathfrak{N}cl(\{x_r\})$ and $1_{\mathcal{P}} - \mathcal{PF}\mathfrak{N}cl(\{x_r\})$ is a $\mathcal{PF}\mathfrak{N}o$ set containing K. Since K is

$\mathcal{PF}\mathfrak{N}Zc$ set, implies $\mathcal{PF}\mathfrak{N}Zcl(K) \subseteq 1_{\mathcal{P}} - \mathcal{PF}\mathfrak{N}cl(\{x_r\})$ which is a contradiction to $x_r \in \mathcal{PF}\mathfrak{N}Zcl(K)$. Therefore, $\mathcal{PF}\mathfrak{N}cl(\{x_r\}) \cap K \neq 0_P$.

Theorem 3.6 *If $\mathcal{PF}\mathfrak{N}ZO(U,A) = \mathcal{PF}\mathfrak{N}ZC(U,A)$, then $\mathcal{PF}\mathfrak{N}ZC(U,A) = P(U)$ is the power set of U.*

Proof. Suppose $K \subseteq O$, where O is $\mathcal{PF}\mathfrak{N}o$ in U. Since every $\mathcal{PF}\mathfrak{N}o$ set is $\mathcal{PF}\mathfrak{N}Zo$, O is $\mathcal{PF}\mathfrak{N}Zo$. By hypothesis, O is $\mathcal{PF}\mathfrak{N}Zc$. Hence, $\mathcal{PF}\mathfrak{N}Zcl(K) \subseteq O$. Therefore, K is a $\mathcal{PF}\mathfrak{N}Zc$ set. Since, K is arbitrary, by Theorem 3.5, every subset of U is $\mathcal{PF}\mathfrak{N}Zc$. Thus $\mathcal{PF}\mathfrak{N}ZC(U,A) = P(U)$.

Definition 3.6 *The intersection of all $\mathcal{PF}\mathfrak{N}o$ subset of U containing K is called the Pythagorean fuzzy nano kernel of K (briefly, $\mathcal{PF}\mathfrak{N}ker(K)$), this means $\mathcal{PF}\mathfrak{N}ker(K) = \cap \{G \in \mathcal{PF}\mathfrak{N}O(U,A): K \subseteq G\}$.*

Theorem 3.7 *A subset K is a $\mathcal{PF}\mathfrak{N}Zc$ set iff $\mathcal{PF}\mathfrak{N}Zcl(K) \subseteq \mathcal{PF}\mathfrak{N}ker(K)$.*

Proof. Suppose K is $\mathcal{PF}\mathfrak{N}Zc$ set, then $\mathcal{PF}\mathfrak{N}Zcl(K) \subseteq O$ whenever $K \subseteq O$ and O is $\mathcal{PF}\mathfrak{N}o$. Let $x_r \in \mathcal{PF}\mathfrak{N}Zcl(K)$. If $x_r \notin \mathcal{PF}\mathfrak{N}ker(K)$, then there exist a $\mathcal{PF}\mathfrak{N}o$ set O containing K such that $x_r \notin O$. Since O is a $\mathcal{PF}\mathfrak{N}o$ set containing K, implies $x_r \notin \mathcal{PF}\mathfrak{N}Zcl(K)$, which is a contradiction. Therefore $x_r \in \mathcal{PF}\mathfrak{N}ker(K)$.

Conversely, let $\mathcal{PF}\mathfrak{N}Zcl(K) \subseteq \mathcal{PF}\mathfrak{N}ker(K)$. If O is a $\mathcal{PF}\mathfrak{N}o$ set containing K, then $\mathcal{PF}\mathfrak{N}ker(K) \subseteq O$, which implies $\mathcal{PF}\mathfrak{N}Zcl(K) \subseteq O$. Therefore, K is a $\mathcal{PF}\mathfrak{N}Zc$ set.

Theorem 3.8 *If $K \subseteq V \subseteq 1_P$ and suppose that K is a $\mathcal{PF}\mathfrak{N}Zc$ set in 1_P, then K is $\mathcal{PF}\mathfrak{N}Zcs$ relative to V.*

Proof. Given that $K \subseteq V \subseteq 1_P$ and let $K \subseteq V \cap O$ where O is $\mathcal{PF}\mathfrak{N}os$ in U. Since K is a $\mathcal{PF}\mathfrak{N}Zc$ set in U, $K \subseteq O$ which implies that $\mathcal{PF}\mathfrak{N}Zcl(K) \subseteq O$. That is, $V \cap \mathcal{PF}\mathfrak{N}Zcl(K) \subseteq V \cap O$, where $V \cap \mathcal{PF}\mathfrak{N}Zcl(K)$ is $\mathcal{PF}\mathfrak{N}Zcl(K)$ in V. Thus K is $\mathcal{PF}\mathfrak{N}Zcs$ relative to V.

Lemma 3.2 *Let K and L be two pfs's of $(U, \tau_{\mathcal{R}}(A))$. Then:*

(i) $1_{\mathcal{P}} - \mathcal{PF}\mathfrak{N}\delta int(K) = \mathcal{PF}\mathfrak{N}\delta cl(1_{\mathcal{P}} - K)$ and $\mathcal{PF}\mathfrak{N}\delta int(1_{\mathcal{P}} - K) = 1_{\mathcal{P}} - \mathcal{PF}\mathfrak{N}\delta cl(K)$,

(ii) $\mathcal{PF}\mathfrak{N}cl(K) \subseteq \mathcal{PF}\mathfrak{N}\delta cl(K)$ (resp. $\mathcal{PF}\mathfrak{N}\delta int(K) \subseteq \mathcal{PF}\mathfrak{N}int(K)$), for any subset K of U,

(iii) $\mathcal{PF}\mathfrak{N}\delta cl(K \cup L) = \mathcal{PF}\mathfrak{N}\delta cl(K) \cup \mathcal{PF}\mathfrak{N}\delta cl(L)$, $\mathcal{PF}\mathfrak{N}\delta int(K \cap L) = \mathcal{PF}\mathfrak{N}\delta int(K) \cap \mathcal{PF}\mathfrak{N}\delta int(L)$.

Proposition 3.1 *Let K be a pfs in a $\mathcal{PFN}ts$ $(U, \tau_{\mathcal{R}}(A))$. Then:*

(i) $\mathcal{PF}\mathfrak{N}\mathcal{P}cl(K) = K \cup \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}int(K))$, $\mathcal{PF}\mathfrak{N}\mathcal{P}int(K) = K \cap \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(K))$,

(ii) $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}cl(1_{\mathcal{P}} - K) = 1_{\mathcal{P}} - \mathcal{PF}\mathfrak{N}\delta\mathcal{S}int(K)$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{S}cl(K \cap L) \subseteq \mathcal{PF}\mathfrak{N}\delta\mathcal{S}cl(K) \cup \mathcal{PF}\mathfrak{N}\delta\mathcal{S}cl(L)$,

Lemma 3.3 *The following hold for a pfs H in a $\mathcal{PFN}ts$ $(U, \tau_{\mathcal{R}}(A))$.*

1. $\mathcal{PF}\mathfrak{N}\mathcal{P}cl(H) = H \cap \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}int(H))$ and $\mathcal{PF}\mathfrak{N}\mathcal{P}int(H) = H \cap \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(H))$,

2. $\mathcal{PFN}\delta\mathcal{S}int(H) = H \cap \mathcal{PFN}cl(\mathcal{PFN}\delta int(H))$ and $\mathcal{PFN}\delta\mathcal{S}cl(H) = H \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(H))$.

Lemma 3.4 *The following hold for a pfs H in a $\mathcal{PFN}ts$ $(U, \tau_{\mathcal{R}}(A))$. $\mathcal{PFN}cl(\mathcal{PFN}\delta int(H)) = \mathcal{PFN}\delta cl(\mathcal{PFN}\ \delta int(H))$ and $\mathcal{PFN}int(\mathcal{PFN}\delta cl(H)) = \mathcal{PFN}\delta int(\mathcal{PFN}\delta cl(H))$.*

Theorem 3.9 *Let $(U, \tau_{\mathcal{R}}(A))$ be a $\mathcal{PFN}ts$. Then:*

(i) If $K \in \mathcal{PFN}\delta O(U, A)$ and $L \in \mathcal{PFN}ZO(U, A)$, then $K \cap L$ is $\mathcal{PFN}Zos$,

(ii) If $K \in \mathcal{PFN}aO(U, A)$ and $L \in \mathcal{PFN}ZO(U, A)$, then $K \cap L \in \mathcal{PFN}Zos$.

Proof. (i) Suppose that $K \in \mathcal{PFN}\delta O(U, A)$. Then $K = \mathcal{PFN}\delta int(K)$. Since $L \in \mathcal{PFN}ZO(U, A)$, then $L \subseteq \mathcal{PFN}cl(\mathcal{PFN}\ \delta int(L)) \cup \mathcal{PFN}int(\mathcal{PFN}cl(L))$ and hence

$$K \cap L \subseteq \mathcal{PFN}\delta int(K) \cap (\mathcal{PFN}cl(\mathcal{PFN}\delta int(L)) \cup \mathcal{PFN}int(\mathcal{PFN}cl(L)))$$

$$= (\mathcal{PFN}\delta int(K) \cap \mathcal{PFN}cl(\mathcal{PFN}\delta int(L))) \cup (\mathcal{PFN}\delta int(K) \cap \mathcal{PFN}int(\mathcal{PFN}cl(L))$$
$$\subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(K) \cap (\mathcal{PFN}\delta int(L))) \cup \mathcal{PFN}int(\mathcal{PFN}int(K) \cap \mathcal{PFN}cl(L))$$
$$\subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(K \cap L)) \cup \mathcal{PFN}int(\mathcal{PFN}cl(K \cap L)).$$

Thus $K \cap L \subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(K \cap L)) \cup \mathcal{PFN}int(\mathcal{PFN}cl(K \cap L))$. Therefore, $K \cap L$ is $\mathcal{PFN}Zos$.

(ii) Suppose that $K \in \mathcal{PFN}aO(U, A)$ and $L \in \mathcal{PFN}ZO(U, A)$, The $K \cap L \subseteq \mathcal{PFN}int\left(\mathcal{PFN}cl\left(\mathcal{PFN}\delta int(K)\right)\right) \cap \left(\mathcal{PFN}cl\left(\mathcal{PFN}\delta int(L)\right) \cup \mathcal{PFN}int\left(\mathcal{PFN}cl(L)\right)\right)$

$$= (\mathcal{PFN}int(\mathcal{PFN}cl(\mathcal{PFN}\delta int(K))) \cap \mathcal{PFN}cl(\mathcal{PFN}\delta int(L))) \cup (\mathcal{PFN}int(\mathcal{PFN}cl$$
$$(\mathcal{PFN}\delta int(K))) \cap \mathcal{PFN}int(\mathcal{PFN}cl(L)))$$

$$\subseteq \mathcal{PFN}cl(\mathcal{PFN}cl(\mathcal{PFN}\delta int(K)) \cap \mathcal{PFN}\delta int(L)) \cup \mathcal{PFN}int(\mathcal{PFN}cl(\mathcal{PFN}\delta int(K))$$
$$\cap \mathcal{PFN}int(\mathcal{PFN}cl(L)))$$

$$\subseteq \mathcal{PFN}cl\left(\mathcal{PFN}cl\left(\mathcal{PFN}\delta int(K) \cap \mathcal{PFN}\delta int(L)\right)\right) \cup \mathcal{PFN}int\left(\mathcal{PFN}cl\left(\mathcal{PFN}\delta int(K)\right) \cap \mathcal{PFN}int\left(\mathcal{PFN}cl(L)\right)\right)$$

and hence

$$K \cap L \subseteq (K \cap \mathcal{PFN}cl(\mathcal{PFN}\delta int(K) \cap \mathcal{PFN}\delta int(L))) \cup (K \cap \mathcal{PFN}int(\mathcal{PFN}cl(\mathcal{PFN}\delta int(K)) \cap \mathcal{PFN}int(\mathcal{PFN}cl(L))))$$

$$\subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(K) \cap \mathcal{PFN}\delta int(L)) \cup \mathcal{PFN}int(\mathcal{PFN}cl(\mathcal{PFN}\delta int(K) \cap \mathcal{PFN}int(\mathcal{PFN}cl(L))))$$

$$\subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(K) \cap \mathcal{PFN}\delta int(L)) \cup \mathcal{PFN}int(\mathcal{PFN}cl(\mathcal{PFN}\delta int(K) \cap \mathcal{PFN}cl(L)))$$

$$\subseteq \mathcal{PFN}cl(\mathfrak{N}\delta int(K) \cap \mathcal{PFN}\delta int(L)) \cap \mathcal{PFN}int(\mathcal{PFN}cl(\mathcal{PFN}\delta int(K) \cap L)).$$

Since $\mathcal{PFN}\delta int(K) \cap \mathcal{PFN}\delta int(L) \subseteq \mathcal{PFN}\delta int(K) \subseteq K$ which is $\mathcal{PFN}\delta o$ in K, Then

$$K \cap L \subseteq \mathcal{PF}\mathfrak{N}cl\mathcal{PF}\mathfrak{N}\delta int(\mathcal{PF}\mathfrak{N}\delta int(K) \cap \mathcal{PF}\mathfrak{N}\delta int(L)) \cup \mathcal{PF}\mathfrak{N}int(K \cup \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(K) \cap L))$$

$$\subseteq \mathcal{PF}\mathfrak{N}cl\mathcal{PF}\mathfrak{N}\delta int(K \cap L) \cup \mathcal{PF}\mathfrak{N}int\mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(K) \cap L)$$

$$\subseteq \mathcal{PF}\mathfrak{N}cl\mathcal{PF}\mathfrak{N}\delta int(K \cap L) \cup \mathcal{PF}\mathfrak{N}int\mathcal{PF}\mathfrak{N}cl(K \cap L).$$

Therefore $K \cap L \in \mathcal{PF}\mathfrak{N}ZO(U, A)$.

Proposition 3.2 *Let $(U, \tau_{\mathcal{R}}(A))$ be a $\mathcal{PF}\mathfrak{N}ts$. Then the closure of a $\mathcal{PF}\mathfrak{N}Zo$ set of A is $\mathcal{PF}\mathfrak{N}Sos$.*

Proof. Let $K \in \mathcal{PF}\mathfrak{N}ZO(U, A)$.

Then $\mathcal{PF}\mathfrak{N}cl(K) \subseteq \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(K)) \subseteq \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(K))) \subseteq \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(K)) \cup \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(K))) = \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}cl(K)))$.

Therefore, $\mathcal{PF}\mathfrak{N}cl(K)$ is $\mathcal{PF}\mathfrak{N}Sos$.

Proposition 3.3 *Let K be a $\mathcal{PF}\mathfrak{N}Zo$ set of a $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_{\mathcal{R}}(A))$ and $\mathcal{PF}\mathfrak{N}\delta int(K) = 0_P$. Then K is $\mathcal{PF}\mathfrak{N}Pos$.*

Proof. Obvious.

Theorem 3.10 *Let K and L be two subsets of a $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_{\mathcal{R}}(A))$. Then the following are hold:*

(i) $\mathcal{PF}\mathfrak{N}Zcl(1_{\mathcal{P}} - K) = 1_{\mathcal{P}} - \mathcal{PF}\mathfrak{N}Zint(K)$,

(ii) $\mathcal{PF}\mathfrak{N}Zint(1_{\mathcal{P}} - K) = 1_{\mathcal{P}} - \mathcal{PF}\mathfrak{N}Zcl(K)$,

(iii) If $K \subseteq L$, then $\mathcal{PF}\mathfrak{N}Zcl(K) \subseteq \mathcal{PF}\mathfrak{N}Zcl(L)$ and $\mathcal{PF}\mathfrak{N}Zint(K) \subseteq \mathcal{PF}\mathfrak{N}Zint(L)$,

(iV) $l \in \mathcal{PF}\mathfrak{N}Zcl(K)$ iff for each $\mathcal{PF}\mathfrak{N}Zo$ set A contains l, $A \cap K \neq 0_P$,

(V) $l \in \mathcal{PF}\mathfrak{N}Zint(K)$ iff there exist a $\mathcal{PF}\mathfrak{N}Zo$ set W such that $l \in W \subseteq K$,

(Vi) $\mathcal{PF}\mathfrak{N}Zcl(\mathcal{PF}\mathfrak{N}Zcl(K)) = \mathcal{PF}\mathfrak{N}Zcl(K)$ and $\mathcal{PF}\mathfrak{N}Zint(\mathcal{PF}\mathfrak{N}Zint(K)) = \mathcal{PF}\mathfrak{N}Zint(K)$,

(Vii) $\mathcal{PF}\mathfrak{N}Zcl(K) \cup \mathcal{PF}\mathfrak{N}Zcl(L) \subseteq \mathcal{PF}\mathfrak{N}Zcl(K \cup L)$ and $\mathcal{PF}\mathfrak{N}Zint(K) \cup \mathcal{PF}\mathfrak{N}Zint(L) \subseteq \mathcal{PF}\mathfrak{N}Zint(K \cup L)$,

(Viii) $\mathcal{PF}\mathfrak{N}Zint(K \cap L) \subseteq \mathcal{PF}\mathfrak{N}Zint(K) \cap \mathcal{PF}\mathfrak{N}Zint(L)$ and $\mathcal{PF}\mathfrak{N}Zcl(K \cap L) \subseteq \mathcal{PF}\mathfrak{N}Zcl(K) \cap \mathcal{PF}\mathfrak{N}Zcl(L)$.

Proof. (i) It follows from Definition 3.5.

Remark 3.5 *By the following example we show that the inclusion relation in parts (vii) and (viii) of the above theorem cannot be replaced by equality.*

Example 3.4 *In Example 3.1, the sets*

1. $A = \left\{\left\langle\frac{s_1, s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.4}\right\rangle\right\}$ and $B = \left\{\left\langle\frac{s_1, s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.4}\right\rangle, \left\langle\frac{s_3}{0.6}\right\rangle\right\}$, then $A \vee B = \left\{\left\langle\frac{s_1, s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.6}\right\rangle\right\}$.

$\mathcal{PF}\mathfrak{R}Zint(A) = \left\{\left\langle\frac{s_1,s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.3}\right\rangle, \left\langle\frac{s_3}{0.4}\right\rangle\right\}$, $\mathcal{PF}\mathfrak{R}Zint(B) = \left\{\left\langle\frac{s_1,s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.4}\right\rangle, \left\langle\frac{s_3}{0.6}\right\rangle\right\}$ and $\mathcal{PF}\mathfrak{R}Zint(A \vee B) = \left\{\left\langle\frac{s_1,s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.6}\right\rangle\right\}$. Thus $\mathcal{PF}\mathfrak{R}Zint(A \vee B) \not\leq \mathcal{PF}\mathfrak{R}Zint(A) \vee \mathcal{PF}\mathfrak{R}Zint(B)$.

2. $C = \left\{\left\langle\frac{s_1,s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.7}\right\rangle\right\}$ and $D = \left\{\left\langle\frac{s_1,s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.4}\right\rangle, \left\langle\frac{s_3}{0.6}\right\rangle\right\}$, then $C \wedge D = \left\{\left\langle\frac{s_1,s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.4}\right\rangle, \left\langle\frac{s_3}{0.6}\right\rangle\right\}$. $\mathcal{PF}\mathfrak{R}Zint(C) = \left\{\left\langle\frac{s_1,s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.7}\right\rangle\right\}$, $\mathcal{PF}\mathfrak{R}Zint(D) = \left\{\left\langle\frac{s_1,s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.4}\right\rangle, \left\langle\frac{s_3}{0.6}\right\rangle\right\}$ and $\mathcal{PF}\mathfrak{R}Zint(C \wedge D) = \left\{\left\langle\frac{s_1,s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.3}\right\rangle, \left\langle\frac{s_3}{0.4}\right\rangle\right\}$.

Thus $\mathcal{PF}\mathfrak{R}Zint(C) \wedge \mathcal{PF}\mathfrak{R}Zint(D) \not\leq \mathcal{PF}\mathfrak{R}Zint(C \wedge D)$.

Theorem 3.11 *Let $(U, \tau_{\mathcal{R}}(A))$ be a $\mathcal{PF}\mathfrak{R}ts$ and $K \subseteq S$. Then K is a $\mathcal{PF}\mathfrak{R}Zo$ set iff $K = \mathcal{PF}\mathfrak{R}\delta\mathcal{S}int(K) \cup \mathcal{PF}\mathfrak{R}\mathcal{P}int(K)$.*

Proof. Let K be a $\mathcal{PF}\mathfrak{R}Zo$ set.

Then $K \subseteq \mathcal{PF}\mathfrak{R}cl(\mathcal{PF}\mathfrak{R}\delta int(K)) \cup \mathcal{PF}\mathfrak{R}int(\mathcal{PF}\mathfrak{R}cl(K))$ and hence by Proposition 3.1 and Lemma 3.3,

$\mathcal{PF}\mathfrak{R}\delta\mathcal{S}int(K) \cup \mathcal{PF}\mathfrak{R}\mathcal{P}int(K) = (S \cap \mathcal{PF}\mathfrak{R}cl(\mathcal{PF}\mathfrak{R}\delta int(K))) \cup (S \cap \mathcal{PF}\mathfrak{R}int(\mathcal{PF}\mathfrak{R}cl(K))) = K \cap (\mathcal{PF}\mathfrak{R}cl(\mathcal{PF}\mathfrak{R}\delta \quad int(K)) \cup \mathcal{PF}\mathfrak{R}int(\mathcal{PF}\mathfrak{R}cl(K))) = K$. The Converse, it follows from Proposition 3.1 and Lemma 3.3.

Proposition 3.4 *Let $(U, \tau_{\mathcal{R}}(A))$ be a $\mathcal{PF}\mathfrak{R}ts$ and $K \subseteq S$. Then K is a $\mathcal{PF}\mathfrak{R}Zc$ set iff $K = \mathcal{PF}\mathfrak{R}\delta\mathcal{S}cl(K) \cap \mathcal{PF}\mathfrak{R}\mathcal{P}cl(K)$.*

Proof. It follows from Theorem 3.11.

Theorem 3.12 *Let K be a subset of a space $(U, \tau_{\mathcal{R}}(A))$. Then:*

(i) $\mathcal{PF}\mathfrak{R}Zcl(K) = \mathcal{PF}\mathfrak{R}\delta\mathcal{S}cl(K) \cap \mathcal{PF}\mathfrak{R}\mathcal{P}cl(K)$,

(ii) $\mathcal{PF}\mathfrak{R}Zint(K) = \mathcal{PF}\mathfrak{R}\delta\mathcal{S}int(K) \cup \mathcal{PF}\mathfrak{R}\mathcal{P}int(K)$.

Proof. (i) It is easy to see that $\mathcal{PF}\mathfrak{R}Zcl(K) \subseteq \mathcal{PF}\mathfrak{R}\delta\mathcal{S}cl(K) \cap \mathcal{PF}\mathfrak{R}\mathcal{P}cl(K)$. Also, $\mathcal{PF}\mathfrak{R}\delta\mathcal{S}cl(K) \cap \mathcal{PF}\mathfrak{R}\mathcal{P}cl(K) = (K \cup \mathcal{PF}\mathfrak{R}int(\mathcal{PF}\mathfrak{R}\delta cl(K)) \cap (K \cup \mathcal{PF}\mathfrak{R}cl(\mathcal{PF}\mathfrak{R}int(K)) = K \cup (\mathcal{PF}\mathfrak{R}int(\mathcal{PF}\mathfrak{R}\delta cl(K)) \cap \mathcal{PF}\mathfrak{R}cl(\mathcal{PF}\mathfrak{R}int(K)))$. Since $\mathcal{PF}\mathfrak{R}Zcl(K)$ is $\mathcal{PF}\mathfrak{R}Zcs$, then $\mathcal{PF}\mathfrak{R}Zcl(K) \subseteq \mathcal{PF}\mathfrak{R}int(\mathcal{PF}\mathfrak{R}\delta cl(\mathcal{PF}\mathfrak{R}Zcl(K))) \cap \mathcal{PF}\mathfrak{R}cl(\mathcal{PF}\mathfrak{R}int(\mathcal{PF}\mathfrak{R}Zcl(K))) \supseteq \mathcal{PF}\mathfrak{R}int(\mathcal{PF}\mathfrak{R}\delta cl(K)) \cap \mathcal{PF}\mathfrak{R}cl(\mathcal{PF}\mathfrak{R}int(K))$.

Thus $K \cup (\mathcal{PF}\mathfrak{R}int(\mathcal{PF}\mathfrak{R}\delta cl(K)) \cap \mathcal{PF}\mathfrak{R}cl(\mathcal{PF}\mathfrak{R}int(K))) \subseteq K \cup \mathcal{PF}\mathfrak{R}Zcl(K) = \mathcal{PF}\mathfrak{R}Zcl(K)$ and hence, $\mathcal{PF}\mathfrak{R}\delta\mathcal{S} \quad cl(K) \cap \mathcal{PF}\mathfrak{R}\mathcal{P}cl(K) \subseteq \mathcal{PF}\mathfrak{R}Zcl(K)$. So, $\mathcal{PF}\mathfrak{R}Zcl(K) = \mathcal{PF}\mathfrak{R}\delta\mathcal{S}cl(K) \cap \mathcal{PF}\mathfrak{R}\mathcal{P}cl(K)$.

(ii) It follows from (i).

Theorem 3.13 *Let K be a Pythagorean fuzzy subset of a space $(U, \tau_{\mathcal{R}}(A))$ Then*

(i) K is a $\mathcal{PF}\mathfrak{R}Zo$ set iff $K = \mathcal{PF}\mathfrak{R}Zint(K)$,

(ii) K is a $\mathcal{PF}\mathfrak{R}Zc$ set iff $K = \mathcal{PF}\mathfrak{R}Zcl(K)$.

Proof. (i) It follows from Theorems 3.11 & 3.12.

Lemma 3.5 *Let* K *be a* pfs *of a* $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_{\mathcal{R}}(A))$. *Then the following statement are hold:*

(i) $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}int(\mathcal{PF}\mathfrak{N}\mathcal{P}cl(K)) = \mathcal{PF}\mathfrak{N}\mathcal{P}cl(K) \cap \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\delta cl(K))$,

(ii) $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(\mathcal{PF}\mathfrak{N}\mathcal{P}int(K)) = \mathcal{PF}\mathfrak{N}\mathcal{P}int(K) \cup \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(K))$.

Proof. (i) By Lemma 3.4,

$\mathcal{PF}\mathfrak{N}\delta\mathcal{P}int(\mathcal{PF}\mathfrak{N}\mathcal{P}cl(K)) = \mathcal{PF}\mathfrak{N}\mathcal{P}cl(K) \cap \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\delta cl(\mathcal{PF}\mathfrak{N}\mathcal{P}cl(K))) = \mathcal{PF}\mathfrak{N}\mathcal{P} cl(K) \cap \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\delta cl(K \cup \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N} cl(K)))) = \mathcal{PF}\mathfrak{N}\mathcal{P}cl(K) \cap \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\delta cl(K))$.

(ii) It follows from (i).

Proposition 3.5 *Let* K *be a* pfs *of a* $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_{\mathcal{R}}(A))$. *Then:*

(i) $\mathcal{PF}\mathfrak{N}Zcl(K) = K \cup \mathcal{PF}\mathfrak{N}\delta\mathcal{P}int(\mathcal{PF}\mathfrak{N}\mathcal{P}cl(K))$,

(ii) $\mathcal{PF}\mathfrak{N}Zint(K) = K \cap \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(\mathcal{PF}\mathfrak{N}\mathcal{P}int(K))$.

Proof. (i) By Lemma 3.5, $K \cup \mathcal{PF}\mathfrak{N}\delta\mathcal{P}int(\mathcal{PF}\mathfrak{N}\mathcal{P}cl(K)) = K \cup (\mathcal{PF}\mathfrak{N}\mathcal{P}cl(K) \cap \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\delta cl(K))) = (K \cup \mathcal{PF}\mathfrak{N}\mathcal{P}cl(K)) \cap (K \cup \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\delta cl(K))) = \mathcal{PF}\mathfrak{N}\mathcal{P}cl(K) \cap \mathcal{PF}\mathfrak{N}\delta\mathcal{S}cl(K) = \mathcal{PF}\mathfrak{N}Zcl(K)$.

(ii) It follows from (i).

Theorem 3.14 *Let* K *be a fuzzy subset of a* $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_{\mathcal{R}}(A))$. *Then the following are equivalent:*

(i) K is a $\mathcal{PF}\mathfrak{N}Zo$ set,

(ii) $K \subseteq \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(\mathcal{PF}\mathfrak{N}\mathcal{P}int(K))$,

(iii) there exists $O \in \mathcal{PF}\mathfrak{N}\mathcal{P}O(A)$ such that $O \subseteq K \subseteq \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(O)$,

(iV) $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(K) = \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(\mathcal{PF}\mathfrak{N}\mathcal{P}int(K))$.

Proof. (i) ⇒ (ii): Let K be a $\mathcal{PF}\mathfrak{N}Zo$ set. Then by Theorem 3.13, $K = \mathcal{PF}\mathfrak{N}Zint(K)$ and by Proposition 3.5, $K = K \cap \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(\mathcal{PF}\mathfrak{N}\mathcal{P}int(K))$ and hence, $K \subseteq \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(\mathcal{PF}\mathfrak{N}\mathcal{P}int(K))$.

(ii) ⇒ (i): Let $K \subseteq \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(\mathcal{PF}\mathfrak{N}\mathcal{P}int(K))$. Then by Proposition 3.5, $K \subseteq K \cap \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(\mathcal{PF}\mathfrak{N}\mathcal{P}int(K)) = \mathcal{PF}\mathfrak{N}Zint(K)$, and hence $K = \mathcal{PF}\mathfrak{N}Zint(K)$. Thus K is $\mathcal{PF}\mathfrak{N}Zos$.

(ii) ⇒ (iii): It follows from putting $O = \mathcal{PF}\mathfrak{N}\mathcal{P}int(K)$,

(iii) ⇒ (ii). Let there exists $O \in \mathcal{PF}\mathfrak{N}\mathcal{P}O(A)$ such that $O \subseteq K \subseteq \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(O)$. Since $O \subseteq K$, then $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(O) \subseteq \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(\mathcal{PF}\mathfrak{N}\mathcal{P}int(K))$, therefore $K \subseteq \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(O) \subseteq \mathcal{PF}\mathfrak{N}\delta\mathcal{P}cl(\mathcal{PF}\mathfrak{N}\mathcal{P}int(K))$.

(ii) ⇔ (iv): It is clear.

Theorem 3.15 *Let* K *be a* pfs *of a* $\mathcal{PF}\mathfrak{N}ts$ $(U, \tau_{\mathcal{R}}(A))$. *Then the following are equivalent:*

(i) K is a $\mathcal{PF}\mathfrak{N}Zc$ set,

(ii) $\mathcal{PFN}\delta\mathcal{P}int(\mathcal{PFNP}cl(K)) \subseteq K$,

(iii) there exists $O \in \mathcal{PFNPC}(A)$ such that $\mathcal{PFN}\delta\mathcal{P}int(O) \subseteq K \subseteq O$,

(iV) $\mathcal{PFN}\delta\mathcal{P}int(K) = \mathcal{PFN}\delta\mathcal{P}int(\mathcal{PFNP}cl(K))$.

Proof. It follows from Theorem 3.14.

Proposition 3.6 *If K is a $\mathcal{PFN}Zo$ set of a $\mathcal{PFN}ts$ $(U, \tau_{\mathcal{R}}(A))$ such that $K \subseteq L \subseteq \mathcal{PFN}\delta\mathcal{P}cl(K)$, then L is $\mathcal{PFN}Zo$.*

Proof. It is clear.

Definition 3.7 *A set K of a $\mathcal{PFN}ts$ $(U, \tau_{\mathcal{R}}(A))$ is said to be locally $\mathcal{PFN}Zcs$ if $K = O \cap S$, where $O \in O(A)$ and $S \in \mathcal{PFN}ZC(A)$.*

Theorem 3.16 *Let H be a pfs of a $\mathcal{PFN}ts$ $(U, \tau_{\mathcal{R}}(A))$. Then H is locally $\mathcal{PFN}Zc$ iff $H = O \cap \mathcal{PFN}Zcl(H)$.*

Proof. Since H is a locally $\mathcal{PFN}Zc$ set, then $H = O \cap S$, where $O \in O(A)$ and $S \in \mathcal{PFN}ZC(A)$ and hence $H \subseteq \mathcal{PFN}Zcl(H) \subseteq \mathcal{PFN}Zcl(S) = S$. Thus $H \subseteq O \cap \mathcal{PFN}Zcl(H) \subseteq O \cap \mathcal{PFN}Zcl(S) = H$.

Therefore $H = O \cap \mathcal{PFN}Zcl(H)$. The Converse is clear.

Theorem 3.17 *Let K be a locally $\mathcal{PFN}Zc$ set of a space $(U, \tau_{\mathcal{R}}(A))$. Then the following statements are hold:*

(i) $\mathcal{PFN}Zcl(K) - K$ is a $\mathcal{PFN}Zc$ set,

(ii) $(K \cup (1_{\mathcal{P}} - \mathcal{PFN}Zcl(K)))$ is a $\mathcal{PFN}Zo$,

(iii) $K \subseteq \mathcal{PFN}Zint(K \cup (1_{\mathcal{P}} - \mathcal{PFN}Zcl(K)))$.

Proof. (i) If K is a locally $\mathcal{PFN}Zc$ set, then there exists an $\mathcal{PFN}o$ set O such that $K = O \cap \mathcal{PFN}Zcl(K)$. Hence, $\mathcal{PFN}Zcl(K) - K = \mathcal{PFN}Zcl(K) - (O \cap \mathcal{PFN}Zcl(K)) = \mathcal{PFN}Zcl(K) \cap (1_{\mathcal{P}} - (O \cap \mathcal{PFN}Zcl(K))) = \mathcal{PFN}Zcl(K) \cap (1_{\mathcal{P}} - O) \cup (1_{\mathcal{P}} - \mathcal{PFN}Zcl(K))) = \mathcal{PFN}Zcl(K) \cap (1_{\mathcal{P}} - O)$ which is $\mathcal{PFN}Zc$.

(ii) Since $\mathcal{PFN}Zcl(K) - K$ is $\mathcal{PFN}Zc$, then $1_{\mathcal{P}} - (\mathcal{PFN}Zcl(K) - K)$ is a $\mathcal{PFN}Zo$ set. Since $1_{\mathcal{P}} - (\mathcal{PFN}Zcl(K) - K) = ((1_{\mathcal{P}} - \mathcal{PFN}Zcl(K)) \cup (1_{\mathcal{P}} \cap K)) = (K \cup (1_{\mathcal{P}} - \mathcal{PFN}Zcl(K)))$, then $K \cup (1_{\mathcal{P}} - \mathcal{PFN}Zcl(K))$ is $\mathcal{PFN}Zos$.

(iii) It follows from (ii).

Definition 3.8 *A Pythagorean fuzzy set K of a space $(U, \tau_{\mathcal{R}}(A))$ is said to be Pythagorean fuzzy nano $D(c, z)$ (briefly, $\mathcal{PFN}D(c, z)$) iff $\mathcal{PFN}int(K) = \mathcal{PFN}Zint(K)$.*

Remark 3.6 *One may notice that the concepts of $\mathcal{PFN}Zo$ and $\mathcal{PFN}D(c, z)$ are independent and by we show this the following example.*

Example 3.5 *In Example 3.1, the nano sets*

1. $\left\{\left\langle\frac{s_1,s_4}{0.3}\right\rangle,\left\langle\frac{s_2}{0.5}\right\rangle,\left\langle\frac{s_3}{0.7}\right\rangle\right\}$ is a $\mathcal{PF}\mathfrak{N}Zos$ but not $\mathcal{PF}\mathfrak{N}D(c,z)$.
2. $\left\{\left\langle\frac{s_1,s_4}{0.1}\right\rangle,\left\langle\frac{s_2}{0.3}\right\rangle,\left\langle\frac{s_3}{0.5}\right\rangle\right\}$ is a $\mathcal{PF}\mathfrak{N}D(c,z)$ but not $\mathcal{PF}\mathfrak{N}Zos$.

Theorem 3.18 *Let K be a pfs of $\mathcal{PF}\mathfrak{N}ts$ $(U,\tau_{\mathcal{R}}(A))$. Then the following are equivalent:*

1. K is an $\mathcal{PF}\mathfrak{N}o$ set,
2. K is $\mathcal{PF}\mathfrak{N}Zo$ and $\mathcal{PF}\mathfrak{N}D(c,z)$.

Proof. Obvious.

4 Application

Entropy as a measure of fuzziness was first proposed by Zadeh [21]. Later many mathematicians defined several entropy measures. In this section, we focus on defining an entropy measure for pfs that connects the degree of membership and non-membership. As an example, we have applied the proposed entropy measure in decision making.

Definition 4.1 *Let $A=\{<x,\mu_A(x),\lambda_A(x)|x\in X\}$ be a pfs in X. The new entropy measure for A denoted by $\varepsilon_{pfs}(A)$, is a function, $\varepsilon_{pfs}:\tau_{pfs}(X)\to[0,1]$ and is defined as $\varepsilon_{pfs}(A)=1-\frac{1}{n}\sum_{i=1}^{n}(\mu_A-\lambda_A)^2;\ forevery`x_i\in A$, where $\tau_{pfs}(X)$ denote the family of all pfs's on X.*

Example 4.1 *The association of the tourism wants to announce that best "Hotel of the year", for each year. The actual problem is, they want to select the best hotel based on reviews and ratings. There are four nominees namely Hotel 1, Hotel 2, Hotel 3 and Hotel 4 for this award and they have to reviewed based on the four criteria namely Ambiance, Good food, Clean and Tidy, Cyber security facility. Here we use entropy measure to find the best hotel by the overall entropy measure with the Pythagorean fuzzy sets.*

Table 1. Reviews of the Hotels based on the Criteria

	Criteria 1 (C_1)	Criteria 2 (C_2)	Criteria 3 (C_3)	Criteria 4 (C_4)
Hotel 1 (H_1)	$<H_1,C_1;0.9,0.3>$	$<H_1,C_2;0.7,0.6>$	$<H_1,C_3;0.5,0.8>$	$<H_1,C_4;0.6,0.4>$
Hotel 2 (H_2)	$<H_2,C_1;0.7,0.1>$	$<H_2,C_2;0.9,0.2>$	$<H_2,C_3;0.8,0.1>$	$<H_2,C_4;0.6,0.3>$
Hotel 3 (H_3)	$<H_3,C_1;0.8,0.4>$	$<H_3,C_2;0.7,0.5>$	$<H_3,C_3;0.6,0.2>$	$<H_3,C_4;0.7,0.5>$
Hotel 4 (H_4)	$<H_4,C_1;0.7,0.2>$	$<H_4,C_2;0.8,0.2>$	$<H_4,C_3;0.8,0.4>$	$<H_4,C_4;0.6,0.6>$

Clearly, all values in the Table 1 are pfs's. Now we calculate the ε_{pfs} of each Hotel.

Table 2. Entropy measure of each Hotel.

	$\varepsilon_{pfs}(H_i)$
H_1	0.87

H_2	**0.64**
H_3	0.9
H_4	0.81

From Table 2, Clearly that $\varepsilon_{pfs}(H_2) < \varepsilon_{pfs}(H_4) < \varepsilon_{pfs}(H_1) < \varepsilon_{pfs}(H_3)$.

Hence we conclude that H_2 is the best Hotel of the year with less fuzziness.

5 Conclusion

In this paper, we have studied a new class of sets called Pythagorean fuzzy nano Z-open sets in Pythagorean fuzzy nano topological spaces and their properties, and also discussed about Pythagorean fuzzy nano Z-closure, Pythagorean fuzzy nano Z-interior and their relations with already existing well known fuzzy sets. In future, this can be extended to Pythagorean fuzzy nano Z continuous function, Pythagorean fuzzy nano Z open mapping, Pythagorean fuzzy nano Z closed mapping and Pythagorean fuzzy nano Z homeomorphic functions.

References

[1] D. Ajay and J. Joseline Charisma, *Pythagorean nano topological space*, International Journal of Recent Technology and Engineering, **8** (2020), 3415-3419.

[2] D. Ajay and J. Joseline Charisma, *On weak forms of Pythagorean nano open sets*, Advances in Mathematics: Scientific Journal, **9** (2020), 5953-5963.

[3] X. Arul Selvaraj and U. Balakrishna, *Z-open sets in nano topological spaces*, AIP Conference Proceedings, **2364** (2021), 020037.

[4] K. T. Atanassov, *Intuitionistic fuzzy sets*, VII ITKR's Session, Sofia (deposed in Central Sci.-Technical Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian), (1983).

[5] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182-190.

[6] D. Coker, *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy sets and systems, **88** (1997), 81-89.

[7] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, Journal of Mathematical analysis and applications, **56** (3) (1976), 621â€"633.

[8] A. I. EL-Magharabi and A. M. Mubarki, *Z-open sets and Z-continuity in topological spaces*, International Journal of Mathematical Archive, **2** (10) (2011), 1819-1827.

[9] M. Lellis Thivagar and C. Richard, *On nano forms of weekly open sets*, International journal of mathematics and statistics invention, **1** (1) (2013), 31-37.

[10] M. Lellis Thivagar, S. Jafari, V. Sutha Devi and V. Antonysamy, *A novel approach to nano topology via neutrosophic sets*, Neutrosophic Sets and Systems, **20** (2018), 86-94.

[11] Murat Olgun, Mehmet Unver and Seyhmus Yardimci (2019), *Pythagorean fuzzy topological*

spaces, Complex & Intelligent Systems, 177-183. https://doi.org/10.1007/s40747-019-0095-2.

[12] A. Padma, M. Saraswathi, A. Vadivel and G. Saravanakumar, *New Notions of Nano M-open Sets*, Malaya Journal of Matematik, **S** (1) (2019), 656-660.

[13] V. Pankajam and K. Kavitha, *δ-open sets and δ-nano continuity in δ-nano topological spaces*, International Journal of Innovative Science and Research Technology, **2** (12) (2017), 110-118.

[14] Shiventhiradevi Sathaananthan, S. Tamilselvan, A. Vadivel and G. Saravanakumar, *Fuzzy $\mathcal{Z}$ closed sets in double fuzzy topological spaces,* AIP Conf Proc., **2277** (2020), 090001.

[15] Shiventhiradevi Sathaananthan, A. Vadivel, S. Tamilselvan and G. Saravanakumar, *Generalized fuzzy $\mathcal{Z}$ closed sets in double fuzzy topological spaces,* Adv. Math: Sci. J., **9** (4) (2020), 2107-2112.

[16] R. Thangammal, M. Saraswathi, A. Vadivel and C. John Sundar, *Fuzzy Nano Z-open Sets in Fuzzy Nano Topological Spaces*, Journal of Linear and Topological Algebra Vol. 11, No. 01, 2022, 27- 38.

[17] A. Vadivel, C. John Sundar, K. Kirubadevi and S. Tamilselvan, *More on Neutrosophic Nano Open Sets*, International Journal of Neutrosophic Science (IJNS), **18** (4) (2022), 204-222.

[18] R. R. Yager (2013), *Pythagorean membership grades in multicriteria decision making*, In: Technical report MII-3301. Machine Intelligence Institute, Iona College, New Rochelle.

[19] R. R. Yager (2013), *Pythagorean fuzzy subsets*, In: Proceedings of the joint $IFSA$ world congress $NAFIPS$ annual meeting, 57-61.

[20] R. R. Yager (2014), *Pythagorean membership grades in multicriteria decision making*, $IEEE$ Trans Fuzzy Syst. **22** (4), 958-965.

[21] L. A. Zadeh, *Fuzzy sets*, Information and control, **8** (1965), 338-353.